Explaining the spin, Zeeman effect and Lamb shift in Sommerfeld's atomic model

Jorma Jormakka
Vantaa, Finland
jorma.o.jormakka@gmail.com

Abstract: It is claimed in Quantum Mechanics that Sommerfeld's atomic model is inable to describe the full spectrum of hydrogen. This claim is justified by stating that Sommerfeld's atomic model cannot describe spin in the fine structure formula and that it cannot explain the Zeeman effect. Both claims are shown incorrect in the article. The article points to two serious errors in the Quantum Mechanical derivation of the fine structure spectrum formula and the Zeeman effect. The first is that there is a confusion of angular momentum and radial momentum: these effects are effects of radial momentum. The operators J and L in Quantum Mechanics, given as angular momentums, are actually radial momentums both in the fine structure spectrum formula and in the Zeeman effect energy levels. The second error is in the calculation of eigenvalues of a sum of operators as the sum of the eigenvalues of the summand operators. The last section comments on the last claimed gap in Sommerfeld's atomic model: that it cannot explain the Lamb shift. It is argued that the explanation of the Lamb shift in Quantum Mechanics is not a valid explanation as Quantum Field Theory is shown incorrect by the anomalous magnetic moment of an electron. In Sommerfeld's atomic model the states $2S_{1/2}$ and $2P_{1/2}$ have different eccentricity and therefore different energy level.

Keywords: Sommerfeld, fine structure spectrum, spin, Zeeman effect, Lamb shift.

1. Inserting spin into Sommerfeld's fine structure formula

In [1] there is a full derivation of Sommerfeld's fine structure formula. It is based in the derivation in [2]. The formula is derived in [1] for hydrogen ($Z = 1$), it can be generalized for any Z by replacing α by αZ. Sommerfeld's formula is therefore

$$E_{n,n_\theta} = E - m_e c^2 = -\frac{\alpha^2 Z^2}{2n^2} - \frac{\alpha^4 Z^4}{8 n_\theta n^4}(4n - 3n_\theta) + O(\alpha^6). \tag{1}$$

The fine structure spectrum formula from Quantum Mechanics is

$$E_{n,j} = E - mc^2 = E_n \left(1 + \frac{Z^2\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)\right) + O(\alpha^6). \tag{2}$$

where

$$E_n = -\frac{m_e c^2 \alpha^2 Z^2}{2n^2} \tag{3}$$

is the Bohr energy. Writing the Quantum Mechanical formula in a similar form as (1) gives

$$E_{n,j} = -\frac{\alpha^2 Z^2}{2n^2} - \frac{\alpha^4 Z^4}{8n_\theta n^4}\left(4n\frac{n_\theta}{j+\frac{1}{2}} - 3n_\theta\right) + O(\alpha^6). \tag{4}$$

The Quantum Mechanical formula describes the fine structure spectrum correctly and Sommerfeld's formula describes the fine structure also correctly except that it does not include spin, but it has a very convincing derivation, see [1]. Therefore both formulas are correct and must give the same answer.

As $j\hbar$ is the eigenvalue of total angular momentum in the Quantum Mechanic model and $n_\theta\hbar$ is quantized angular momentum of an elliptic orbit in Sommerfeld's model, we are tempted to make the identification

$$n_\theta = j + \frac{1}{2}. \tag{5}$$

This is wrong. There is another possibility for the identification. When n is fixed, n_θ ranges from 1 to n and $n_r = n - n_\theta$ ranges from 0 to $n-1$. We must identify n_r with l where $n = l \pm s$ is the orbital angular momentum of Quantum Mechanics, which ranges from 0 to $n-1$. All spectrum lines are correctly obtained by the identification $n_r = l$. It is the correct identification for spin taken as $-1/2$ and the confusion is in Quantum Mechanics: l is not orbital angular momentum, n_r is a quantum number obtained by integrating radial momentum over r, so it is work done by the magnetic field of a proton in giving the orbit of an electron precession.

