# Negative Proof of The Continuum Hypothesis

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Abstract: The continuum hypothesis states that the set of real numbers is the power set of the countable set of natural numbers, the smallest uncountable set, and its cardinality is greater than that of natural numbers. There is no set whose cardinality is absolutely greater than that of countable sets and absolutely less than that of the set of real numbers.

This paper proposes a method for constructing a subset of real numbers and proves that its cardinality exceeds that given by the continuum hypothesis. This result refutes the continuum hypothesis and pushes the cardinality of real numbers to a higher order of magnitude.

Keywords: Set, Cardinality, Natural Numbers, Real Numbers, Continuum Hypothesis, Hilbert's First Problem

## 1. The Continuum Hypothesis and Hilbert's first problem

Cantor first proposed the concept of cardinality in the 1870s and classified infinity into levels, calling the set of natural numbers countable and the smallest set of infinity, thus advancing human understanding of the concept of infinity. Cantor proved that the cardinality of the power set of a set is greater than that of the original set, and the cardinality of the set of real numbers is greater than that of the natural numbers. He proposed the continuum hypothesis, arguing that there is no set whose cardinality is absolutely greater than that of a countable set and absolutely less than that of the set of real numbers. That is, the set of real numbers is the smallest uncountable set. [1,2]

At the International Congress of Mathematicians in Paris in 1900, Hilbert proposed his famous 23 open problems, listing the continuum hypothesis (CH) as the first problem. For over a century, this conjecture has remained unproven or undisproven. [3,4]

## 2. Cardinality and power set of sets

The cardinality of a set, also known as its power, is a fundamental concept in set theory, used to characterize the size of the set or represent the number of elements in it. The power set of a set is the set of all its subsets, including the empty set and the set itself.

Cantor proposed a "one-to-one correspondence" method for comparing the sizes of infinite sets. If a one-to-one correspondence can be established between the elements of two sets, then their cardinality is equal or equivalent. In this sense, it can be deduced that the set of odd or even numbers is equivalent to the set of natural numbers, rational numbers are equivalent to natural numbers, any finite closed interval of real numbers is equivalent to the entire set of real numbers represented on the number line, and so on.

Cantor proposed using the subscripted Hebrew letter $\aleph_k$, starting from zero, to represent the size of infinite sets, $\aleph_0$, $\aleph_1$, $\aleph_2$, ..., ..., where each successor is the cardinality of the power set of the previous one, $\aleph_0$ corresponds to the cardinality of the set of natural numbers, $\aleph_1$ corresponds to the cardinality of the set of real numbers, and so on. Therefore, the continuum hypothesis is expressed as

$$\aleph_1 = 2^{\aleph_0}$$

The Generalized Continuum Hypothesis (GCH) states:

$$\aleph_{k+1} = 2^{\aleph_k}, k = 0,1,2,...$$

That is, the successor of each $\aleph_k$ is the power set of the previous one.

To the best of our knowledge, no results have been reported of cardinality of any set of $\aleph_2$ so far. Apart from $\aleph_0$ as the cardinality of natural numbers, and $\aleph_1$ as the cardinality of real numbers (as per the continuum hypothesis), which remains unconfirmed, $\aleph_2$ and subsequent $\aleph_k$ remain only conceptual, with no known construction methods or instances. Therefore, the definition of $\{\aleph_k\}, k = 0,1,2,...,...,$ as a sequence is still undetermined.

## 3. Construction of a subset of real numbers with cardinality 2ℵ

Following general convention, let N denote the set of natural numbers, C denote the set of real numbers, and $|S|$ denote the cardinality of set S.

$$N = \{1,2,3,...\}$$

$$|N| = \aleph_0$$

The power set $N^p$ of N is defined as the set of all subsets of N, denoted as $2^N$.

$$\mathrm{N^p} = 2^N = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \ldots, \ldots, N\}$$

where $\emptyset$ is the empty set。

If we define ordinary addition within the elements of $N^p$, i.e., subsets of N, and assign the elements of the power set to natural numbers and the empty set to zero, the resulting set is called $N^{p+}$, then it is obviously equivalent to N except for having an extra zero.

$$\mathrm{N^{p+}} = \{0,1,2,3,4,5,6, \ldots, \ldots\} = 0 \cup N$$

Now assume that the operation of radical of order k is performed on each element of natural numbers of N, and the resulting set is called $N_k$,

$$N_k = \left\{\sqrt[k]{1}, \sqrt[k]{2}, \sqrt[k]{3}, \ldots\right\}$$

where k>2, a prime number.

