Classification of Infinite Sets and Negative Proof of the Continuum Hypothesis

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Abstract:
This paper constructs a series of transcendental subsets of real numbers by creating a construction method based on infinite sets of transcendental numbers. It also proposes the concept that infinite sets can be divided into one-dimensional, two-dimensional, and multi-dimensional sets according to the number of degrees of freedom required for the expansion of their elements. For example, the set of natural numbers, the set of even numbers, and the set of odd numbers belong to one-dimensional infinite sets, while rational numbers belong to two-dimensional infinite sets, and so on. When the number of degrees of freedom is infinite, it becomes an infinite-dimensional set, and several infinite-dimensional sets can constitute a hyperdimensional set.

This paper proves that the set of real numbers belongs to the hyperdimensional set, and its cardinality, $\aleph_C$, is far greater than the infinite power set of the cardinality of the set of natural numbers, $\aleph_0$; between the set of natural numbers and the set of real numbers, there exist infinitely many number sets. The result of this paper disproves the continuum hypothesis.

Keywords:
Set, Cardinality, Transcendental number, Real number, Dimension, Continuum Hypothesis, Hilbert's first problem

1.Introduction

Georg Cantor first introduced the concept of set cardinality in the 1870s and was the first to classify infinity, thereby expanding human understanding of infinite sets. Cantor proved that the cardinality of the power set of a set is greater than that of the original set, and he proposed the continuum hypothesis, stating that there is no set whose cardinality is absolutely greater than that of the countable set and absolutely less than that of the real number set. [1,2]

Hilbert proposed the famous 23 unsolved problems at the International Congress of Mathematicians held in Paris in 1900, and listed the Continuum Hypothesis (CH) as the first problem. [3] Up to now, this hypothesis remains to be proven or disproven.

The hierarchical classification of infinite sets is an important and difficult problem. According to set theory, there are countable infinity represented by natural numbers, and uncountable infinity represented by real numbers. Whether sets with other cardinal numbers exist remains unclear, with no relevant theories or examples found.

Proofs or refutations of the continuum hypothesis mostly stem from set theory axioms, involving logic or philosophy. There has been no discussion based on first principles that directly addresses the construction of subsets of real numbers.

This paper constructs a series of transcendental subsets of real numbers by creating a construction method based on the infinite set of transcendental numbers. It also proposes the concept that infinite sets can be divided into one-dimensional, two-dimensional, and multi-dimensional sets according to the number of degrees of freedom required for the expansion of their elements. For example, the set of natural numbers, the set of even numbers, and the set of odd numbers belong to one-dimensional infinite sets, while rational numbers belong to two-dimensional sets, and so on. When the number of degrees of freedom is infinitely large, it becomes an infinite-dimensional set, and several infinite-dimensional sets can constitute a hyperdimensional set.

This paper proves that the set of real numbers belongs to the hyperdimensional set, and its cardinality, $\aleph_C$, is far greater than the infinite power set of the cardinality of the set of natural numbers, $\aleph_0$; between the set of natural numbers and the set of real numbers, there exist infinitely many number sets. The result of this paper disproves the continuum hypothesis.

2. Cardinality of sets and the continuum hypothesis

The cardinality of a set, also known as the potential of a set, is a fundamental concept in set theory, used to characterize the size of the set or to denote the number of elements in the set.

Cantor proposed using the zero-based indexed Hebrew letters $\aleph_{k}$ to represent the size of an infinite set, $\aleph_{0}$, $\aleph_{1}$, $\aleph_{2}$, ..., ..., where each successor is the cardinality of the power set of the previous one, $\aleph_{0}$ corresponds to the cardinality of the set of natural numbers, $\aleph_{1}$ to the cardinality of the set of real numbers, and so on. Thus, the continuum hypothesis is expressed as

$$\aleph_1 = 2^{\aleph_0}$$

The Generalized Continuum Hypothesis (GCH) states that

$$\aleph_{k+1} = 2^{\aleph_k}, k = 0,1,2,...$$

That is, $\aleph_{k+1}$, the successor of every $\aleph_{k}$, is the cardinality of the power set of the predecessor.

To the author's knowledge, apart from $\aleph_{1}$, the cardinality of real numbers under the continuum hypothesis to be proven, no reports have been found of infinite sets with cardinality $\aleph_{2}$ or larger. Clearly, research on the cardinality of infinite sets is still in its early stages.

3. Construction of sets of transcendental numbers and negative proof of the continuum hypothesis

3.1 Construction of set A of transcendental numbers

According to the definition, transcendental numbers within the real numbers are not roots of any algebraic equation with integer coefficients, and they retain their transcendental nature even after being subjected to finite elementary function operations. Proving whether a number is transcendental is a challenging problem, and currently, only a few transcendental numbers are known. Nevertheless, we can utilize the properties of transcendental numbers to construct sets with special characteristics.

Define the set A as consisting of the transcendental number e plus natural numbers, with its elements being e+1, e+2, e+3, ..., e+n. Here, n is a natural number that tends towards infinity.

$$A = \{a_k | \ e + k, k = 1,2,3,...\}$$

Clearly, due to the existence of the transcendental number e, set A is a set of transcendental numbers, which is a subset of real numbers. Its cardinality is $\aleph_0$, consistent with that of natural numbers.

