

Non-inertial Relativity Theory, MOND and Generalized Gravity in Curved Phase Spaces *

Carlos Castro Perelman
Bahamas Advanced Science Institute and Conferences
Ocean Heights, Stella Maris, Long Island, the Bahamas
perelmanc@hotmail.com

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Abstract

We pursue further our work on (Born Reciprocal) Non-inertial Relativity theory. Starting with a brief review of the theory, and how Non-inertial Relativity redefines the notion of mass, the phase space particle trajectories in $D = 2 + 2$ are revisited by emphasizing the key difference between a truly $U(1, 1)$ -invariant mass \mathcal{M} and the Lorentz-invariant mass m . The derivation of the generalization of the special relativistic expression of $E = m(1 - v^2)^{-1/2}$ ($c = 1$) to the non-inertial relativistic case follows. This, in turn, leads to the non-inertial relativistic version of an *analog* of Milgrom's modified Newtonian dynamics (MOND) law. Subsequent modifications to Newton's law of motion in the Galilean limit $v \ll 1$ are derived. By recurring to the tools of Finsler geometry, we finalize by writing the generalized gravitational field equations in curved phase space in the presence of matter sources, like particles and cosmic strings. As a result, both spacetime and momentum space are curved. We conclude with some remarks as to why curved momentum space should play an important role in quantum gravity. In particular, why the fusion of non-inertial relativity with quantum mechanics should lead to a novel formulation of thermal quantum field theory (TQFT), and in turn, should cast some light into the quantization of gravity due to the equivalence principle.

Keywords : Born Reciprocal Relativity; Non-inertial Relativity; Modified Newtonian dynamics; Curved Phase Space; Strings; Quantum Gravity.

*Dedicated to the memory of Carlos Terol

1 Brief Introduction of Non-inertial Relativity Theory

In this section we shall review very briefly the basic ideas behind Non-inertial relativity theory. The principle behind the concept of “Born reciprocal relativity theory”, or non-inertial relativity to be more precise¹, was advocated by [3], [4], [5] and it was based on the idea proposed long ago by [1] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. Hence, a *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [5].

The principle of maximal acceleration was advocated earlier on by [7]. A chapter on Reciprocity Theory and Born’s Quantum Metric Operator appeared early on in the book by [2]. The concept of Born reciprocity in order to provide a new point of view on string theory in which spacetime is a derived dynamical concept was advanced by [14].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the *flat* 8D Phase space coordinates, where $X^i, t, E, P^i; i = 1, 2, 3$ are \mathbf{c} -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [3] based on the group $U(1, 3)$ and which is the Born version of the Lorentz group $SO(1, 3)$.

Adopting the units $\hbar = c = 1$, the $U(1, 3) = SU(1, 3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -dt \wedge dE + \delta_{ij} dX^i \wedge dP^j; i, j = 1, 2, 3$ and also the following Born-Green line interval in the *flat* 8D phase-space

$$(d\omega)^2 = (dt)^2 - (dX)^2 - (dY)^2 - (dZ)^2 + \frac{1}{b^2} ((dE)^2 - (dP_x)^2 - (dP_y)^2 - (dP_z)^2), \quad (c = 1) \quad (1.1)$$

The maximal proper force is set to be given by b . The symplectic group is relevant because $U(1, 3) = Sp(8, R) \cap O(2, 6)$; $U(3, 1) = Sp(8, R) \cap O(6, 2)$, and $U(2, 2) = Sp(8, R) \cap O(4, 4)$.

These transformations can be *simplified* drastically when the velocity and force (acceleration) boosts are both parallel to the x -direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1, 1) = SU(1, 1) \times U(1) \subset U(1, 3)$ which leaves invariant the following phase space line interval

$$(d\omega)^2 = (dt)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

¹We thank one of the referees of a previous article for highlighting this fact in order to clarify the point that Born did not propose a reciprocal relativity theory

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 + \frac{\mathcal{F}^2}{F_{max}^2} \right) =$$

$$(d\tau)^2 \left(1 - \frac{F^2}{F_{max}^2} \right), \quad \mathcal{F}^2 = -F^2 < 0, \quad F_{max} = b \quad (1.2)$$

where one has factored out the non-vanishing proper time infinitesimal $(d\tau)^2 = dt^2 - dX^2 \neq 0$ in eq-(1.2). The numerical quantity F^2 is positive by definition. The proper force-squared on a massive particle is $\mathcal{F}^2 = m^2 a^2$, where a^2 is the proper acceleration-squared $a^2 = a_\mu a^\mu$, and m is the rest mass. We refrained from factoring out $(dt)^2$ in (1.2) because it is not Lorentz invariant, whereas $(d\tau)^2$ is Lorentz invariant.

Due to the orthogonality condition $u_\mu a^\mu = 0$ resulting from differentiating the normalization condition $u_\mu u^\mu = \pm 1$ of timelike/spacelike velocities, when the velocity is timelike (subluminal particle) one has $(d\tau)^2 > 0$, so the acceleration is spacelike $a^2 = a_\mu a^\mu < 0$. Therefore $m^2 a^2 < 0$ since $m^2 > 0$. And viceversa, when the velocity is spacelike (superluminal particle) one has $(d\tau)^2 < 0$, so the acceleration is timelike $a^2 = a_\mu a^\mu > 0$. Therefore $m^2 a^2 < 0$ since $m^2 < 0$ (tachyonic particle). Consequently, the proper force squared $\mathcal{F}^2 \equiv (\frac{dE}{d\tau})^2 - (\frac{dP}{d\tau})^2 = \frac{(dE)^2 - (dP)^2}{(d\tau)^2} = m^2 a^2 < 0$ is always negative. Therefore, one may rewrite the negative definite \mathcal{F}^2 as $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$, so that the factorization can always be rewritten as $(d\tau)^2 (1 + \frac{\mathcal{F}^2}{F_{max}^2}) = (d\tau)^2 (1 - \frac{F^2}{b^2})$, with $F^2 > 0$. When $m = 0$, one has $(d\tau)^2 = (dE)^2 - (dP)^2 = 0$ so that $(d\omega)^2 = 0$. No factorization is needed.

Consequently, the *negative* sign appearing inside the parenthesis in eqs-(1.2) furnishes the analog of the Lorentz relativistic factor in special relativity and it involves the ratio of the square of two *proper* forces. These results can be generalized to the $8D$ -dim phase space (and to higher dimensions)

The $U(1, 1)$ group transformations involving the velocity and force boosts (along the X direction) acting on the phase-space coordinates X, t, P, E and which leave invariant the interval (1.2) are given by [3], [4]

$$t' = t \cosh \xi + (\xi_v x + \frac{\xi_a P}{b}) \frac{\sinh \xi}{\xi} \quad (1.3a)$$

$$E' = E \cosh \xi + (b \xi_a X + \xi_v P) \frac{\sinh \xi}{\xi} \quad (1.3b)$$

$$X' = X \cosh \xi + (\xi_v t + \frac{\xi_a E}{b}) \frac{\sinh \xi}{\xi} \quad (1.3c)$$

$$P' = P \cosh \xi + (\xi_v E + b \xi_a t) \frac{\sinh \xi}{\xi} \quad (1.3d)$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. The rapidity parameters ξ_a, ξ_v, ξ are defined, respectively, in terms of the spatial velocity $v = dx/dt$, and proper force $F = ma$, as follows

$$\tanh(\xi_v) = v; \quad \tanh(\xi_a) = \frac{F}{F_{max}}, \quad F_{max} = b, \quad \xi = \sqrt{(\xi_v)^2 + (\xi_a)^2} \quad (1.3e)$$

When $\xi_v \rightarrow \infty \Rightarrow v \rightarrow c = 1$. And $\xi_a \rightarrow \infty \Rightarrow F \rightarrow F_{max} = b$.

It is straight-forward to verify that the transformations of eqs-(1.3) leave invariant the phase space interval $(dt)^2 - (dX)^2 + ((dE)^2 - (dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dt^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination* is truly left invariant under force (acceleration) boosts

$$(d\omega)^2 = (d\tau)^2 \left(1 + \frac{\mathcal{F}^2}{F_{max}^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F_{max}^2} \right), \quad F_{max} = b^2 \quad (1.4)$$

where $\mathcal{F}^2 \equiv m^2 a^2 = -F^2 < 0$, with $F^2 > 0$. The transformations of eqs-(1.3a-1.3d) also leave invariant the symplectic 2-form (phase space areas) $\Omega = -dt \wedge dE + dX \wedge dP$, see [3], [4] for full details.

