

# A mathematical proof about spatial symmetry invariance of solution of the three-dimensional Navier-Stokes equations

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## Abstract

We prove such a mathematical theorem of spatial symmetry invariance: solution of the three-dimensional incompressible Navier-Stokes equations under periodic boundary condition preserves the same spatial symmetry  $\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t)$  for all  $t > 0$  as its smooth initial condition, where  $x'_i = x_i + \pi$  and/or  $x'_i = -x_i$  ( $i = 1, 2, 3$ ), if there exist the spatial symmetry  $\mathbf{u}'(\mathbf{x}', 0) = \mathbf{u}(\mathbf{x}, 0)$  for initial condition and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$  for steady-state external force, and besides the temporal Taylor series of  $\mathbf{u}(\mathbf{x}, t)$  exists and has a non-zero radius of convergence for  $t \geq 0$ . This theorem of spatial symmetry invariance can be used to check the correctness and accuracy of numerical simulation of the Navier-Stokes equations for turbulent flows.

**Keyword** Navier-Stokes equations, invariance of spatial symmetry

**MSC** 76F02, 76F65, 65M25

## 1 Introduction

Consider the dimensionless three-dimensional (3D) incompressible Navier-Stokes equations [1]:

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f}, \quad (2)$$

under periodic boundary condition in a square domain  $\mathbf{x} = (x_1, x_2, x_3) \in [0, 2\pi]^3$ , where  $t \in [0, +\infty)$  denotes the time,  $\mathbf{u} = (u_1, u_2, u_3)$  is the velocity of fluid,  $p$  is the pressure,  $\mathbf{f}(\mathbf{x})$  is an steady-state external force,  $Re$  is the Reynolds number,  $\nabla$  is the Hamilton operator, and  $\Delta = \nabla \cdot \nabla$  is the Laplace operator, respectively. Let  $O - x_1 x_2 x_3$  denote its corresponding coordinate system.

In this paper I give the mathematical proof of the following theorem:

**Theorem of Spatial Symmetry Invariance** *The solution  $\mathbf{u}$  of the incompressible Navier-Stokes equations (1) and (2) under periodic boundary condition has the same spatial symmetry  $\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t)$  for all  $t > 0$  as its smooth initial condition, where  $x'_i = x_i + \pi$  for translation and/or  $x'_i = -x_i$  ( $i = 1, 2, 3$ )*

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for rotation, if there exist the spatial symmetry  $\mathbf{u}'(\mathbf{x}', 0) = \mathbf{u}(\mathbf{x}, 0)$  for initial condition and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$  for external force, and besides the temporal Taylor series of  $\mathbf{u}(\mathbf{x}, t)$  exists and has a non-zero radius of convergence for  $t \geq 0$ .

## 2 Spatial symmetry invariance of solution

Define

$$\mathbf{u}^{(m)}(\mathbf{x}, t) = \frac{1}{m!} \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t^m}, \quad p^{(m)}(\mathbf{x}, t) = \frac{1}{m!} \frac{\partial p(\mathbf{x}, t)}{\partial t^m}, \quad m \geq 0, \quad (3)$$

with

$$\mathbf{u}^{(0)}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t), \quad p^{(0)}(\mathbf{x}, t) = p(\mathbf{x}, t), \quad (4)$$

for integer  $m \geq 0$ .

From (2), it holds

$$\frac{\partial(\nabla \cdot \mathbf{u})}{\partial t} + \nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] = -\Delta p + \frac{1}{Re} \Delta(\nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{f}, \quad (5)$$

which gives using the continuation equation (1) that

$$\Delta p = \nabla \cdot \mathbf{f} - \nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}]. \quad (6)$$

Differentiating (2) and (6)  $m$  times with respect to  $t$  and then dividing by  $m!$ , we have

$$(m+1)\mathbf{u}^{(m+1)} + \sum_{i=0}^m (\mathbf{u}^{(i)} \cdot \nabla) \mathbf{u}^{(m-i)} = -\nabla p^{(m)} + \frac{1}{Re} \Delta \mathbf{u}^{(m)}, \quad (7)$$

$$\Delta p^{(m)} = -\sum_{i=0}^m \nabla \cdot [(\mathbf{u}^{(i)} \cdot \nabla) \mathbf{u}^{(m-i)}], \quad (8)$$

for  $m \geq 1$  and  $t \geq 0$ .