The source of this error in Quantum Mechanics (QM) is in QM derivation of the fine structure spectrum formula. There the spin-orbit effect is incorrectly understood as the magnetic field of the proton doing work that changes the orbit of an electron radially. There is a radial force caused by the magnetic field of a proton, but this force cannot make any work. We know this because electrons of an atom do not radiate though their orbit is curved. This is because the orbits are quantized and cannot get gradually smaller. The only way for an electron to get to a different orbit is to make a jump and it requires emission or absorbtion of a photon. The only component of the force that the magnetic field of a proton causes to an electron on an elliptic orbit that can do work is angular: this force gives the orbit precession. All orbits, also the circular orbits, in Sommerfeld's model have precession, unlike in a planetary system where planet orbits do not need to precess (some do, like Mercuty's orbit, it is caused by the effects of other planets). Planets do not have a charge and the magnetic field of the Sun is weak, this is why the orbits do not need to precess. In an atom, electrons have a charge and the magnetic field of a proton is not that weak. It causes all orbits (circular or elliptic) to precess. This precession requires work and the force cannot accelerate the precession speed beyond some limit. The result is a stable precessing orbit where the precession speed has kinetic energy. This kinetic energy changes the energy levels of the orbits and a photon that is emitted has energy corresponding to an energy level difference between two orbits: we can see the precession kinetic energy as a correction to the energy levels in the spectrum of the atom.

There is a simple way to conclude that the identification $n_\theta = j + 1/2$ must be wrong and the identification $n_r = l$ must be correct for the spin taken as $s = -1/2$ in j. A circular orbit has the highest angular momentum L of all elliptic orbits and when the eccentricity ϵ grows, the angular momentum decreases as $L\sqrt{1-\epsilon^2}$. Eccentricity is zero in a circle and it grows towards 1 in more eccentric orbits. It is generally understood that S-states have a rather circular orbits. Certainly it cannot be so that S-states have maximally eccentric orbits. The identification can only be $n_r = l$ and the naming in Quantum Mechanics is totally inverse: J is not total angular momentum, it is total radial momentum, and L is not orbital angular momentum, it is orbital radial momentum.

The reason for stating that $n_r = l$ for spin taken as $s = -1/2$ is that there is nothing in Sommerfeld's derivation of the fine structure spectrum formula that models spin. Therefore the formula does not describe both possible values of spin $\pm 1/2$ in $j = |l \pm s|$. When $l = 0$, the state is S and the orbit must be a circle, so $n_r = 0$. Then we must have $j = |0 - 1/2|$. It follows that the mapping $n_r = l$ is for the value $-1/2$. If $n > 0$, then $n_r = j + 1/2$ gives two relations $n_r = l + 1$ and $n_r = l$. It follows that for $j = l + 1/2$ we must set $n_r = l + 1$.

We see that there is no difficulty in inserting spin into Sommerfeld's fine structure spectrum formula: if $j = |l - 1/2|$, then $n_r = l$ while if $j = l + 1/2$, then $n_r = l + 1$.

Let us continue to the Zeeman effect, we see the same confusion in Quantum Mechanics also there, but also notice another serious error at the end of Section 2.

2. The error in the derivation the Zeeman effect in Quantum Mechanics

The usual Quantum Mechanical derivation of the Zeeman effect is given in [3]. The Zeeman effect, the splitting of spectral lines in the presence of an external magnetic field **B**, is explained in [3] by the effect of the field on orbital and spin magnetic moments

$$\mu_l = -\frac{e}{2m_e}\mathbf{L} \qquad \mu_\mathrm{s} = -\mathrm{g_e}\frac{\mathrm{e}}{2\mathrm{m_e}}\mathbf{S} \tag{6}$$

where the anomalous magnetic moment of an electron is ignored and g_e is set to to 2. This choice does not cause a significant error. The contribution to the Hamiltonian (i.e., the change in energy levels due to the Zeeman effect) is

$$\delta H_{Zeeman} = -(\mu_l + \mu_s)\cdot\mathbf{B} = \frac{\mathrm{e}}{2\mathrm{m_e}}(\mathbf{L} + \mathbf{2S})\cdot\mathbf{B}. \tag{7}$$