Related to the importance of the symplectic 2-form, one should add that linear and quadratic Casimir operators corresponding to representations of the Linear Canonical Transformations (LCT) group $Sp(2, 8)$, have been constructed by [23]. $Sp(2, 8)$ emerges naturally as the symmetry group of the $10D$ relativistic quantum phase space associated with $5D$ spacetime dimensions. The symplectic group also plays an important role in the formulation of metaparticles and the metastring [14].

To finalize this introduction, it is warranted to explore the “dual” limit $b \rightarrow 0$ to the $b \rightarrow \infty$ limit, in the same vein that the Carrollian limit $c \rightarrow 0$ is the “dual” version of the Galilean limit $c \rightarrow \infty$ in special relativity.

2 Non-inertial Relativity redefines notion of Mass

In this section we will explore further the physical consequences of non-inertial relativity. Starting with a summary of the recent work in [15] on phase space trajectories in $D = 2 + 2$, and after explaining the key differences between a truly $U(1, 1)$ -invariant mass \mathcal{M} with the Lorentz-invariant mass m , we proceed with deriving the generalization of the special relativistic expression of $E = m(1 - v^2)^{-1/2}$ ($c = 1$) to the non-inertial relativistic case

$$E = \mathcal{M} \frac{1}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - \frac{F^2}{b^2}}}, \quad (c = 1) \quad (2.1)$$

A similar relation (but not the same) and based on the principle of maximal proper acceleration, instead of maximal proper force, has been found by [16]. And finally, we proceed with the derivation of the non-inertial relativistic version of an *analog* of Milgrom’s modified Newtonian dynamics (MOND) law [17],

Given the motion of a massive particle subjected to a proper acceleration g , the phase space infinitesimal interval (1.2) is given by $d\omega = \sqrt{1 - \frac{m^2 g^2}{b^2}} d\tau$. Due to the $U(1, 1)$ invariance of $d\omega$, under force boosts transformations one has the relation

$$d\omega = \sqrt{1 - \frac{(mg)^2}{b^2}} d\tau = \sqrt{1 - \frac{(m'g')^2}{b^2}} d\tau' \quad (2.2)$$

which reveals that $F = mg \neq F' = m'g'$ and $\tau \neq \tau'$. Namely, the proper force mg and the proper time τ are *not* fully $U(1, 1)$ invariant, they are ($SO(1, 1)$) Lorentz invariant. The explicit transformations relating $m'g'$ with mg , and relating τ' with τ , in terms of the force-boost rapidity parameter ξ_a in the particular case of particles exhibiting a uniform proper acceleration and following Rindler hyperbolic trajectories can be found in [6].

Therefore, ω is the truly $U(1, 1)$ -invariant evolution parameter that must be used in order to describe the phase space trajectories of a particle instead of the proper time τ which is *not* $U(1, 1)$ invariant. Consequently, one must have expressions for the phase space coordinates defined in terms of ω as follows

$$Z_I(\omega) \equiv \{ t(\omega), x(\omega), \frac{1}{b}E(\omega), \frac{1}{b}p(\omega) \}, \quad I = 1, 2, 3, 4 \quad (2.3a)$$

The analog of the condition $V_\mu V^\mu = 1$ in phase space is

$$\dot{Z}_I \dot{Z}^I = \left(\frac{dt}{d\omega} \right)^2 - \left(\frac{dx}{d\omega} \right)^2 + \frac{1}{b^2} \left(\frac{dE}{d\omega} \right)^2 - \frac{1}{b^2} \left(\frac{dp}{d\omega} \right)^2 = 1 \quad (2.3b)$$

The analog of the condition $a_\mu a^\mu < 0$ in phase space is

$$\ddot{Z}_I \ddot{Z}^I = \left(\frac{d^2 t}{d\omega^2} \right)^2 - \left(\frac{d^2 x}{d\omega^2} \right)^2 + \frac{1}{b^2} \left(\frac{d^2 E}{d\omega^2} \right)^2 - \frac{1}{b^2} \left(\frac{d^2 p}{d\omega^2} \right)^2 = -\mathcal{A}^2(\omega) \quad (2.4)$$

where $\mathcal{A}(\omega)$ is the phase space analog of the proper spacetime acceleration and must not be confused with $g(\tau)$.

Before we discuss the energy and momentum it is very important to invoke the construction of the quadratic Casimir invariants of the Quaplectic group studied by Low [3], [4]. The Quaplectic group in four spacetime dimensions (eight phase space dimensions) is the semi-direct product of $U(1, 3)$ with the translations in phase space and including the unit central element associated with the Weyl-Heisenberg algebra. The relevance of the quadratic Casimir is that it correctly defines the analog of proper mass \mathcal{M} in phase space. Therefore, upon defining

$$\mathcal{M} \frac{dt}{d\omega} = E, \quad \mathcal{M} \frac{dx}{d\omega} = p \quad (2.5)$$

where \mathcal{M} is the $U(1,1)$ -invariant proper mass in phase space, and which is not the same as m , the two eqs-(2.3,2.4) become a system of two simultaneous differential equations for the two functions $E(\omega), p(\omega)$ given by

$$\left(\frac{E}{\mathcal{M}}\right)^2 - \left(\frac{p}{\mathcal{M}}\right)^2 + \frac{1}{b^2} \left(\frac{dE}{d\omega}\right)^2 - \frac{1}{b^2} \left(\frac{dp}{d\omega}\right)^2 = 1 \quad (2.6)$$

$$\left(\frac{1}{\mathcal{M}}\right)^2 \left(\frac{dE}{d\omega}\right)^2 - \left(\frac{1}{\mathcal{M}}\right)^2 \left(\frac{dp}{d\omega}\right)^2 + \frac{1}{b^2} \left(\frac{d^2E}{d\omega^2}\right)^2 - \frac{1}{b^2} \left(\frac{d^2p}{d\omega^2}\right)^2 = -\mathcal{A}^2(\omega) \quad (2.7)$$

Solutions to eqs-(2.6,2.7) were found in [15] in the special case that \mathcal{A} is constant. Setting $\mathcal{A} = \mathcal{A}(\kappa)$ to be a one-parameter family of accelerations *independent* of the phase space evolution parameter ω , we found a one-parameter family of solutions to eqs-(2.6,2.7) is given by

$$t(\omega; \kappa) = \frac{\kappa}{\mathcal{A}(\kappa)} \sinh[\mathcal{A}(\kappa) \omega], \quad x(\omega; \kappa) = \frac{\kappa}{\mathcal{A}(\kappa)} \cosh[\mathcal{A}(\kappa) \omega] \quad (2.8a)$$

$$E(\omega; \kappa) = \kappa \mathcal{M} \cosh[\mathcal{A}(\kappa) \omega], \quad p(\omega; \kappa) = \kappa \mathcal{M} \sinh[\mathcal{A}(\kappa) \omega] \quad (2.8b)$$

where κ is a numerical parameter. From eq-(2.8a) one infers that as $\omega \rightarrow \infty$ the particle reaches the speed of light $\frac{dx}{dt} = \frac{(dx/d\omega)}{(dt/d\omega)} = \tanh[\mathcal{A}(\kappa)\omega] \rightarrow 1$ ($c = 1$).

Inserting the solutions given by eqs-(2.8) into eqs-(2.6,2.7) lead to the relations

$$\frac{M^2 \mathcal{A}^2(\kappa)}{b^2} = 1 - \frac{1}{\kappa^2}, \quad \mathcal{M} \mathcal{A}(\kappa) = b \sqrt{1 - \frac{1}{\kappa^2}} \leq b, \quad \kappa \geq 1 \quad (2.9)$$

In order to avoid complex values for $\mathcal{M}\mathcal{A}(\kappa)$, one must choose $\kappa \geq 1$. The restriction $\kappa \geq 1$ is required to keep the phase-space acceleration real and bounded, and that $\kappa < 1$ has no clear physical interpretation in the present framework. A value of $\kappa = 1$ yields $\mathcal{M}\mathcal{A}(\kappa = 1) = 0$. The condition $\mathcal{M}\mathcal{A}(\kappa) \leq b$ in eq-(2.9) is also a sign of consistency such that the maximal upper bound of b is not exceeded. In the limit $\kappa \rightarrow \infty$ one has $\mathcal{M}\mathcal{A}(\kappa) \rightarrow b$ and the upper bound b is saturated.