Let  $\mathbf{u}'(\mathbf{x}', t)$ ,  $p'(\mathbf{x}', t)$ ,  $\mathbf{f}'(\mathbf{x}')$  denote the velocity and pressure of fluid, and the external force in the coordinate system  $O' - x'_1 x'_2 x'_3$ , where  $\mathbf{x}' \in [0, 2\pi]^3$ . Certainly, they also satisfy (2) and (6)-(8) for  $t \geq 0$ , thus

$$\mathbf{u}'^{(1)} + (\mathbf{u}' \cdot \nabla') \mathbf{u}' = -\nabla' p' + \frac{1}{Re} \Delta' \mathbf{u}' + \mathbf{f}', \quad (9)$$

$$(m+1)\mathbf{u}'^{(m+1)} + \sum_{i=0}^m (\mathbf{u}'^{(i)} \cdot \nabla') \mathbf{u}'^{(m-i)} = -\nabla' p'^{(m)} + \frac{1}{Re} \Delta' \mathbf{u}'^{(m)}, \quad m \geq 1, \quad (10)$$

and

$$\Delta' p' = \nabla' \cdot \mathbf{f}' - \nabla' \cdot [(\mathbf{u}' \cdot \nabla') \mathbf{u}'], \quad (11)$$

$$\Delta' p'^{(m)} = -\sum_{i=0}^m \nabla' \cdot [(\mathbf{u}'^{(i)} \cdot \nabla') \mathbf{u}'^{(m-i)}], \quad m \geq 1. \quad (12)$$

In general case,  $\mathbf{u}'(\mathbf{x}', t) \neq \mathbf{u}(\mathbf{x}, t)$ . However, if there exists  $\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t)$ , then there exists the spatial symmetry under the coordinate translation  $x'_i = x_i + \pi$  and/or the coordinate rotation  $x'_i = -x_i$ .

**Lemma A** *Under the coordinate translation  $x'_i = x_i + \pi$  and/or the coordinate rotation  $x'_i = -x_i$ , it holds*

$$\nabla' = \nabla, \quad \Delta' = \Delta. \quad (13)$$

Proof: Let  $\mathbf{e}_i, \mathbf{e}'_i$  denote the axis unit vector for  $x_i, x'_i$ , respectively, where  $i = 1, 2, 3$ . For the coordinate translation  $x'_i = x_i + \pi$ , it holds

$$\mathbf{e}_i = \mathbf{e}'_i, \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_i} \frac{\partial x'_i}{\partial x_i} = \frac{\partial}{\partial x'_i}, \quad i = 1, 2, 3, \quad (14)$$

thus

$$\nabla' = \sum_{i=1}^3 \mathbf{e}'_i \frac{\partial}{\partial x'_i} = \sum_{i=1}^3 \mathbf{e}_i \frac{\partial}{\partial x_i} = \nabla. \quad (15)$$

Similarly, for the coordinate rotation  $x'_i = -x_i$ , where  $i = 1, 2, 3$ , it holds

$$\mathbf{e}_i = -\mathbf{e}'_i, \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_i} \frac{\partial x'_i}{\partial x_i} = -\frac{\partial}{\partial x'_i}, \quad (16)$$

thus,

$$\nabla' = \sum_{i=1}^3 \mathbf{e}'_i \frac{\partial}{\partial x'_i} = \sum_{i=1}^3 (-\mathbf{e}_i) \left( -\frac{\partial}{\partial x_i} \right) = \sum_{i=1}^3 \mathbf{e}_i \frac{\partial}{\partial x_i} = \nabla. \quad (17)$$

So, under  $x'_i = x_i + \pi$  and/or  $x'_i = -x_i$ , it holds

$$\Delta' = \nabla' \cdot \nabla' = \nabla \cdot \nabla = \Delta. \quad (18)$$

This ends the proof of Lemma A.

**Lemma B** *For the incompressible Navier-Stokes equations (1) and (2) under periodic boundary condition, if there exist the spatial symmetry  $\mathbf{u}'(\mathbf{x}', t_0) = \mathbf{u}(\mathbf{x}, t_0)$  at  $t = t_0$  and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$ , where  $x'_i = x_i + \pi$  and/or  $x'_i = -x_i$  ( $i = 1, 2, 3$ ), and besides there exists the temporal Taylor series of  $\mathbf{u}(\mathbf{x}, t)$  at  $t = t_0$ , then there exist the spatial symmetry  $\mathbf{u}'^{(m+1)}(\mathbf{x}', t_0) = \mathbf{u}^{(m+1)}(\mathbf{x}, t_0)$  and  $p'^{(m)}(\mathbf{x}', t_0) = p^{(m)}(\mathbf{x}, t_0)$  at  $t = t_0$  for all integer  $m \geq 0$ .*

Proof. Lemma B can be proved by means of recursive method.

(A) When  $m = 0$ , (11) becomes at  $t = t_0$  that

$$\Delta p' = \nabla \cdot \mathbf{f} - \nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}], \quad (19)$$

since  $\Delta' = \Delta, \nabla' = \nabla$  according to Lemma A, and besides there exist the spatial symmetry  $\mathbf{u}'(\mathbf{x}', t_0) = \mathbf{u}(\mathbf{x}, t_0)$  and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$ . Comparing (19) with (6), we have  $\Delta p = \Delta p'$ , that gives  $p'(\mathbf{x}', t_0) = p(\mathbf{x}, t_0)$ , i.e.  $p'^{(0)}(\mathbf{x}', t_0) = p^{(0)}(\mathbf{x}, t_0)$ .