The magnetic field is taken to be in the z-direction. As $\mathbf{J} = \mathbf{L} + \mathbf{S}$

$$\delta H_{Zeeman} = \frac{e}{2m_e}(\mathbf{J} + \mathbf{S})\cdot\mathbf{B}. \tag{8}$$

The projection of the moment S on $\mathbf{J}$ is

$$\mathbf{S_z} = (\mathbf{S} \cdot \tilde{\mathrm{e_j}})\tilde{\mathrm{e_j}} \tag{9}$$

but [3] writes it with operators as

$$\mathbf{S_z} = \frac{\mathbf{1}}{\mathbf{J^2}}(\mathbf{S} \cdot \mathbf{J})\mathbf{J}. \tag{10}$$

The eigenvalue of $\mathbf{J^2}$ is $j(j+1)$, [3] inserts the eigenvalue

$$\mathbf{S}_\mathrm{z} = \frac{1}{\mathrm{j(j+1)}}(\mathbf{S} \cdot \mathbf{J})\mathbf{J}. \tag{11}$$

Then [3] uses the relation

$$\mathbf{S} \cdot \mathbf{J} = \frac{1}{2}\left(\mathbf{J^2} - \mathbf{L^2} + \mathbf{S^2}\right) \tag{12}$$

and inserts the eigenvalues $l(l+1)$ and $s(s+1)$ of $\mathbf{J^2}$, $\mathbf{L^2}$ and $\mathbf{S^2}$ respectively. The result is

$$\delta H_{Zeeman} = \frac{e}{2m_e}\left(1 + \frac{1}{2}\frac{j(j+1) - l(l+1) + s(s+1)}{j(j+1)}\right)\mathbf{J} \cdot \mathbf{B}. \tag{13}$$

The term

$$g_j = 1 + \frac{1}{2}\frac{j(j+1) - l(l+1) + s(s+1)}{j(j+1)} \tag{14}$$

is called the Landé factor. The projection of $\mathbf{J}$ in the z-direction is quantized by the magnetic quantum number m_j, thus

$$\mathbf{J} \cdot \mathbf{B} = \mathrm{m_j B}. \tag{15}$$

The energy changes caused by the Zeeman effect are

$$\delta H_{Zeeman} = \frac{e}{2m_e} g_j m_j B. \tag{16}$$

The Landé factor and this formula have been verified to a very high precision, the formula is correct, however the derivation is incorrect. The operators $\mathbf{J^2}$, $\mathbf{L^2}$ and $\mathbf{S^2}$ have the eigenvalues $j(j+1)$, $l(l+1)$ and $s(s+1)$ according to Quantum Mechanics. However the eigenvalue of the operator $\mathbf{J^2} - \mathbf{L^2} + \mathbf{S^2}$ is not obtained by summing the eigenvalues of the individual operators in it. An eigenvalue b for an operator is obtained from the eigenvalue equation

$$(\mathbf{J^2} - \mathbf{L^2} + \mathbf{S^2})|\Psi> = \mathrm{b}|\Psi> . \tag{17}$$

The wavefunction Ψ ranges over all wavefunctions in the allowed wavefunction space and the eigenvalue equation is only satisfied by few wavefunctions, the eigenvectors. With an eigenvector Ψ giving the eigenvalue k we have

$$< \Psi|\mathbf{J^2} - \mathbf{L^2} + \mathbf{S^2}|\Psi> = \mathrm{k}. \tag{18}$$