The initial positions of the trajectories in eq-(2.8a) is described by the functions

$$x(\omega = 0; \kappa) = x_o(\kappa) = \frac{\kappa}{\mathcal{A}(\kappa)} = \frac{\kappa \mathcal{M}}{\mathcal{M}\mathcal{A}(\kappa)} = \frac{\kappa \mathcal{M}}{b\sqrt{1 - (1/\kappa^2)}} \quad (2.10)$$

$x_o(\kappa)$ is proportional to the throat-sizes of the elliptic-hyperboloids.

When $\kappa = 1$ and $\infty \Rightarrow x_o = \infty$. The minimum value of x_o is obtained by solving $\frac{dx_o}{d\kappa} = 0$. After some algebra one arrives at a quadratic equation for κ : $1 - \frac{2}{\kappa^2} = 0 \Rightarrow \kappa = \sqrt{2} > 1$. Inserting this value of κ into (2.10) yields

$$(x_o)_{min} = \frac{2\mathcal{M}}{b} = \frac{2GM}{Gb} \quad (2.11)$$

If one sets the maximal proper force b to be equal to the Planck mass-squared $b = M_P^2$, in units of $\hbar = c = 1$, then $Gb = GM_P^2 = L_P^2 M_P^2 = 1$, with L_P being the Planck scale in $4D$, such that the minimal initial position turns out to be $2GM$ which coincides precisely with the numerical value of the horizon radius of the Schwarzschild black hole in four spacetime dimensions (which must not be confused with the four dimensions of the phase space we have been working with signature $(2, 2)$). One must emphasize that this is just a numerical coincidence and that a rigorous derivation would require working in a curved gravitational background in order to invoke the horizon radius of the Schwarzschild black hole.

The solutions in eqs-(2.8) describe a one-parameter family of elliptic-hyperboloids in four dimensions defined by the algebraic equation

$$x^2 + \frac{p^2}{b^2} - t^2 - \frac{E^2}{b^2} = \frac{1}{\mathcal{A}(\kappa)^2} = \frac{\mathcal{M}^2}{b^2(1 - 1/\kappa^2)} \quad (2.12a)$$

The above algebraic equation for the elliptic-hyperboloid is $U(1, 1)$ -invariant. Namely, because the evolution parameter ω and \mathcal{M} are $U(1, 1)$ -invariant, then under $U(1, 1)$ transformations of the trajectories $Z^I(\omega) \rightarrow Z'^I(\omega)$ in phase space, one ends up with the same analytical (functional) form for the elliptic-hyperboloids

$$x'^2 + \frac{p'^2}{b^2} - t'^2 - \frac{E'^2}{b^2} = \frac{1}{\mathcal{A}(\kappa)^2} = \frac{\mathcal{M}^2}{b^2(1 - 1/\kappa^2)} \quad (2.12b)$$

After having reviewed the above basic results of [15] pertaining accelerating particle trajectories in a $D = 2 + 2$ -dim flat phase space, we proceed with the derivation of the generalization of the expression relating mass with energy. Focusing on eq-(2.6), after some straightforward algebra, using the special relativistic relation $E^2 - p^2 = m^2$, and $(d\omega)^2 = (d\tau)^2(1 - F^2/b^2)$, it furnishes the relationship between m and \mathcal{M}

$$E^2 - p^2 = m^2 = \frac{\mathcal{M}^2}{1 - \frac{F^2}{b^2}} \Rightarrow m = m(F) = \frac{\mathcal{M}}{\sqrt{1 - \frac{F^2}{b^2}}} \quad (2.13)$$

Eq-(2.13) leads to a force-dependent mass $m = m(F)$, and a modified mass-energy expression of the form

$$E = \mathcal{M} \frac{dt}{d\omega} = \mathcal{M} \frac{dt}{d\tau \sqrt{1 - \frac{F^2}{b^2}}} = \mathcal{M} \frac{1}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - \frac{F^2}{b^2}}} \quad (2.14)$$

which is a generalization of the relativistic expression $E = m(1 - v^2)^{-1/2}$ in special relativity with m being the invariant rest mass (proper mass) of the particle. In the $b \rightarrow \infty$ limit, eq-(2.13) yields $m = \mathcal{M}$ as expected and one recovers in eq-(2.14) the standard relation $E = m(1 - v^2)^{-1/2}$ of special relativity.

One of the most salient features of the force-dependent mass relation $m^2 = m(F)^2 = \mathcal{M}^2(1 - F^2/b^2)^{-1}$, is that the definition $-F^2 = m^2 a^2 < 0$ yields an expression of the form $F = \sqrt{-m(F)^2 a^2}$, and which could be viewed as a non-inertial relativistic “extension” of the Milgrom’s modified Newtonian dynamics (MOND) law $\vec{f} = m(|\vec{a}|\vec{a})$ [17]. One should emphasize that the non-inertial relativistic extension of the Milgrom’s modified Newtonian dynamics described here is very *different* than the various theoretical attempts made to effectively embed the modifications of Newtonian dynamics within a relativistic theory of gravity [18], [19].

Given $-F^2 = m^2 a^2 < 0$, after defining $g^2 \equiv -a^2 \Rightarrow F^2 = m^2 g^2$, from eq-(2.13) one ends up with the following relation among the real-valued masses m, \mathcal{M}

$$m = \frac{\mathcal{M}}{\sqrt{1 - \frac{m^2 g^2}{b^2}}} \Leftrightarrow \frac{m^2}{\mathcal{M}^2} = \frac{1 \pm \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \quad (2.15)$$

Firstly we are going to examine the minus sign choice in front of the square root, and then the plus sign choice.

$$\frac{m^2}{\mathcal{M}^2} = \frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \Rightarrow m = \mathcal{M} \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} \quad (2.16)$$

Another more transparent way to rewrite eq-(2.16), after defining $F = mg$, and $\tilde{F} = \mathcal{M}g$, is

$$F = \tilde{F} \left(\frac{1 - \sqrt{1 - \frac{4\tilde{F}^2}{b^2}}}{\frac{2\tilde{F}^2}{b^2}} \right)^{1/2} \quad F = mg, \quad \tilde{F} = \mathcal{M}g \quad (2.17)$$

in other words, the ratio F/\tilde{F} is a function of the ratio \tilde{F}/b involving the maximal proper force $b : F/\tilde{F} = f(\tilde{F}/b)$, such $b \rightarrow \infty \Rightarrow \tilde{F} \rightarrow F$ since $f(0) = 1$. We shall return to eq-(2.17) when we analyze its Galilean limit $v \ll 1$ in eq-(2.28) leading to modified Newtonian dynamics.

Once again, it is important to emphasize that g is *not* $U(1,1)$ invariant as one can infer from the relation

$$d\omega = d\tau \sqrt{1 - \frac{m^2 g^2}{b^2}} = d\tau' \sqrt{1 - \frac{m'^2 g'^2}{b^2}} \quad (2.18)$$

\mathcal{M} and the maximal proper force b are truly $U(1,1)$ invariant as shown in [6]. The fact that $b \neq \infty$ does not imply that there is a maximal proper acceleration. On the contrary, the maximal proper acceleration could be $g = \infty$ when the mass $m = 0$ such that $mg = b$. The invariance of the maximal proper force b under $U(1,1)$ transformations, implies that $mg = m'g' = m''g'' = \dots = b$,

therefore there is a *flow* of values of m, g along the hyperbola described by the algebraic equation $mg = b = \text{constant}$. In particular, as $m \rightarrow \infty, g \rightarrow 0$, and vice versa, as $m \rightarrow 0, g \rightarrow \infty$.

A careful inspection of eqs-(2.15, 2.16) reveals that

(i) When $\mathcal{M} = 0 \Leftrightarrow m = 0$.

(ii) When $\mathcal{M} \neq 0; b = \infty$ and/or $g = 0 \Rightarrow m = \mathcal{M}$ after using L'Hopital's rule. This result is also consistent with the one obtained from eq-(2.13) after setting $b = \infty$, and/or $g = 0$.

(iii) One has the following bounds $mg \leq b$ and $\mathcal{M}g \leq \frac{b}{2}$.

(iv) When $\mathcal{M} \neq 0$, and $mg = b$, eqs-(2.13, 2.15) lead to $m = \infty, g = 0$.

(v) When $m^2 < 0 \Leftrightarrow \mathcal{M}^2 < 0$. $m^2 > 0 \Leftrightarrow \mathcal{M}^2 > 0$.

(vi) And when $0 < mg < b$, then m is finite and $m > \mathcal{M}$.

To sum up, the relation $m(F) = \mathcal{M}(1 - F^2/b^2)^{-1/2}$ is the non-inertial relativistic version of the relation $m(v) = m(1 - v^2)^{-1/2}$ in special relativity where m is the rest mass, and such that $m(F = 0) = \mathcal{M}$. Therefore, the $U(1, D - 1)$ -invariant mass \mathcal{M} is just the zero-force mass in $D = (d + 1)$ -spacetime dimensions.

