An Expository Application of Lagrange Interpolation to Diophantine Sequences and the Collatz Conjecture

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Abstract

A common challenge in discrete mathematics and numerical analysis is translating discrete, rule-based algorithms into continuous algebraic functions. While the foundations of this translation lie in the historic work of Joseph-Louis Lagrange, applying these principles to modern piecewise systems offers valuable geometric insights. This expository paper demonstrates an intuitive, step-by-step construction of continuous indicator polynomials to model the trivial sequence of a symmetric Diophantine equation and provides a novel polynomial alternative to Marc Chamberland's trigonometric extension of the Collatz (3x+1) Conjecture.

1 Introduction

In mathematical analysis, sequences generated by piecewise logic or discrete conditions are often difficult to analyze using standard calculus due to their discontinuous nature. The goal of this paper is to explore a methodical algebraic approach to converting finite discrete sequences into continuous polynomial functions without the use of programming logic (such as "if/then" statements).

We rely on the foundational mechanics of Lagrange Basis Polynomials [1], demonstrating how a product series of differences, normalized by factorials, can act as a mathematical "switch." We will then apply this switch to two distinct areas of number theory: the sum-product Diophantine equation and the $3x + 1$ problem.

2 The Algebraic Switch (Indicator Polynomials)

To construct a continuous function that outputs specific values for a set of consecutive integers, we must first build an indicator term that equals 1 at a target integer j, and 0 for all other integers in a given set $\{1, 2, \ldots, m\}$.

By utilizing the Zero Product Property, we define a product series that purposefully skips the target index j:

$$P_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^{m} (x - k) \tag{1}$$

For any integer input $x \in \{1, \ldots, m\}$ where $x \neq j$, the product evaluates to zero. To normalize this product so that it evaluates to exactly 1 when $x = j$, we divide the polynomial by its own value at j. For a sequence of consecutive integers, this denominator elegantly simplifies into a factorial expression:

$$L_j(x) = \frac{\prod_{k\neq j}(x-k)}{\prod_{k\neq j}(j-k)} \tag{2}$$

This construct mirrors the classical Lagrange basis polynomial $l_j(x)$. By manipulating this framework, we can force continuous mathematical functions to obey strict discrete rules.

3 Application I: A Symmetric Diophantine Sequence

Consider the symmetric Diophantine equation where the sum of n variables equals their product:

$$\sum_{i=1}^{n} x_i = \prod_{i=1}^{n} x_i \tag{3}$$

It is an established trivial solution in number theory [3] that for positive integers, this equation is satisfied by a set containing $(n-2)$ ones, a single 2, and the number n. For example, when $n = 4$, the set is $\{1, 1, 2, 4\}$.

We wish to construct a single continuous function $f(i)$ that generates this exact set for $i \in \{1, \ldots, n\}$. Because the majority of the sequence consists of 1s, we define our function as $f(i) = 1 + $ [correction term].

Using our indicator polynomial logic, the correction term must be 0 for the first $n-2$ terms. We define:

$$f(i) = 1 + \frac{\prod_{k=1}^{n-2}(i-k)}{(n-2)!} \tag{4}$$

Proof of behavior:

1. For any integer $i \leq n-2$, the numerator contains $(i-i) = 0$. Thus, $f(i) = 1$.
2. For $i = n-1$, the numerator evaluates to $(n-2)!$. Thus, $f(n-1) = 1 + \frac{(n-2)!}{(n-2)!} = 2$.
3. For $i = n$, the numerator evaluates to $(n-1)!$. Thus, $f(n) = 1 + \frac{(n-1)!}{(n-2)!} = 1 + (n-1) = n$.

This yields a perfect, continuous algebraic curve that inherently contains the discrete Diophantine sequence.

4 Application II: A Polynomial Extension of the Collatz Conjecture

The Collatz Conjecture operates on two discrete rules: $x/2$ for even integers and $3x+1$ for odd integers [4]. Marc Chamberland successfully extended this to the real number line using trigonometric switches, specifically $\cos^2(\frac{\pi x}{2})$ and $\sin^2(\frac{\pi x}{2})$, which result in highly chaotic Julia sets upon iteration [2].

Using the indicator polynomials developed in Section 2, we can offer an alternative: a finite, continuous polynomial extension for any given subset of Collatz inputs. Let us model the domain $x \in \{1, 2, 3, 4\}$. The target outputs are $4, 1, 10$, and 2 respectively.

We construct four separate normalized switches, $L_1(x)$ through $L_4(x)$. For example, the switch for $x = 1$ is:

$$L_1(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} = -\frac{(x-2)(x-3)(x-4)}{6} \tag{5}$$

The complete continuous Collatz function for this domain is the sum of these switches multiplied by their respective target outputs:

$$C(x) = 4L_1(x) + 1L_2(x) + 10L_3(x) + 2L_4(x) \tag{6}$$

Unlike Chamberland's bounded trigonometric wave, this polynomial $C(x)$ is a cubic curve. By the Intermediate Value Theorem, because $C(3) = 10$ and $C(4) = 2$, the polynomial must intersect the line $y = x$ at some real number $c \in (3, 4)$. While no non-trivial integer fixed points exist in the standard conjecture, this polynomial mapping proves the geometric existence of fractional "phantom" fixed points bridging the even and odd rules.

5 Conclusion

By utilizing normalized product series, we successfully bypass the need for algorithmic "if/then" statements when translating discrete sequences into continuous mathematics. While the foundation of this technique relies on Lagrange interpolation, applying it directly to piece-wise dynamical systems like the Collatz conjecture provides a purely algebraic, polynomial-driven geometry to contrast against existing trigonometric methods.

References

[1] Burden, R. L., & Faires, J. D. (2010). *Numerical Analysis* (9th ed.). Brooks/Cole.

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[3] Guy, R. K. (2004). *Unsolved Problems in Number Theory* (3rd ed.). Springer.

[4] Lagarias, J. C. (1985). The $3x + 1$ problem and its generalizations. *The American Mathematical Monthly*, 92(1), 3-23.