However, if this operator is divided into three parts

$$< \Psi|\mathbf{J^2}|\Psi > - < \Psi|\mathbf{L^2}|\Psi > + < \Psi|\mathbf{S^2}|\Psi > \tag{19}$$

there is the same wavefunction Ψ in all three parts. The operators $\mathbf{J^2}$, $\mathbf{L^2}$ and $\mathbf{S^2}$ do not (at least usually) have a common eigenvector that gives the eigenvalues $j(j+1)$, $l(l+1)$ and $s(s+1)$. The equation that [3] tries to get is

$$j(j+1) - l(l+1) + s(s+1) =< \Psi_j|\mathbf{J^2}|\Psi_{\mathrm{j}} > - < \Psi_{\mathrm{l}}|\mathbf{L^2}|\Psi_{\mathrm{l}} > + < \Psi_{\mathrm{s}}|\mathbf{S^2}|\Psi_{\mathrm{s}} > \tag{20}$$

but it is not pssible to get this equation by taking the expectation value of the operator $\mathbf{J^2} - \mathbf{L^2} + \mathbf{S^2}$.

3. Derivation of the Zeeman effect in Sommerfeld's model

Only the weak Zeeman effect makes any sense in the Sommerfeld atomic model because in the strong Zeeman effect the external magnetic field is so strong that an atom does not exist in the usual sense. If so, then there is no point in trying to explain the effect with Sommerfeld's atomic model where an atom exists in a very clear sense. But we will not continue to analyse the weak Zeeman effect with a perturbation series. The Zeeman effect is given as a reason why Sommerfeld's atomic model is wrong: supposedly it cannot explain the Zeeman effect. Our goal is only to check if the Zeeman effect and especially the Landé g-factor can be obtained from Sommerfeld's atomic mode.

We follow the previous derivation, but with vectors, not operators, and correct the basic misunderstanding of Quantum Mechanics of the magnetic force acting radially rather than angularly. It is the angular force that can do work. The radial force from the magnetic field cannot make any work because the orbits cannot be made smaller or larger due to quantization.

The Wilson-Sommerfeld quantization rules are

$$\int_0^{2\pi} p_\theta d\theta = n_\theta h \tag{21}$$

and

$$\int_{r_{min}}^{r_{max}} p_r dr = n_r h. \tag{22}$$

We see that the radial moment is quantized by n_r and it is already integrated over r giving the property that we need for energy.

The magnetic moments are similar in a semiclassical model:

$$\mu_r = -\frac{e}{2m_e}\vec{p_r} \qquad \mu_s = -g_e\frac{e}{2m_e}\vec{S}. \tag{23}$$

See the radial momentum vector p_r in the derivation of Sommerfeld's fine structure spectrum formula in [1]. This vector is in the direction $\vec{e_r}$ and has the quantized length $n_r\hbar$. We will follow Quantum Mechanics and violate the classical naming convention by renaming $\vec{p_r}$ as $\vec{L_1}$ and n_r as l_1. This we do in order to make comparison with the Quantum Mechanical (false) derivation of the Zeeman energy levels straightforward.

As [3], we set $g_e = 2$. Similarly as in Section 2

$$\delta H_{Zeeman} = -(\mu_l + \mu_s) \cdot \vec{B} = \frac{e}{2m_e}(\vec{L_1} + 2\vec{S}) \cdot \vec{B}. \tag{24}$$

The magnetic field is taken to be in the z-direction.

We define $\vec{J_1} = \vec{L_1} + \vec{S}$, thus

$$\delta H_{Zeeman} = \frac{e}{2m_e}(\vec{J_1} + \vec{S}) \cdot \vec{B}. \tag{25}$$

In the vector model $\vec{J_1} = \vec{L_1} + \vec{S_1}$, $\vec{J_1} = j_1\vec{e_j}$, $\vec{L_1} = l_1\vec{e_r}$ and $\vec{S} = s\vec{e_s}$. There is nothing like that $\mathbf{J^2}$ has the eigenvalues $j(j+1)$, or ladder operators and so on as in Quantum Mechanics. These are ordinary vectors.