3.2 Construction of set B of transcendental numbers

Define set B as a set consisting of the radical operations in the elements of set A of transcendental numbers, where the degree of radical operations is chosen as corresponding to the natural number.

$$B = \{b_L| \sqrt[L]{e+k}, k = 1,2,3,...; L = 2,3,4,...\}$$

To fully display the elements in set B, two-dimensional parameters need to be expanded and arranged into an infinite table, where the row and column indices are natural numbers, as detailed in Table 1.

Table 1: Element 1 of Set B.

	1	2	3	...
1	$\sqrt[2]{e+1}$	$\sqrt[2]{e+2}$	$\sqrt[2]{e+3}$	...
2	$\sqrt[3]{e+1}$	$\sqrt[3]{e+2}$	$\sqrt[3]{e+3}$	...
3	$\sqrt[4]{e+1}$	$\sqrt[4]{e+2}$	$\sqrt[4]{e+3}$	...
...				

The significance of the set B, chosen in this way, lies in the fact that its elements are transcendental numbers. For any element, there is no possibility of escaping from radical expressions, and their algebraic sum cannot undergo "fusion". Clearly, the cardinality of set B is $\aleph_0$^2.

3.3 Construction of set F of transcendental numbers

The elements of the defined set F are composed of elements from set B divided by 10^n, where n is a natural number.

$$F = \{f_n| \frac{\sqrt[L]{e+k}}{10^n}, n = 1,2,3,...; k = 1,2,3,...; L = 2,3,4,...\}$$

To fully display the elements of set F, it requires the expansion of parameters in three dimensions. The method is to present each row in the table of the transcendental number subset B as an infinite table of the transcendental number subset F, where both row and column indices are natural numbers.

A total of n tables are required, such as Table 2, Table 3, ..., Table n, ..,. ... where n is a natural number. The results are presented in the first row of Table B, as detailed in Table 2.

Table 2: Presentation of the first row in Table 1 of Set B

	1	2	3	...
1	$\frac{\sqrt[2]{e+1}}{10}$	$\frac{\sqrt[2]{e+2}}{10}$	$\frac{\sqrt[2]{e+3}}{10}$	...
2	$\frac{\sqrt[2]{e+1}}{100}$	$\frac{\sqrt[2]{e+2}}{100}$	$\frac{\sqrt[2]{e+3}}{100}$	...
3	$\frac{\sqrt[2]{e+1}}{1000}$	$\frac{\sqrt[2]{e+2}}{1000}$	$\frac{\sqrt[2]{e+3}}{1000}$	...
...				

Clearly, the set F is a set of transcendental numbers, a subset of the real numbers, and its cardinality is $\aleph_0$^$\aleph_0$.

3.4 Negative proof of the continuum hypothesis

According to the power operation of cardinality in set theory, the cardinality of set F is $\aleph_F$=$\aleph_0$^$\aleph_0$≥ $\aleph_1$, where $\aleph_1$ is the cardinality of the power set of $\aleph_0$.

Based on set F, repeating the above operations yields a subset of transcendental numbers with larger cardinality, constructing a subset of transcendental numbers that is a power set of infinite cardinal superpositions containing the transcendental number e, such as $\aleph_0$^$\aleph_0$^$\aleph_0$, $\aleph_0$^$\aleph_0$^$\aleph_0$^$\aleph_0$, ..., ...,, $\aleph_0$^$\aleph_0$^$\aleph_0$ ... ^$\aleph_0$ or $\aleph_n$^$\aleph_n$, where n is a natural number. Clearly, choosing different transcendental numbers allows the construction of different subsets of transcendental numbers that are power sets of infinite cardinal superpositions.

Since transcendental numbers belong to the real numbers, and the real numbers include an uncountable number of transcendental numbers, it is proven that the cardinality of the set of real numbers $\aleph_c$>>$\aleph_0$^$\aleph_0$^$\aleph_0$ ... ^$\aleph_0$, that is, the cardinality $\aleph_c$ of the set of real numbers is a superimposed set of infinite powers of the cardinality $\aleph_0$ of the set of natural numbers.

Therefore, the above reasoning proves that there exist infinitely many sets of numbers between the set of natural numbers and the set of real numbers. The continuum hypothesis states that there are no other sets between the set of natural numbers and the set of real numbers， whose cardinality lies between the two. Therefore, the above result negates the continuum hypothesis. [4,5,6]

4. Discussions and Conclusions

The hierarchical classification of infinite sets is an important and challenging problem. According to the classification method of set theory, there can be countable infinity represented by natural numbers, and there can be uncountable infinity represented by real numbers. Whether there are sets of other cardinalities has not been theoretically proven or exemplified. Apart from the need to prove the cardinality of real numbers $\aleph_1$ under the continuum hypothesis, there are no reports of infinite sets with cardinality $\aleph_2$ or larger.

Regarding the proof or disproof of the continuum hypothesis, discussions often start from set-theoretical axioms and involve logic or philosophy. However, there has yet to be a discussion based on first principles that directly starts from the construction of subsets of real numbers.

This paper constructs a series of transcendental subsets of real numbers by creating a construction method based on infinite sets of transcendental numbers. It also proposes the concept that infinite sets can be divided into one-dimensional, two-dimensional, and multi-dimensional sets according to the number of degrees of freedom required for the expansion of their elements. For example, the set of natural numbers, the set of even numbers, and the set of odd numbers belong to one-dimensional infinite sets, while rational numbers belong to two-dimensional infinite sets, and so on. When the degrees of freedom are infinite, it becomes an infinite-dimensional set, and several infinite-dimensional sets can constitute a hyperdimensional set.

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