Since contained in the radical are all natural numbers, for any k, some natural number n in $N_k$ may escape out of the radical and become a complete natural number. For later application, add a transcendental number α to any natural number $\mathrm{n_i}$ in $N_k$, making it $(\mathrm{n_i} + \alpha)$. For brevity, the set after adding a transcendental number to each element is still denoted as $N_k$,

$$N_k = \left\{\sqrt[k]{1+\alpha}, \sqrt[k]{2+\alpha}, \sqrt[k]{3+\alpha}, \ldots\right\}$$

For simplicity, let it be

$$N_k = \{1_k, 2_k, 3_k, \ldots\}$$

Its power set is

$$N_k{}^{\mathrm{p}} = 2^{N_k} = \{\emptyset, \{1_k\}, \{2_k\}, \{3_k\}, \{1_k, 2_k\}, \{1_k, 3_k\}, \{2_k, 3_k\}, \{1_k, 2_k, 3_k\}, \ldots, \ldots\}$$

If we define ordinary addition within the elements of $N_k{}^{\mathrm{p}}$, i.e., the subsets $N_k$, and establish a correspondence between each subset and the real number representing the sum of its contents, with the empty set corresponding to zero, the resulting set is called $N_k{}^{\mathrm{p+}}$, then we have:

$$N_k{}^{\mathrm{p+}} = \{0, 1_k, 2_k, 3_k, (1_k + 2_k), (1_k + 3_k), (2_k + 3_k), (1_k + 2_k + 3_k), \ldots, \ldots\}$$

Obviously, $N_k{}^{\mathrm{p+}} \subset \mathrm{C}$ is a subset of the set of real numbers, and due to radical operations involving transcendental numbers, addition in each subset of $N_k{}^{\mathrm{p}}$ defines its own independent real number; no two elements in $N_k{}^{\mathrm{p+}}$ are identical. Therefore, for a fixed k, the cardinality of $N_k{}^{\mathrm{p+}}$ is $\aleph_1$:

$$\left|N_k{}^{\mathrm{p+}}\right| = 2^{\aleph_0} = \aleph_1$$

Now repeat the above process. Choose prime numbers j>2, j ≠ k, and perform the operation of radical of order j on each element of $N_k{}^{\mathrm{p+}}$, generate a set $M_j$. Construct its power set $M_j{}^{\mathrm{p}}$, and do addition within the elements of each subset of this power set, mapping them to real numbers to form $M_j{}^{\mathrm{p+}}$. Then $M_j{}^{\mathrm{p+}} \subset \mathrm{C}$ is still a subset of real numbers, and obviously, due to the different orders of the radicals, the sums generated by the addition mapping are all independent real numbers. Since $M_j^{\mathrm{p}+}$ is generated from $N_k^{\mathrm{p}+}$ with a cardinality of $\aleph_1$, the cardinality of $M_j^{\mathrm{p}+}$ is $\aleph_2$, i.e.

$$|M_j^{\mathrm{p}+}| = 2^{\aleph_1} = \aleph_2$$

## 4. Expansion of the subset and the cardinality of real numbers

The above process can continue, with each loop consisting of three steps, forming the following algorithm:

(1) Perform new radical operations of different prime numbers on each element in the preorder set, generating a set of multi-level radical numbers;

(2) Generate the power set of the set of the multi-level radical numbers;

(3) Complete the addition mapping from the elements of subset of the new power set to real numbers, transforming the subset elements into real numbers, forming a new set of numbers.

The cardinality of the newly generated set is the cardinality of the power set of the preorder set. Repeating this process yields a series of subsets of real numbers whose elements are multi-level radical numbers and whose cardinality continuously expands.

Thus, we have established a method for constructing sets whose cardinality is the sequence $\{\aleph_{\mathrm{k}}\}$. Note that the sets defined above are always only subset of real numbers, and the above results indicate the cardinality of real numbers,

$$|\mathrm{C}| = \aleph_{\mathrm{p}}, \mathrm{p} \to \infty$$

## 5.Discussions and conclusions

Cantor proved using the diagonal method that the cardinality of the set of real numbers is greater than that of the set of natural numbers, but he did not explicitly state a difference in magnitude; the continuum hypothesis states that there are no other sets between natural and real numbers whose cardinality lies between the two.

The method for constructing subsets of real numbers presented in this paper exceeds the cardinality conjecture of the continuum hypothesis regarding the cardinality of the set of real numbers, thus refuting the continuum hypothesis.

The results of this paper show that there exist infinitely many sets of numbers between the set of natural and real numbers whose cardinality can be described by the sequence $\{\aleph_k\}$; the entire set of real numbers corresponds to the limit of this infinite sequence. This far exceeds the continuum hypothesis's conjecture regarding the cardinality of the set of real numbers.

## 6.References


[1] Set theory - Wikipedia. https://en.wikipedia.org/wiki/Set_theory

[2] Continuum hypothesis - Wikipedia. https://en.wikipedia.org/wiki/Continuum_hypothesis

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