Let us look at the Landé g-factor. Alfred Landé gave his factor as an empirical formula in 1925. It was found correct by Miguel Calalán and Ernst Back. It was given a proof by Quantum Mechanics, but Section 2 shows that the proof is incorrect and Section 1 explains why the quantum numbers in the g-factor are not angular momentums. Yet, the formula is correct and we can find important insight from it.

Define

$$f(l,k) = \frac{(l+k)(l+k+1) - l(l+1) + 3/4}{(l+k)(l+k+1)} = \frac{j(j+1) - l(l+1) + 3/4}{j(j+1)} \tag{26}$$

in the last formula $j = l + k$, and define

$$h(l,k) = \frac{1}{(l+k)(l+k+1)} = \frac{1}{j(j+1)} \tag{27}$$

in the last formula $j = l + k$.

Then

$$f(l,-1/2) = \frac{1-l}{(l-1/2)(l+1/2)} \tag{28}$$

$$f(l-1,1/2) = \frac{1+l}{(l-1/2)(l+1/2)} \tag{29}$$

$$h(l,-1/2) = h(l-1,1/2) = \frac{1}{(l-1/2)(l+1/2)} \tag{30}$$

Lange's g-factor $g_j(l,k)$ where $j = l + k$ where $k = \pm 1/2$ is the spin satisfies the equation

$$g_j(l,-1/2) - 1 - h(l,-1/2) = -(g_j(l-1,1/2) - 1 - h(l-1,1/2)) \tag{31}$$

$$= -\frac{l}{(l-1/2)(l+1/2)}. \tag{32}$$

We see in this equation two symmetric expressions that have the opposite sign. This is what we would expect if the spin is taken as $-1/2$ in one and $1/2$ in the other. But l is not quite symmetric: if the spin changes direction, l changes to $l-1$. This suggests that we must have a similar mapping of n_r to l: for one spin value in $j = l \pm 1/2$ the mapping is $n_r = l$ and for the other $n_r = l+1$. In Section 1 we already identified such a rule: if $j = l - 1/2$, then $n_r = l$ while if $j = l + 1/2$, then $n_r = l + 1$.

Theorem 1: Let the numbers j_1, l and k satisfy

$$j_1 = n_r + k \quad k \in \{-1/2, 1/2\} \tag{33}$$

$$\text{if } k = \mathit{1/2} \quad \text{then} \quad \mathrm{l = n_r - 1} \tag{34}$$

$$\text{if } k = -\mathit{1/2} \quad \text{then} \quad \mathrm{l = n_r}. \tag{35}$$

Then

$$\vec{S} \cdot \vec{J} = j_1^2 - n_r^2 + \frac{1}{4} \tag{36}$$

gives the same value as

$$\mathbf{S} \cdot \mathbf{J} = \mathrm{j(j+1) - l(l+1)} + \frac{3}{4} \tag{37}$$

Proof:

Case 1: $k = 1/2$, thus $l = n_r - 1$ and $j = n_r - 1/2$. Then

$$\mathbf{S} \cdot \mathbf{J} = \left(n_\theta - \frac{1}{2}\right)\left(n_\theta + \frac{1}{2}\right) - (n_\theta - 1)n_\theta + \frac{3}{4} = n_\theta + \frac{1}{2} \tag{38}$$

and the other formula with $j_1 = n_r + 1/2$ gives the same

$$\vec{S} \cdot \vec{J} = \left(n_r + \frac{1}{2}\right)^2 - n_r^2 + \frac{1}{4} = n_r + \frac{1}{2}. \tag{39}$$

Case 2: $k = -1/2$, thus $l = n_r$ and $j = n_r - 1/2$. Then

$$\mathbf{S} \cdot \mathbf{J} = \left(n_r - \frac{1}{2}\right)\left(n_r + \frac{1}{2}\right) - n_r(n_r + 1) + \frac{3}{4} = -n_r + \frac{1}{2} \tag{40}$$

and the other formula with $j_1 = n_r - 1/2$ gives the same

$$\vec{S} \cdot \vec{J} = \left(n_r - \frac{1}{2}\right)^2 - n_r^2 + \frac{1}{4} = -n_r + \frac{1}{2}. \tag{41}$$

We need an old vector theorem.