A Taylor expansion of eq-(2.16) leads to

$$m^2 = \mathcal{M}^2 \left(1 + \left(\frac{\mathcal{M}g}{b}\right)^2 + 2 \left(\frac{\mathcal{M}g}{b}\right)^4 + \dots \right) \quad (2.19)$$

and must *not* be confused with the modified dispersion relations in Doubly Special Relativity (DSR) [21].

If one had a relation of the form

$$E^2 - p^2 = \frac{m^2}{1 - \frac{F^2}{b^2}} = m^2 \left(1 + \frac{F^2}{b^2} + \left(\frac{F^2}{b^2}\right)^2 + \left(\frac{F^2}{b^2}\right)^3 + \dots \right), \quad \frac{F^2}{b^2} \leq 1 \quad (2.20)$$

it would resemble the modified dispersion relations in Doubly Special Relativity (DSR) [21] leading to an effective mass $m_{eff} = m(1 - F^2/b^2)^{-1/2}$. But as we have emphasized, $m \neq \mathcal{M}$, and what we have instead is $E^2 - p^2 = m^2 = \mathcal{M}^2(1 - F^2/b^2)^{-1}$.

Let us examine now the plus sign choice in front of the square root

$$m^2 = \frac{1 + \sqrt{1 - \frac{4\mathcal{M}^2g^2}{b^2}}}{\frac{2g^2}{b^2}} \quad (2.21)$$

A close inspection of eq-(2.21) reveals that in the $b \rightarrow \infty$ limit, then the mass $m \rightarrow \infty$ blows up, and consequently, eq-(2.21) does not have a well behaved special relativistic limit. However, if $b \neq \infty$, one finds a very interesting case behind eq-(2.21) when $\mathcal{M} = 0$:

(vii) Given $\mathcal{M} = 0 \Rightarrow m^2 = \frac{b^2}{g^2}$ and one finds that $mg = b$ reaches the *invariant* maximal proper force b . Plugging these values into eq-(2.13) gives

$$m = \frac{\mathcal{M}}{\sqrt{1 - \frac{F^2}{b^2}}} = \frac{0}{0}, \quad \mathcal{M} = 0, \quad F = mg = b \quad (2.22)$$

Because the ratio $\frac{0}{0}$ is undetermined one finds a continuum of values of m and g such that $mg = m'g' = m''g'' = \dots = b$ along the hyperbola described by the equation $mg = \text{constant} = b$. Under force (acceleration) boosts the values of (m, g) flow along the hyperbola to $(m', g') = (m'', g'') \dots$ such that $mg = b$, since the maximal proper force b remains invariant under $U(1, 1)$ transformations in non-inertial relativity theory [6], like the speed of light c is invariant in special relativity. In particular, a zero mass particle like a photon, can experience an infinite value of g such that $mg = 0 \times \infty = b$ (finite and non-zero). This occurs for Rindler particles being subjected to a uniform proper acceleration $g = g_0$ and following hyperbolic trajectories in the Rindler wedge regions (left/right). In the limiting case when $g = g_0 = \infty$ the hyperbolas evolve into the light-like lines comprising the Rindler horizon and corresponding to massless photon trajectories.

To sum up, if one requires to have a well behaved special relativistic limit $b \rightarrow \infty$, then one is forced to choose the *minus* sign in eq-(2.15) leading to the functional relation $m = m(\mathcal{M}, g)$ displayed by eq-(2.16). And, in doing so, the expression for the energy (2.14) becomes

$$E = \mathcal{M} \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} (1 - v^2)^{-1/2}, \quad c = 1 \quad (2.23)$$

and clearly involves a velocity and proper acceleration dependence via the force $\vec{F} = \mathcal{M}g$ and the standard Lorentz dilation factor. Eq-(2.23) can be rewritten in the condensed form $E = m(\mathcal{M}, \vec{F})(1 - v^2)^{-1/2}$ and which is another manifestation of the non-inertial relativistic extension of the MOND law $\vec{f} = m(|\vec{a}|)\vec{a}$. Taking the $b \rightarrow \infty$ limits in eqs-(2.13, 2.23) furnish the special relativistic result $E = m(1 - v^2)^{-1/2}$ since $m \rightarrow \mathcal{M}$ in that limit. Since the speed of light c was set to unity, one may keep b *finite* while taking the Galilean limit $v \ll 1$, such that one will obtain the modified Newtonian corrections ² to the kinetic energy $T = \Delta E$ given by

$$T = \Delta E = E(v) - E(v = 0) = \frac{1}{2} \mathcal{M} \left(\frac{1 - \sqrt{1 - \frac{4\mathcal{M}^2 g^2}{b^2}}}{\frac{2\mathcal{M}^2 g^2}{b^2}} \right)^{1/2} v^2 \quad (2.24)$$

A Taylor expansion of eq-(2.16) was provided in eq-(2.19). Inserting this expansion into eq-(2.24) yields

²We thank Igor Kanatchikov for reminding us that the $b \rightarrow \infty$ limit does not lead to (modified) classical Newtonian physics

$$T = \frac{1}{2} \mathcal{M} v^2 \left(1 + \frac{1}{2} \left(\frac{\mathcal{M}g}{b} \right)^2 + \left(\frac{\mathcal{M}g}{b} \right)^4 + \dots \right) \quad (2.25)$$

In the Galilean limit, $\tau \sim t$ so that the magnitude of the proper acceleration $g \sim |\vec{a}|$ tends to the magnitude of the spatial acceleration. Hence, eq-(2.25) becomes

$$T = \frac{1}{2} \mathcal{M} v^2 \left(1 + \frac{1}{2} \left(\frac{\mathcal{M}|\vec{a}|}{b} \right)^2 + \left(\frac{\mathcal{M}|\vec{a}|}{b} \right)^4 + \dots \right) \quad (2.26)$$

In the $b \rightarrow \infty$ limit, $\mathcal{M} \rightarrow m$, and eq-(2.26) leads to $T = \frac{1}{2} m v^2$ as expected.

Upon defining the Ω function as

$$\Omega\left(\frac{M|\vec{a}|}{b}\right) \equiv \left(1 + \frac{1}{2} \left(\frac{M|\vec{a}|}{b} \right)^2 + \left(\frac{M|\vec{a}|}{b} \right)^4 + \dots \right) \quad (2.27)$$

when $b \neq \infty$ one can then modify Newton's law of motion by writing

$$\vec{f} = \mathcal{M} \vec{a} \Omega\left(\frac{M|\vec{a}|}{b}\right), \quad \frac{M|\vec{a}|}{b} < 1 \quad (2.28)$$

and which follows directly from eq-(2.17) simply by replacing $F \rightarrow |\vec{f}|$ and $\tilde{F} = \mathcal{M}g \rightarrow \mathcal{M}|\vec{a}|$ in the Galilean limit and performing a Taylor series expansion. One should emphasize that eq-(2.28) does *not* have the same functional form as Milgrom's modified Newton's law. The interpolating functions are *different*. To be more precise, after defining $\vec{f} \equiv \mathcal{M} \vec{a}'$ in eq-(2.28), it gives $\vec{a}' = \vec{a} \Omega\left(\frac{M|\vec{a}|}{b}\right)$, which bears an analogous relation between \vec{a}' and \vec{a} as the relation between the Milgromian a_M and Newtonian a_N accelerations [18]. Note, once more, the former relations involves *different* functions than the μ, ν interpolating functions relating the Milgromian a_M and Newtonian a_N accelerations [18] : $a_N = \mu(a_M/a_o)a_M \leftrightarrow a_M = \nu(a_o/a_N)a_N$.

The reader may ask : what is the role of Milgrom's acceleration constant a_o in all of this ? To answer this question one may notice that one can rewrite the maximal proper force b in terms of the Planck mass M_P , the Planck scale L_P ; as well as the observable mass of the Universe M_U , and the Hubble Radius R_H as follows $b = m_P(c^2/L_P) = M_U(c^2/R_H)$, and such that c^2/R_H is closer to Milgrom's acceleration constant $a_o \sim 1.2 \times 10^{-10} m/s^2$.

