Pi's Irrationality:
Geometry and Logic

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Abstract

We give a proof of π's irrationality that references principles of set theory and cardinality within the context of basic geometric properties of a circle.

Introduction

There have been many proofs of the irrationality of π [2, 4, 7]. The first is attributed to Lambert. It's long and complicated. In 1947 Niven gave an entirely different shockingly short half a page proof [9, 10]. Still his proof made various unacknowledged (hence obscure) references to the techniques of Hermite in his transcendence of e proof [8]; difficult. In both proofs the natural connection of π to the circle is quite remote.

The proof here makes this connection. It is geometric in nature. Other geometric proofs of note are Sondow's proof of the irrationality of e [11] and Hardy's of the square root of five [5]. These might be thought of as curiosities, not destined for standard analysis textbooks. But, I suggest, π's origins in geometry might make a geometric proof of its irrationality more natural and attractive (classy) to students and mathematicians.

Of course all these words are premised on the proof being correct. It uses an atypical argument. If lines consist of two types ones with defined slopes and ones with undefined slopes and all defined slopes includes all slopes having rational number values then given all radii specified by arc lengths on a unit circle are lines, then a line with an undefined slope can't have a rational slope associated with it, but it can have a rational arc length unless they've been exhausted by some clever (if I do say so myself) trick.

Background

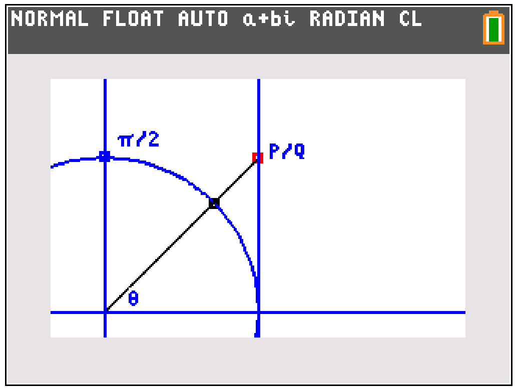

Figure 1: If every rational p/q exists like this, what must $\pi/2$ be?

Figure 1: A unit circle has a tangent starting at $(1, 0)$. The tangent line has one representative of all rational numbers; the red box at length p/q from $(1, 0)$. A line is drawn connecting this p/q point with the origin of the circle. This line intersects the circle and generates an arc length, a radian measure, θ of the angle formed. The intersection point is a black box on the circle. The p/q on the line thus generates a slope and an arc length. The intersection of the positive y-axis and the circle is given by a blue box. The radius associated with this intersection has an arc length of $\pi/2$ and doesn't have a defined slope. This contrasts with the other radii of the first quadrant generated by both red boxes residing on the tangent line.

With this diagram we can prove π is irrational. It's a cardinality (set theory) as well as a geometric proof.

Proof

All lines have defined and undefined slopes. All defined slopes have rational or irrational slopes. All lines with defined slopes have a unique associated arc on a unit circle. All lines with undefined slopes (vertical lines) have an associated unique arc. The arc associated with a line with an undefined slope (a vertical line) is different than the arcs associated with non-vertical lines.

In Figure 1, all lines with defined slopes are given as points on the tangent line: p/q represents a line with an rational slope. As the function $\tan$ is

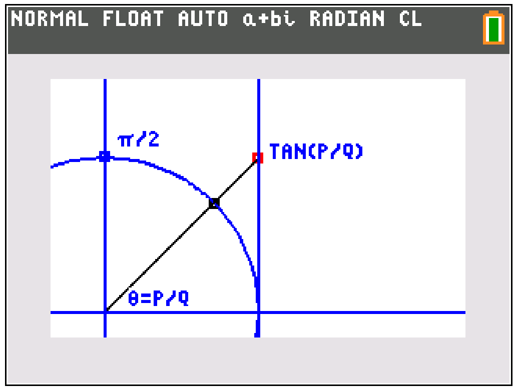

Figure 2: Every positive rational arc so exists like P/Q depicted here.

one to one and maps $[0, \pi/2)$ to $\mathbf{R}$ these positive real numbers on the tangent correspond to $\tan([0, \pi/2))$.

The arcs generated by these $\tan(\theta)$s correspond to lines with defined rational and irrational slopes and have associated arcs θ: Figure 2. The arc associated with the vertical line defined by the polar equation $\theta = \pi/2$ is not included. There is an arc associated with this line, but it can't be a rational, say p/q because all lines with $\tan(p/q)$ slopes exist as has been shown. The arc must be irrational, if it is not rational.

Conclusion

Figure 2 is a *proof without words*. The pigeon hole principle [6] says that pigeons (dwelling in holes on sides of cliffs) are such that if five pigeons are distributed into four holes at least one hole must have more than one pigeon in it. Given every rational slope from the line can fly into a rational hole on the circle by way of the tangent function, every rational valued arc is occupied [1][1]; that is it has a pigeon from the line occupying it. If we assume $\pi/2$ is rational, it would have to double up in one hole. But each arc length is unique, so that can't happen. It must be $\pi/2$ is irrational.[2]

This proof also shows that $\tan(p/q)$ is irrational.

[1] Benardete develops the paradox of an infinite hotel (holes) without any vacancies (pigeon convention) via a story he attributes to Hilbert.

[2] Note sums of rational numbers are implied by an irrational number. So an infinite sum of rational numbers (slopes from the line) can add up to a vertical, a slope-less line given by arc length $\pi/2$.

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