Theorem 2: Vectors $\vec{J_1}$, $\vec{L_1}$ and $\vec{S}$ with $\vec{J_1} = \vec{L_1} + \vec{S}$ satisfy the equation

$$\vec{S} \cdot \vec{J_1} = \frac{1}{2}(j_1^2 - l_1^2 + s^2). \tag{42}$$

Proof: We can express $\vec{S} \cdot \vec{J_1}$ in two ways:

$$\vec{S} \cdot \vec{J} = (\vec{J_1} - \vec{L_1}) \cdot \vec{J} = \vec{J_1} \cdot \vec{J_1} - \vec{L_1} \cdot (\vec{J_1} + \vec{S}) \tag{43}$$

$$= j_1^2 - l_1^2 - \vec{L_1} \cdot \vec{S} \quad \rightarrow \quad \vec{L_1} \cdot \vec{S} = -\vec{S} \cdot \vec{J_1} + j_1^2 - l_1^2 \tag{44}$$

and

$$\vec{S} \cdot \vec{J_1} = \vec{S} \cdot (\vec{L_1} + \vec{S_1}) = \vec{S_1} \cdot \vec{L_1} + s^2 \tag{45}$$

gives the result

$$\vec{S} \cdot \vec{J_1} = -\vec{S} \cdot \vec{J_1} + j^2 - l^2 + s^2. \tag{46}$$

Combining Theorems 1 and 2 gives

$$\vec{S} \cdot \vec{J_1} = \frac{1}{2}\left(j(j+1) - l(l+1) + s(s+1)\right). \tag{47}$$

Next we have to get a denominator $j(j+1)$ as in g_j.

The projection of the moment S on any $\vec{V} = v\vec{e_z}$ is

$$\vec{S}_z = (\vec{S} \cdot \vec{e_z})\vec{e_z} = \frac{1}{v}(\vec{S} \cdot \vec{V})\vec{e_z}. \tag{48}$$

Especially, J_1 has the length $j+1$ if $j = l + 1/2$. If so, then

$$\vec{S}_z = (\vec{S} \cdot \vec{J_1})\vec{e_z} = \frac{1}{j+1}(\vec{S} \cdot \vec{J_1})\vec{e_z}. \tag{49}$$

$$\vec{S}_z = (\vec{S} \cdot \vec{J_1})\vec{J_1} = \frac{1}{(j+1)(j+1)}(\vec{S} \cdot \vec{J_1})\vec{J_1}. \tag{50}$$

We can write

$$\delta H_{Zeeman} = \frac{e}{2m_e}\left(1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2(j+1)^2}\right)\vec{J_1} \cdot \vec{B}. \tag{51}$$

Notice that with these simple vectors

$$\vec{J_1} = \frac{j+1}{j}\vec{J}. \tag{52}$$

This does not seem obvious because we have kept different vectors $\vec{e}_r$, $\vec{e}_s$, but in reality these vectors are the same.

The spin vector of an electron circulating an elliptic orbit in Sommerfeld's model should have its spin (almost) to the direction of the tangent of the ellipse, while $\vec{e_r}$ is orthogonal to the angular vector. So, initially it appears that the vectors $\vec{e_r}$ and $\vec{e_s}$ must point to different directions, but consider what component of the vector is important in this situation.