If the maximal proper force b acting on a fundamental particle is set to be M_P^2 , where M_P is the Planck mass in $D = 4$ spacetime dimensions, in $\hbar = c = 1$ units, it is clear that one cannot set \mathcal{M} to be a huge planetary mass, unless the magnitude of the acceleration $|\vec{a}|$ is *very* small such that the ratio $\frac{M|\vec{a}|}{b} < 1$ remains small. Hence, for very small accelerations one could still use eq-(2.28) provided \mathcal{M} is very large. In the case of fundamental particles whose masses are very small compared to the Planck mass M_P , due to the fact that they can acquire very large accelerations, the ratio $\frac{M|\vec{a}|}{b} < 1$ could still be small.

Therefore, one of the most novel and relevant findings of this work is that eq-(2.28) has a wide range of validity, both for very large *and* very small accelerations as well by choosing the appropriate range of mass values. In other words,

non-inertial relativity theory has applications over a wide range of accelerations, large or small, corresponding to small and large masses, respectively.

To sum up, non-inertial relativity furnishes modifications to Newton's law of motion given by eq-(2.28) in the Galilean limit $v \ll 1$, and which differ from Milgromian MOND [17] due to the difference in the interpolating functions. Since $m \geq \mathcal{M}$, whether or not this discrepancy between m and \mathcal{M} might shed some light into the physics behind dark matter (missing mass) is unknown at the moment.

The invariance of $d\omega$ under $U(1, 1)$ transformations

$$d\omega = d\tau \sqrt{1 - \frac{F^2}{b^2}} = d\tau' \sqrt{1 - \frac{F'^2}{b^2}} \quad (2.29a)$$

and the relations

$$m = \frac{\mathcal{M}}{\sqrt{1 - \frac{F^2}{b^2}}}, \quad m' = \frac{\mathcal{M}}{\sqrt{1 - \frac{F'^2}{b^2}}} \quad (2.29b)$$

lead to the important result

$$\frac{d\tau}{m} = \frac{d\tau'}{m'} \Rightarrow m' = m \frac{d\tau'}{d\tau} \quad (2.30)$$

In [6] we have shown that the scaling factor $\frac{d\tau'}{d\tau} = \lambda(F, F'_{obs})$ relating m and m' is a function of the proper force F experienced by the particle moving with respect to a reference frame S , and the proper force F'_{obs} experienced by an accelerated observer S' moving with respect to the reference frame S .

Contrary to what occurs in Lorentz transformations, under force (acceleration) boost transformations $d\tau \neq d\tau' \Rightarrow m \neq m'$, consequently the mass m is no longer a non-inertial-relativistic invariant, as expected. When $m = 0 \Leftrightarrow d\tau = 0$. We showed in [6] that under force (acceleration) boost transformations one has cases where $d\tau = d\tau' = 0$, so $m = m' = 0$. But also, there are cases when $d\tau' \neq 0$ and consequently $m' \neq 0$, hence a massless photon could appear massive in a non-inertial frame of reference [6].

One should not confuse a reparametrization $\tau \rightarrow \tau'$ with a force boost transformation. Under a reparametrization, the point particle action in special relativity can be rewritten as $S = - \int m d\tau = - \int m \frac{d\tau}{d\tau'} d\tau'$ and one may reinterpret the quantity $m \frac{d\tau}{d\tau'} = m' = m'(\tau')$ as a "reparametrized" mass such that $m d\tau = m' d\tau'$. One may note that this relation clearly differs from $\frac{d\tau}{m} = \frac{d\tau'}{m'}$ obtained from $U(1, 1)$ transformations.

When $g(\tau) = g_0$ (constant), $m = m_0$ (constant), under velocity boosts (Lorentz) transformations one has $m_0 = m'_0; g_0 = g'_0$ since the proper mass and proper acceleration are relativistic invariants in special relativity. However, under force (acceleration) boosts $m_0 \rightarrow m'_0 \neq m_0, g_0 \rightarrow g'_0 \neq g_0$ [6]. This is the key difference and the reason why one can have a *flow* of values of $(m_0, g_0); (m'_0, g'_0), \dots$ along a hyperbola obeying the equation $mg = b = \text{constant}$

The relation between $F = m_0 g_0$, and $F' = m'_0 g'_0$ under force (acceleration) boosts whose rapidity parameter is ξ was found to be [6]

$$\frac{F'^2}{b^2} = \frac{\left(\frac{m_0 g_0}{b} \cosh(\xi) + \sinh(\xi)\right)^2}{\left(\cosh(\xi) + \frac{m_0 g_0}{b} \sinh(\xi)\right)^2}, \quad F' = m'_0 g'_0 \quad (2.31)$$

By inspection of eq-(2.31) one has that if $m_0 g_0 = b$, one finds that $F' = m'_0 g'_0 = b$ also, for *all* values of the force boost rapidity parameter ξ , and which is consistent with the fact that $F = F' = b$ must remain invariant since they coincide with the maximal (and invariant) proper force.

In essence, eq-(2.31) is the analog of the “addition” of velocities in special relativity. The physical interpretation of eq-(2.31) is the following. One has a massive particle of proper mass m_0 moving with respect to a reference frame S with a uniform proper acceleration and force given by g_0 and $F = m g_0$, respectively. A typical example is the Rindler particle sweeping hyperbolic trajectories. A new observer S' comes into the picture³ moving with respect to S with a uniform proper force \mathcal{F} associated to a force boost rapidity parameter ξ given by $\tanh(\xi) = \frac{\mathcal{F}}{b}$. Therefore, the expression in eq-(2.31) depicts the “addition” of two *proper* forces F and \mathcal{F} describing the net proper force F' experienced by the original massive particle with respect to the second observer S' .

To finalize this section we should mention that one must have particles with a uniform proper acceleration g and subjected to a uniform force $F = mg$. If one has a variable proper acceleration $g(\tau)$, as a consequence of eqs-(2.15,2.16), it would lead to a variable $m(\tau)$ and $\mathcal{M}(\tau)$ leading to complications. For instance, the force $\mathcal{F}^\mu = \frac{d}{d\tau}(m u^\mu) = m a^\mu + u^\mu (dm/d\tau)$ would acquire extra terms and one could no longer ensure that $\mathcal{F}^2 = m^2 a^2 + (dm/d\tau)^2$ is negative-definite for time-like trajectories ($m^2 > 0, a^2 < 0$). In particular, one could have temporal domains with $\mathcal{F}^2 > b^2$. And in the extreme case, one could envision a value of τ_* such that $(dm/d\tau)^2(\tau_*) = \infty \Rightarrow \mathcal{F}^2 = \infty$ and leading to an unbounded proper-force squared.

It is not farfetched to contemplate the idea of point particles with a variable mass $m(\tau)$. Strings with a *dynamical* (variable) tension $T(\sigma_0, \sigma_1)$ along the world-sheet parametrized by the coordinates σ_0, σ_1 have been extensively studied by Guendelman [20] over the years. In the modified measure formulation of strings/branes, the tension appear as an additional dynamical degree of freedom. There are many important physical consequences of these variable tension models of strings and branes. Recently, Guendelman has reviewed how the model avoids the Swampland constraints making treatments for Dark energy and inflation more realistic and how strings with a different tension appear as Dark Matter to us. We refer to [20] and the many references therein for specific details.

³The new observer S' could be represented by a physical apparatus of mass M moving with a uniform proper acceleration a such that $\mathcal{F} = Ma$

3 Generalized Gravity in Curved Phase Space

Most of the contents of this section are based on our prior work [10] and the results of [12]. Additions are made at the end where we include the matter actions of point particles and strings in phase space.

Let us begin with the Sasaki-Finsler metric of the cotangent space of a d -dim manifold T^*M_d , and which is given by the following metric in *block diagonal* form [8], [9]

$$(d\omega)^2 = g_{ij}(x^k, p_a) dx^i dx^j + h^{ab}(x^k, p_c) \delta p_a \delta p_b = g_{ij}(x^k, p_a) dx^i dx^j + h_{ab}(x^k, p_c) \delta p^a \delta p^b \quad (3.1)$$

The range of the base manifold indices is $i, j, k = 0, 1, 2, 3, \dots, d-1$; whereas the range of the fiber indices is $a, b, c = 0, 1, 2, 3, \dots, d-1$. The standard coordinate basis frame has been replaced by the following anholonomic non-coordinate basis frame comprised of the following elongated and ordinary derivatives, respectively,

$$\delta_i = \delta/\delta x^i = \partial_{x^i} + N_{ia} \partial^a = \partial_{x^i} + N_{ia} \partial_{p_a}; \quad \partial^a \equiv \partial_{p_a} = \frac{\partial}{\partial p_a} \quad (3.2)$$

The signature is chosen to be Lorentzian $(-, +, +, +, \dots, +)$ for both g_{ij} and h_{ab} . It is important to emphasize that one does *not* have a theory with two times because the energy coordinate is not time. One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a . The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \delta p_a = dp_a - N_{ja} dx^j, \delta p^a = dp^a - N_j^a dx^j \quad (3.3)$$

where the N -coefficients define a nonlinear connection, N-connection structure.

