Pi's Irrationality:
Geometry and Logic

Timothy W. Jones

March 27, 2026

Abstract

We give a proof of Pi's irrationality using geometry and some logic.

Introduction

There have been many proofs of the irrationality of π [2, 3, 6]. The first is attributed to Lambert. It's long and complicated. In 1947 Niven gave an entirely different shockingly short half a page proof [8, **?**]. Still his proof made various unacknowledged (hence obscure) references to the techniques of Hermite in his transcendence of e proof [7]; difficult. In both proofs the natural connection of π to the circle is quite remote.

The proof here makes this connection. It is geometric in nature. Other geometric proofs of note are Sondow's proof of the irrationality of e [9] and Hardy's of the square root of five [4]. These might be thought of as curiosities, not destined for standard analysis textbooks. But, I suggest, π's origins in geometry might make a geometric proof of its irrationality more natural and attractive (classy) to students and mathematicians.

Of course all these words are premised on the proof being correct. It uses an atypical argument. If lines consist of two types ones with defined slopes and ones with undefined slopes and all defined slopes includes all slopes having rational number values then given all radii specified by arc lengths on a unit circle are lines, then a line with an undefined slope can't have a rational slope associated with it, but it can have a rational arc length unless they've been exhausted by some clever (if I do say so myself) trick.

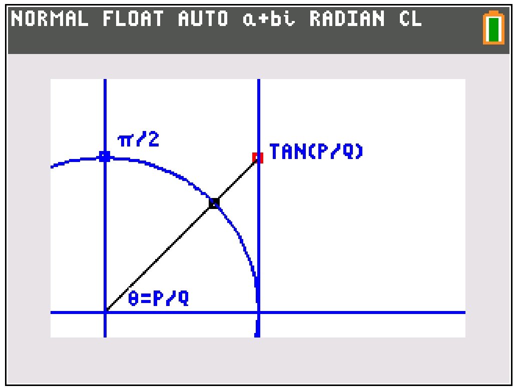

Figure 1: Every positive rational arc so exists like P/Q depicted here.

Proof

Figure 1: A unit circle has a tangent at $(1, 0)$. As the tangent function is such that $\tan([0, \pi/2)) = \mathbf{R}$, all $\tan(P/Q)$, for rational P/Q exist on this line.

Lines with defined slopes have the familiar slope-intercept Cartesian coordinate form: $y = mx + b$. For the line going through the origin in Figure 1, this form is $y = mx$.

We can solve for m. Using similar triangles,

$$\frac{\sin(P/Q)}{\cos(P/Q)} = \frac{?}{1} \tag{1}$$

and this implies that the question mark in (1) is $\tan(P/Q)$. The polar form of a line with an arc length (or radian measure of the angle formed) of P/Q is specified by $\theta = P/Q$. Using $x = r\cos\theta$ and $y = r\sin\theta$, we do the conversion:

$$\tan\theta = \tan(P/Q) \implies y = \tan(P/Q)x.$$

This is the line formed from the rational arc length of P/Q depicted in Figure 1.

Assuming $\pi/2$ is a rational arc length, the radius given by $\theta = \pi/2$ would have the form $y = \tan(\pi/2)x$, but $\tan(\pi/2)$ is not defined, a contradiction.

Conclusion

Using a slightly different angle, Figure 1 is a *proof without words*. The pigeon hole principle [5] says that pigeons (dwelling in holes on sides of cliffs) are such that if five pigeons are distributed into four holes at least one hole must have more than one pigeon in it. Given every rational slope from the line can fly into a rational hole on the circle by way of the tangent function, every rational valued arc is occupied [1][1]; that is it has a pigeon from the line occupying it. If we assume $\pi/2$ is rational, it would have to double up in one hole. But each arc length is unique, so that can't happen. It must be $\pi/2$ is irrational.[2]

References

[1] J. A. Benardete, *Infinity: An Essay in Metaphysics*, Clarendon Press, 1964

[2] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[3] P. Eymard and J.-P. Lafon, *The Number* π, American Mathematical Society, Providence, RI, 2004.

[4] G. H. Hardy and E. M. Wright, Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[5] I. N. Herstein, *Topics in Algebra*, 2nd ed., Wiley, New York, 1975.

[6] T. W. Jones, The Powers of Pi are Irrational (2010), available at `https://vixra.org/abs/1102.0058`.

[7] ——, Discovering and Proving Pi is Irrational, *Monthly*, **53** (2010) 553-557.

[8] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947) 509.

[9] J. Sondow (2006), A geometric proof that e is irrational and a new measure of its irrationality. *Amer. Math. Monthly* 113(7): 637–641.

[1] Benardete develops the paradox of an infinite hotel (holes) without any vacancies (pigeon convention) via a story he attributes to Hilbert.

[2] Note sums of rational numbers are implied by an irrational number. So an infinite sum of rational numbers (slopes from the line) can add up to a vertical, a slope-less line given by arc length $\pi/2$.