In an elliptic orbit the electron has momentum in the radial and orbital directions and a static external magnetic field **B** creates a force that also has an orbital component. Yet, the radial component of this force does not do any work. All radial forces are ignored because the quantization rules forbid the orbits from gradually changing their sizes. This implies that the effective part of the spin vector of an electron is radial: $\vec{e_s} = \vec{e_r} = \vec{e_j}$. These vectors are pointing to the same direction. Thus, for $s = 1/2$ the equation is

$$\delta H_{Zeeman} = \frac{e}{2m_e}\left(1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right)\vec{J}\cdot\vec{B} \tag{53}$$

$$\delta H_{Zeeman} = \frac{e}{2m_e}g_J\vec{J}\cdot\vec{B}. \tag{54}$$

We can generalize this equation to the spin where $j = l - 1/2$ by (31). In (31) there is the term $1 + h(l, -1/2) = 1 + h(l-1, 1/2)$ in both sides. There must be a symmetry in (31): when the spin sign is changed, the effect changes the sign. Therefore the term $2(1 + h(l,k))$ cannot be asymmetrically divided into the sides of the equation (31) and (31) is the unique way to divide $1 + h(l, -1/2) = 1 + h(l-1, 1/2)$. I do not know what this term is but Landé did not measure anything else than $g_j(l, \pm 1/2)$ and also the calculation for $j = l + 1/2$ did not show the term $1 + h(l, 1/2)$. I conclude that the term is cancelled by something and does not contribute to energy. Because for $k = 1/2$ the calculation shows that the energy is as Landé measured, the same must be true also for $k = -1/2$.

Following Quantum Mechanics we introduce a magnetic quantum number m_j to Sommerfeld's atomic model by defining that the projection of $\vec{J}$ in the z-direction is quantized by the magnetic quantum number m_j, thus for $s = 1/2$

$$\vec{J}\cdot\vec{B} = m_j B. \tag{55}$$

The energy changes caused by the Zeeman effect are

$$\delta H_{Zeeman} = \frac{e}{2m_e}g_j m_j B. \tag{56}$$

This is the same result as from Quantum Mechanics.

It is quite possible to explain the Zeeman effect with Sommerfeld's model and the analysis showed two serious errors in the Quantum Mechanical derivation: it is not about angular momentums at all, it is about radial momentums, and one cannot get eigenvalues from sums of operators by summing the eigenvalues of the summand operators.

4. The Lamb shift

The Lamb shift was measured by Willis Lamb and Robert Retherford in 1947, Hans Bethe proposed a solution for it in 1947. It is told that he made his calculation during a train trip. This solution was immediately accepted and it is hailed as having revolutionized Quantum Electrodynamics. Earlier it had been noticed that Dirac's quantum field theory had a serious problem: terms in the perturbation series diverged. This divergence is caused by the four-dimensional Fourier transform. I have discussed this issue in [4][5][6], the error is in considering four-dimensional space and momentum coordinates as conjugate coordinates. Bethe introduced renormalization which means cancelling the divergent terms by adding compensating counter terms. This is not correct: it a method at some step has an error, and divergence of the integrals shows that it has, it is not valid to ignore the problem, as cancelling the divergences does, and continuing further. Bethe did just that.

The Lamb shift is claimed to give the fine structure constant in one part in a million and to be the second verification of Quantum Field Theory (QFT) and the Standard Model. I discussed in [7] the other verification of Quantum Field Theory: the anomalous magnetic moment of an electron. My conclusion of that experiment is the following. The calculation of Feynman diagrams in QFT for explaining the most precise measurement of the anomalous magnetic moment of an electron must include hadronic and weak interactions, else the theoretical value does not match the measured value. But in the experiment a single electron is trapped in a magnetic cage and there are no hadrons it could interact with and there are no possibilities for weak interactions. Therefore in order to explain the measured value of the anomalous magnetic model, the theoretical calculation must not include hadronic and weak interactions. This means, QFT fails. When a theory fails in one experiment, it cannot any more be verified by another experiment. The Lamb shift experiment does not verify the correctness of QFT or the Standard Model. Indeed, I have written many other articles [8]-[11] that show errors in Quantum Physics and even more papers showing errors in the Relativity Theory that is also used in the Standard Model. The Standard Model is based on a theory with many serious errors, it cannot be correct, it cannot be verified by a match between some measurements and theoretical calculations.