An N-linear connection D on T^*M allows to construct covariant derivatives which are compatible with the structure induced by the nonlinear connection and that preserve the horizontal-vertical split of the cotangent bundle. Thus, an N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) = H_{ij}^k \delta_k; \quad D_{\delta_j}(\partial^a) = -H_{bj}^a \partial^b; \quad (3.4a)$$

$$D_{\partial^a}(\delta_i) = C_i^{ka} \delta_k; \quad D_{\partial^a}(\partial^b) = -C_c^{ba} \partial^c \quad (3.4b)$$

where $H_{ij}^k(x, p), H_{bj}^a(x, p), C_i^{ka}(x, p), C_c^{ba}(x, p)$ are the connection coefficients. Our notation for the derivatives is

$$\partial^a = \partial/\partial p_a, \quad \partial_i = \partial_{x^i}, \quad \delta_i = \delta/\delta x^i = \partial_{x^i} + N_{ia} \partial^a \quad (3.4c)$$

The N-connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W_{\alpha\beta}^\gamma \delta_\gamma$. The only nontrivial (nonvanishing) nonholonomy coefficients are

$$W_{ija} = R_{ija}; \quad W_{jb}^a = \partial^a N_{jb} = -W_j^a{}_b \quad (3.5a)$$

and

$$R_{ija} = \delta_j N_{ia} - \delta_i N_{ja} \quad (3.5b)$$

is the nonlinear connection curvature (N-curvature).

Imposing a zero nonmetricity condition of $g_{ij}(x, p), h^{ab}(x, p)$ along the horizontal and vertical directions, respectively, gives

$$D_i g_{jk} = \delta_i g_{jk} - H_{ij}^l g_{lk} - H_{ik}^l g_{jl} = 0, \quad (3.6a)$$

$$D^a h^{bc} = \partial^a h^{bc} + C_d^{ab} h^{dc} + C_d^{ac} h^{bd} = 0 \quad (3.6b)$$

Performing a cyclic permutation of the indices in eqs-(3.6a,3.6b), followed by linear combination of the equations obtained yields the irreducible (horizontal, vertical) h-v-components for the connection coefficients

$$H^i{}_{jk} = \frac{1}{2} g^{in} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}) \quad (3.7)$$

$$C_c^{ab} = -\frac{1}{2} h_{cd} (\partial^b h^{ad} + \partial^a h^{bd} - \partial^d h^{ab}) \quad (3.8)$$

The additional conditions $D_i h^{ab} = 0, D^a g_{ij} = 0$, yield the *mixed* components of the connection coefficients

$$H_{ja}^b = \partial^b N_{ja} + \frac{1}{2} h^{bc} (\delta_j h_{ac} - h_{ad} \partial^d N_{jc} - h_{cd} \partial^d N_{ja}) \quad (3.9)$$

and

$$C_i^{ja} = \frac{1}{2} g^{jk} \partial^a g_{ik} \quad (3.10)$$

For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i = \frac{1}{2} T_{jk}^i dx^j \wedge dx^k + C_j^{ia} dx^j \wedge \delta p_a \quad (3.11a)$$

$$\Omega_a = \frac{1}{2} R_{jka} dx^j \wedge dx^k + P_{aj}^b dx^j \wedge \delta p_b + \frac{1}{2} S_a^{bc} \delta p_b \wedge \delta p_c \quad (3.11b)$$

and the curvature 2-forms are

$$\Omega_j^i = \frac{1}{2} R_{jkm}^i dx^k \wedge dx^m + P_{jk}^{ia} dx^k \wedge \delta p_a + \frac{1}{2} S_j^{iab} \delta p_a \wedge \delta p_b \quad (3.12)$$

$$\Omega_b^a = \frac{1}{2} R_{bkm}^a dx^k \wedge dx^m + P_{bk}^{ac} dx^k \wedge \delta p_c + \frac{1}{2} S_b^{acd} \delta p_c \wedge \delta p_d \quad (3.13)$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i$, $\partial^a = \partial/\partial p_a$ is given by dx^i , $\delta p_a = dp_a - N_{ja} dx^j$.

The distinguished torsion tensors are given by

$$\begin{aligned} T_{jk}^i &= H_{jk}^i - H_{kj}^i; \quad S_c^{ab} = C_c^{ab} - C_c^{ba}; \quad T_j^{ia} = C_j^{ia} = -T^{ia}_j \\ P_b^a{}_j &= H_{bj}^a - \partial^a N_{jb}, \quad P_b^a{}_j = -P_{bj}^a \\ R_{ija} &= \frac{\delta N_{ja}}{\delta x^i} - \frac{\delta N_{ia}}{\delta x^j} \end{aligned} \quad (3.14)$$

And the distinguished tensors of the curvature are

$$R_{kjh}^i = \delta_h H_{kj}^i - \delta_j H_{kh}^i + H_{kj}^l H_{lh}^i - H_{kh}^l H_{lj}^i - C_k^{ia} R_{jha} \quad (3.15)$$

$$P_{cj}^{ab} = \partial^a H_{cj}^b + C_c^{ad} P_{dj}^b - (\delta_j C_c^{ab} + H_{dj}^b C_c^{da} + H_{dj}^a C_c^{bd} - H_{cj}^d C_d^{ab}) \quad (3.16)$$

$$P_{ij}^{ak} = \partial^a H_{ij}^k + C_i^{al} T_{lj}^k - (\delta_j C_i^{ak} + H_{bj}^a C_i^{bk} + H_{lj}^k C_i^{al} - H_{ij}^l C_l^{ak}) \quad (3.17)$$

$$S_d^{abc} = \partial^c C_d^{ab} - \partial^b C_d^{ac} + C_d^{eb} C_e^{ac} - C_d^{ec} C_e^{ab}; \quad (3.18)$$

$$S_j^{ibc} = \partial^c C_j^{bi} - \partial^b C_j^{ci} + C_j^{bh} C_h^{ci} - C_j^{ch} C_h^{bi} \quad (3.19)$$

$$R_{bjk}^a = \delta_k H_{bj}^a - \delta_j H_{bk}^a + H_{bj}^c H_{ck}^a - H_{bk}^c H_{cj}^a - C_b^{ca} R_{jkc} \quad (3.20)$$

A scalar-gravity model was duly studied in [10] and exact nontrivial analytical solutions for the metric and non-linear connection were found, in the very simple case for constant scalar field configurations, that obeyed the generalized gravitational field equations, in addition to satisfying the zero torsion conditions for all of the torsion components. The curved base spacetime manifold and internal momentum space both turned out to be (Anti) de Sitter type. The most salient feature was that the solutions captured both the very early inflationary, and very-late-time de Sitter phases of the Universe. In this work we will study instead a gravity-matter model.

Adopting the units where $\hbar = c = G = 1$ such that the Planck mass and length squared are respectively $M_P^2 = 1, L_P^2 = 1$; and given $g^{AB} \equiv g^{ij}, h^{ab}$, one may construct the simplest gravity action of the form⁴

⁴ $d^4x d^4p = dx^0 \wedge dx^1 \wedge \dots \wedge \delta p_0 \wedge \delta p_1 \wedge \dots = dx^0 \wedge dx^1 \wedge \dots \wedge dp_0 \wedge dp_1 \wedge \dots$

$$\mathcal{S}_G = \frac{1}{2\kappa} \int d^4x d^4p \sqrt{|\det g_{AB}|} \left(g^{ij} R_{(ij)} + h_{ab} S^{(ab)} \right) \quad (3.21)$$

The determinant factorizes $\det(g_{AB}) = \det(g_{ij})\det(h_{ab})$ in an anholonomic basis adapted to the nonlinear connection (the metric assumes the block diagonal form (1)). κ is the gravitational coupling constant. If the phase space action (3.21) is dimensionless, after reintroducing the physical constants that were set to unity, gives $\kappa = 8\pi \rightarrow (8\pi G/c^4)(M_{PC})^4$.

The action of a point particle in phase space is

$$S_{matter} = -\mathcal{M} \int d\omega \quad (3.22)$$

and it can also be written in terms of an auxiliary $E = E(\omega)$ einbein field (whose physical units are those of $mass^{-1}$) as follows

$$S_{matter} = -\frac{1}{2} \int d\omega \left(E^{-1} \left(\frac{dZ^I}{d\omega} \right)^2 + E\mathcal{M}^2 \right) \quad (3.23)$$

Eliminating $E(\omega)$ via its algebraic equation of motion $\mathcal{M}^2 - E^{-2} \left(\frac{dZ^I}{d\omega} \right)^2 = 0$, and inserting its value back into the action one recovers $S_{matter} = -\mathcal{M} \int d\omega$. One could interpret the first term $E^{-1} \left(\frac{dZ^I}{d\omega} \right)^2$ as a “kinetic energy” and the second term $E\mathcal{M}^2$ as a “cosmological constant”.

Inserting delta functions and integrating allows to rewrite the matter action (3.22) as

$$S_{matter} = -\mathcal{M} \int d^8Z \sqrt{|g|} \int d\omega \frac{\delta^8(Z^A - Z^A(\omega))}{\sqrt{|g|}} \quad (3.24)$$

where $|g|$ denotes the absolute value of the determinant $\det(g_{AB}) = \det(g_{ij})\det(h_{ab})$. The indices $A, B = 1, 2, \dots, 8$ span all the coordinates of the 8-dim phase space. The Born interval in an 8-dim *curved* phase space (cotangent space) is given by

$$(d\omega)^2 = g_{AB} dZ^A dZ^B = g_{ij}(x, p) dx^i dx^j + h^{ab}(x, p) (dp_a + N_{ai}(x, p) dx^i) (dp_b + N_{bj}(x, p) dx^j) \quad (3.25)$$

The phase space metric g_{AB} components are comprised of g_{ij}, h^{ab}, N_{ai} . The matter stress energy tensor associated with the matter action (3.24) is defined as

$$T_{AB} = -\frac{2}{\sqrt{|g|}} \frac{\delta \sqrt{|g|} \mathcal{L}_{matter}}{\delta g^{AB}} = \mathcal{M} \int d\omega \frac{dZ_A}{d\omega} \frac{dZ_B}{d\omega} \frac{\delta^8(Z^A - Z^A(\omega))}{\sqrt{|g|}} \quad (3.26)$$

Given the net gravity-matter action $\mathcal{S} \equiv \mathcal{S}_G + S_{matter}$, after a very laborious procedure, the authors [12] have shown that variations of the gravity-matter action

$$\frac{\delta \mathcal{S}}{\delta g^{ij}} = 0, \quad \frac{\delta \mathcal{S}}{\delta h^{ab}} = 0, \quad \frac{\delta \mathcal{S}}{\delta N_{ai}} = 0, \quad (3.27)$$

with respect to g^{ij}, h^{ab}, N_{ai} , respectively, leads to the following field equations

$$R_{(ij)}(x, p) - \frac{1}{2} g_{ij}(x, p) (R + S) + R_{k(ia} C_j^{ka} = 8\pi T_{ij} \quad (3.28)$$

$$S_{(ab)}(x, p) - \frac{1}{2} h_{ab}(x, p) (R + S) = 8\pi T_{ab} \quad (3.29)$$

$$g^{ik} \partial^a H_{kj}^j - g^{kl} \partial^a H_{kl}^i = 8\pi T^{ia} \quad (3.30)$$

where the curvature contractions are given by

$$R_{kh} = R_{kjh}^i \delta_i^j, \quad R = g^{kh} R_{(kh)}, \quad S^{ac} = S_d^{abc} \delta_b^d, \quad S = h_{ac} S^{(ac)} \quad (3.31)$$

after symmetrizing the indices accordingly and denoted by (\cdot) .

Eqs-(3.28,3.29,3.30) are the generalized field equations in curved phase space where the stress energy tensor components are provided by eq-(3.26) by letting the indices run from $1, 2, \dots, 8$. The first four indices span the T_{ij} components of the 4-dim spacetime. The last four span the T_{ab} components of the 4-dim momentum space, and the mixed indices span the mixed T_{ia} components. One finds that the source of curvature in spacetime and momentum space is due to the presence of matter in phase space, and it takes into account the back-reaction of matter on both spacetime and momentum space (back reaction of matter on the geometry).

The generalization of the point particle matter action in phase space to the (cosmic) string case is

$$S_{string} = -T \int d^8 Z \sqrt{|g|} \int d\sigma_0 \sigma_1 \frac{\delta^8(Z^A - Z^A(\sigma_0, \sigma_1))}{\sqrt{|g|}} \sqrt{|\det g_{AB} \partial_\alpha Z^A \partial_\beta Z^B|} \quad (3.32)$$

where $\partial_\alpha, \partial_\beta$ both denote derivatives with respect to the world sheet coordinates σ_0, σ_1 . In this case the string becomes the source of phase space curvature, after evaluating the stress energy tensor components associated with the string action (3.32). It can be generalized to p -branes as well [24]. To find solutions to the generalized field equations in curved phase space given by eqs-(3.28,3.29,3.30) is a very difficult task compared to what was found in [10] for constant scalar field configurations.

4 Concluding Remarks

Starting with a brief review of Non-inertial Relativity Theory, and how it redefines the notion of mass, the phase space particle trajectories in $D = 2 + 2$ were revisited by emphasizing the key difference between a truly $U(1, 1)$ -invariant mass \mathcal{M} and the Lorentz-invariant mass m . This led to the derivation of the

generalization of the special relativistic expression of $E = m(1-v^2)^{-1/2}$ ($c = 1$) to the non-inertial relativistic case given by eq-(2.23). And, in turn, we found the subsequent modifications to Newton's law of motion in the Galilean limit $v \ll 1$ in eq-(2.28) that differ from Milgrom's MOND law [17]. In general one arrives at the condition $m \geq \mathcal{M}$ which might shed some light into the physics behind dark matter (missing mass).

By recurring to the tools of Finsler geometry, the generalized gravitational field equations in curved phase space due to the presence of matter sources were provided in eqs-(3.28,3.29,3.30). As a result, both spacetime and momentum space are curved. We should add that recently the authors [22] investigated how quantum features of spacetime, in particular the curvature of momentum space, can back react on classical gravity in a tractable semiclassical (2+1)-dimensional setting with a negative cosmological constant. As a result of this back reaction, a mass-dependent geodesic motion and a mild violation of the equivalence principle was found. For other physical applications of curved momentum space see [25], [26].

The essence behind why curved momentum space should play an important role in quantum gravity is because noncommutative momentum coordinates $[\hat{p}_\mu, \hat{p}_\nu] \neq 0$ have a correlation to the noncommutativity of the spacetime covariant derivatives $[\nabla_\mu, \nabla_\nu] \neq 0$ due to the curvature (and torsion) in spacetime. And vice versa, noncommutative spacetime coordinates $[\hat{x}_\mu, \hat{x}_\nu] \neq 0$ have a correlation to the noncommutativity of the momentum space covariant derivatives $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \neq 0$ due to the curvature (and torsion) in momentum space. And finally, as stated in [11], the non-vanishing commutator $[\hat{x}_\mu, \hat{p}_\nu] \neq 0$ has a correlation to the base manifold metric components in phase space given by $g_{\mu\nu}(x, p) + h^{ab}(x, p)N_{a\mu}(x, p)N_{b\nu}(x, p)$ involving $g_{\mu\nu}(x, p)$; $h^{ab}(x, p)$, $N_{a\mu}(x, p)$. These ideas warrant further investigation.

Perhaps more importantly is the interplay of non-inertial relativity with thermal quantum field theory (TQFT). The Fulling-Davies-Unruh effect states that a uniformly accelerating observer experiences the vacuum state of a quantum field in Minkowski spacetime as a mixed state in thermodynamic equilibrium. Such mixed state is comprised of a thermal bath (warm gas) of Rindler particles whose temperature is proportional to the proper acceleration. Given the expression of Unruh's temperature in terms of the proper acceleration $T = \frac{g}{2\pi}$, in natural units $\hbar = c = k_B = 1$, after substituting $g = 2\pi T$ in eqs-(2.15,2.16) it leads to a T -dependent mass $m = m(T)$, which resembles the renormalization process of the physical parameters in QFT in terms of the energy scale. Hence, given $m(g = 0) = m(T = 0) = \mathcal{M}$, by "switching" on the acceleration, the renormalized mass will flow from $\mathcal{M} \rightarrow m(g) = m(2\pi T)$.

To conclude, the fusion of special relativity with quantum mechanics led to quantum field theory (QFT). The fusion of non-inertial relativity with quantum mechanics should lead to a novel formulation of thermal quantum field theory (TQFT), and in turn, should cast some light into the quantization of gravity due to the equivalence principle.