Let us briefly discuss whether Sommerfeld's atomic model can explain the Lamb shift. The most visible case of the Lamb shift is the energy level difference between $2S_{1/2}$ and $2P_{1/2}$ states. In these states the principal quantum number $n = 2$. In S-states $l = 0$ and in P-states $l = 1$. The quantum number $j = |l \pm 1/2|$ is the subscript. We see that in $2S_{1/2}$ the absolute value of $|\pm 1/2|$ is 1/2, thus either $j = |-1/2|$ or $j = |1/2|$. In $2P_{1/2}$ we have $j = 1 - 1/2$.

Section 1 identifies the rule: for spin $-1/2$ holds $n_r = l$ and if spin is 1/2, then $p = l + 1$. This means that in $2P_{1/2}$ holds $n_r = l = 1$. In $2S_{1/2}$ we must have $n_r = 0 = l$ because S-states are circles in Sommerfeld's model. Therefore $j = |-1/2|$ in $2S_{1/2}$. In both of these states the spin in j is $-1/2$.

According to Dirac's fine structure spectrum formula quantum states with the same n and j have the same energy, but Dirac's derivation of the fine structure formula is incorrect: there is a confusion between angular and radial momentums. The derivation by Sommerfeld is at least correct calculated, let us look at this formula.

Sommerfeld's formula calculates the kinetic energy of an elliptic orbit. The calculation does not include spin, but we already noticed that both states have spin as $-1/2$, so it should not give a difference.

Sommerfeld's formula does not give the same kinetic energy for $n_r = 1$ and $n_r = 0$. The kinetic energy is larger in a more circular orbit, the kinetic energy of a mass on an elliptic orbit depends inversely proportionally on the major semi-axis. That is, Sommerfeld's formula does predict that $2S_{1/2}$ has a higher energy than $2P_{1/2}$, $n_\theta = n - n_r$ in Sommerfeld's fine structure spectrum formula. I leave the calculation of this energy difference from (1) as a home-work. If it does not match the Lamb shift (which I am not interested in), please, look deeper into Sommerfeld's model, or fix Quantum Mechanics if it seems more promising.

5. References

[1] Jormakka J., "A look at Sommerfeld's fine structure formula", ResearchGate, 2026.

[2] Barley, K., Ruffins, A., Suslov, S.K., Old Quantum Mechanics by Bohr and Sommerfeld from a Modern Perspective, 2025, arxiv.org/html/2506.00408v5.

[3] Zweibach, B., Chapter 2 of lecture notes to Quantum Physics III, Spring 2018. Open lecture course in MIT, available in ocw.mit.edu/courses/8-06-quantum-physics-iii-spring-2018.

[4] Jormakka J., "Three fatal errors in quantum physics", ResearchGate, 2026.

[5] Jormakka J., "A critical view of quantum physics", ResearchGate, 2026.

[6] Jormakka J., "On quantization of a scalar gravity field with Feynman's path integral quantization method", ResearchGate, 2026.

[7] Jormakka J., "No relation between the fine structure constant and electron anomalous magnetic moment", ResearchGate, 2026.

[8] Jormakka J., "Errors in the Quantum Mechanical calculation of the fine structure spectrum of the hydrogen atom", ResearchGate, 2026.

[9] Jormakka J., "Old quantum theory or Quantum Mechanics: a look at the spectrum of the hydrogen atom", ResearchGate, 2026.

[10] Jormakka J., "The Dirac equation is not Lorentz invariant", ResearchGate, 2025.

[11] Jormakka J., "Apparent momentum in Compton scattering",, ResearchGate, 2025.