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References

- [1] M. Born, “Elementary Particles and the Principle of Reciprocity”. *Nature* **163** (1949), p. 207
- [2] E.Prugovečki, *Stochastic Quantum Mechanics and Quantum Spacetime*, Kluwer Academic Publishers, 1984, ISBN 978-9027716170. (Section 4.5 Reciprocity Theory and Born’s Quantum Metric Operator)
- [3] S.Low, “ Canonically relativistic quantum mechanics: Casimir field equations of the Quaplectic group”, arXiv : math-ph/0404077
- [4] S. Low, “ $U(3, 1)$ Transformations with Invariant Symplectic and Orthogonal Metrics” , *Il Nuovo Cimento* **B 108** (1993) 841.
S. Low, “Representations of the canonical group, (the semi-direct product of the unitary and Weyl-Heisenberg groups, acting as a dynamical group on noncommutative extended phase space”, *J. Phys. A : Math. Gen.*, **35** (2002) 5711.
- [5] C. Castro Perelman, “On Born Reciprocal Relativity Theory, The Relativistic Oscillator and the Fulling-Davies-Unruh effect”, to appear in the proceedings of the 9th European Congress of Mathematics (9ECM), the quadrennial Congress of the European Mathematical Society, Seville (Spain) 15 -19 July. 2024 (Springer 2026).
- [6] C. Castro Perelman, “Born Reciprocal Relativity Theory Redefines the notion of Mass : How Massless photons appear Massive in Accelerated Frames”, to appear in the IJGMMP, 2026. .
- [7] E.R. Caianiello, “Is there a Maximal Acceleration”, *Il Nuovo Cim.*, **32** (1981) 65.
H. Brandt, “Finslerian Fields in the Spacetime Tangent Bundle” *Chaos, Solitons and Fractals* **10** (2-3) (1999) 267.
H. Brandt, “Maximal proper acceleration relative to the vacuum” *Lett. Nuovo Cimento* **38** (1983) 522.
M. Toller, “The geometry of maximal acceleration” *Int. J. Theor. Phys* **29** (1990) 963.
- [8] S. Vacaru, “On axiomatic formulation of gravity and matter field theories with MDRs and Finsler-Lagrange-Hamilton geometry on (co)tangent Lorentz bundles” arXiv : 1801.06444.

- S. Vacaru, “Finsler-Lagrange Geometries and Standard Theories in Physics: New Methods in Einstein and String Gravity” [arXiv : hep-th/0707.1524].
- S. Vacaru, P. Stavrinou, E. Gaburov, and D. Gonta, ”*Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity*” (Geometry Balkan Press, 693 pages).
- [9] H. Rund, *The Differential Geometry of Finsler Spaces* (Springer-Verlag, 1959).
- I. Bucataru and R. Miron, *Finsler-Lagrange Geometry* (Editura Academiei Romane, Bucarest, 2007).
- R. Miron, D. Hrimiuc, H. Shimada and S. Sabau, *The Geometry of Hamilton and Lagrange Spaces* (Kluwer Academic Publishers, Dordrecht, Boston, 2001).
- C. Pfeifer, The Finsler spacetime framework: backgrounds for physics beyond metric geometry (Thesis, University of Hamburg, 2013).
- [10] C. Castro Perelman, “Born Reciprocal Relativity Theory, Curved Phase Space, Finsler Geometry and the Cosmological Constant”, *Annals of Physics* **416**, (May 2020) 168143.
- C. Castro Perelman, “On Born Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity”, *Phys. Lett. B* **668** (2008) 442-446.
- [11] C. Castro Perelman, “On Born Reciprocal Relativity, Algebraic Extensions of the Yang and Quaplectic Algebra, and Noncommutative Curved Phase Spaces”, *Universe* **9** (3) (2003) 144.
- [12] A. Triantafyllopoulos and P. Stavrinou, “Weak field equations and generalized FRW cosmology on the tangent Lorentz bundle” arXiv : 1903.12521.
- G. Minas, E. N. Saridakis, P. C. Stavrinou, and A. Triantafyllopoulos, “Bounce cosmology in generalized modified gravities”, arXiv:1902.06558v2 (2019).
- S. Ikeda, E. N. Saridakis, P. C. Stavrinou, and A. Triantafyllopoulos, “Cosmology of Lorentz fiber-bundle induced scalar-tensor theories” arXv : 1907.10950.
- [13] C. Castro Perelman, “On Maximal Acceleration, Strings with Dynamical Tension, and Rindler Worldsheets”. *Phys. Letts. B*. Published online <https://doi.org/10.1016/j.physletb.2022.137102>.
- [14] L. Freidel, R. Leigh, D. Minic, “Born reciprocity in string theory and the nature of spacetime”, *Phys. Letters B* **730**, (2014) 302.
- L. Freidel, R. Leigh, D. Minic, “Modular Spacetime and Metastring Theory”, *Journal of Physics: Conf. Series* **804** (2017) 012032
- L. Freidel, R. Leigh, D. Minic, “Quantum Gravity, Dynamical Phase Space and String Theory”, arXiv : 1405.3949

- [15] C. Castro Perelman, “Born Reciprocal (Non-inertial) Relativity, Phase Space Trajectories and Strings with variable Tension”, to appear in *Mod. Phys. Lett. A*, 2026.
- [16] R. Gallego Torrome, “On the effect of the maximal proper acceleration in the inertia”, *Proceedings of the Royal Society A* **480** : 20230876 (2024).
 R. Gallego Torrome, “On the notion of Kinetic energy in a spacetime of maximal proper acceleration”. <https://www.researchgate.net/publication/399868403>.
 R. Gallego Torromé, “On the Unruh effect for hyperbolic observers in spacetimes of maximal proper acceleration”, *Proceedings of the Royal Society A* **481** 20240931 (2025).
- [17] M. Milgrom, “A modification of the Newtonian dynamics as an alternative to the hidden mass hypothesis”. *Astrophysical Journal*. **270** (1983) 365.
- [18] B. Famaey, and S. McGaugh, “Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions”. *Living Reviews in Relativity* **15** (1) (2012) 10.
- [19] I. V. Kanatchikov and V. A. Kholodnyi, “Observable signatures of pre-canonical quantum gravity”, *Europhysics Letters*, **150**, no. 5 (2025) 59002.
 I. V. Kanatchikov, V. A. Kholodnyi, J. Kozicki, and M. E. Pietrzyk, “Modified Newtonian Dynamics from higher moments of quantum spin connection in precanonically quantized gravity”, *Mod. Phys. Letts A* 542012 (2025); <https://doi.org/10.1142/S0217732325410123>.
- [20] E. I. Guendelman, “Dynamical Tension Strings with Target Scale Symmetry producing DE, DM and why 4D?” *arXiv* : 2508.17333.
 E.I. Guendelman, “Strings and branes with a modified measure”, *Class.Quant.Grav.* **17**, 3673-3680 (2000)
 E.I. Guendelman, A.B. Kaganovich, E.Nissimov, S. Pacheva, “String and brane models with spontaneously or dynamically induced tension”, *Phys.Rev D***66** :046003 (2002)
 E.I. Guendelman, “Superstrings and superbranes with a modified measure”, *Phys.Rev. D* **63** (2001) 046006
- [21] G. Amelino-Camelia, “Testable scenario for relativity with minimum length”, *Phys. Lett. B* **510** (2001) 255.
 G. Amelino-Camelia, “Relativity in space-times with short distance structure governed by an observer independent (Planckian) length scale,” *Int. J. Mod. Phys. D* **11** (2002), 35.
- [22] P. Nandi, M. Roy, L. Horoto, F. G. Scholtz, and B. Chakraborty, “Quantum-Gravitational Backreaction in BTZ via Curved Momentum Space” , *arXiv* : 2509.05713.

- [23] P. M. Randriantsoa, R. T. Ranaivoson, R. Andriambololona, R. Raboanary, W. C. Solofoarisina, A.F.H. Rasamimanana, “Casimir operators for the relativistic quantum phase space symmetry group” arXiv : 2512.18262.
- [24] K. Becker, M. Becker, and J. Schwarz, *String Theory and M-Theory* (Cambridge University Press, 2007).
- [25] D. Ketels, “Intrinsic Regularization via Curved Momentum Space: A Geometric Solution to Divergences in Quantum Field Theory”, arXiv : 2502.14443.
- [26] S. A. Franchino-Viñas, S. Mignemi, J. J. Relancio, “The beauty of curved momentum space”, Proc. Sci. **436** (2023) 340.