Similarly, (9) becomes at  $t = t_0$  that

$$\mathbf{u}'^{(1)} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f}, \quad (20)$$

since  $\Delta' = \Delta, \nabla' = \nabla$  according to Lemma A and there exist the spatial symmetry  $\mathbf{u}'(\mathbf{x}', t_0) = \mathbf{u}(\mathbf{x}, t_0)$ ,  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$  and  $p'(\mathbf{x}', t_0) = p(\mathbf{x}, t_0)$  as mentioned above. Comparing (20) to (2) gives  $\mathbf{u}'^{(1)}(\mathbf{x}', t_0) = \mathbf{u}^{(1)}(\mathbf{x}, t_0)$ .

Thus, Lemma B holds when  $m = 0$ .

(B) Assume that Lemma B holds when  $m = n \geq 0$ , say, there exists the spatial symmetry

$$\mathbf{u}'^{(i+1)}(\mathbf{x}', t_0) = \mathbf{u}^{(i+1)}(\mathbf{x}, t_0), \quad p'^{(i)}(\mathbf{x}', t_0) = p^{(i)}(\mathbf{x}, t_0), \quad 0 \leq i \leq n. \quad (21)$$

Then, (12) becomes at  $t = t_0$  that

$$\Delta p'^{(n+1)} = - \sum_{i=0}^{n+1} \nabla \cdot [(\mathbf{u}^{(i)} \cdot \nabla) \mathbf{u}^{(n+1-i)}], \quad (22)$$

since  $\Delta' = \Delta, \nabla' = \nabla$  according to Lemma A, and there exist the spatial symmetry (21). Comparing (22) to (8) gives  $p'^{(n+1)}(\mathbf{x}', t_0) = p^{(n+1)}(\mathbf{x}, t_0)$ .

Then, according to the spatial symmetry (21), (10) (when  $m = n + 1$ ) becomes at  $t = t_0$  that

$$(n+2)\mathbf{u}'^{(n+2)} + \sum_{i=0}^{n+1} (\mathbf{u}^{(i)} \cdot \nabla) \mathbf{u}^{(n+1-i)} = -\nabla p^{(n+1)} + \frac{1}{Re} \Delta \mathbf{u}^{(n+1)}, \quad (23)$$

since  $\Delta' = \Delta, \nabla' = \nabla$  according to Lemma A and  $p'^{(n+1)}(\mathbf{x}', t_0) = p^{(n+1)}(\mathbf{x}, t_0)$  as proved above. Setting  $m = n + 1$  in (7) gives at  $t = t_0$  that

$$(n+2)\mathbf{u}^{(n+2)} + \sum_{i=0}^{n+1} (\mathbf{u}^{(i)} \cdot \nabla) \mathbf{u}^{(n+1-i)} = -\nabla p^{(n+1)} + \frac{1}{Re} \Delta \mathbf{u}^{(n+1)}. \quad (24)$$

Comparing (23) to (24) gives  $\mathbf{u}'^{(n+2)}(\mathbf{x}', t_0) = \mathbf{u}^{(n+2)}(\mathbf{x}, t_0)$ .

Thus, if Lemma B holds for  $m = n \geq 0$ , it holds for  $m = n + 1$ .

(C) According to (A) and (B), Lemma B holds for arbitrary integer  $m \geq 0$ .

This ends the proof of Lemma B.

**Lemma C** *For the incompressible Navier-Stokes equations (1) and (2) under periodic boundary condition, if there exist the spatial symmetry  $\mathbf{u}'(\mathbf{x}', t_0) = \mathbf{u}(\mathbf{x}, t_0)$  at  $t = t_0$  and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$ , where  $x'_i = x_i + \pi$  and/or  $x'_i = -x_i$  ( $i = 1, 2, 3$ ), and besides there exists the temporal Taylor series of  $\mathbf{u}(\mathbf{x}, t)$  at  $t = t_0$  with a non-zero radius  $\rho$  of convergence, then there exists the spatial symmetry  $\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t)$  in  $t \in [t_0, \rho)$ .*

Proof. According to the Lemma B, the temporal Taylor series of  $\mathbf{u}(\mathbf{x}, t)$  at  $t = t_0$  reads

$$\mathbf{u}(\mathbf{x}, t) = \sum_{m=0}^{+\infty} \mathbf{u}^{(m)}(\mathbf{x}, t_0)(t - t_0)^m = \sum_{m=0}^{+\infty} \mathbf{u}'^{(m)}(\mathbf{x}', t_0)(t - t_0)^m = \mathbf{u}'(\mathbf{x}', t) \quad (25)$$

in  $|t - t_0| < \rho_0$ , which includes  $t \in [t_0, t_0 + \rho_0)$ . This ends the proof of Lemma C.

Finally, the proof of **Theorem of Spatial Symmetry Invariance** mentioned in § 1 is given below.

Proof. Since there exists the spatial symmetry  $\mathbf{u}'(\mathbf{x}', 0) = \mathbf{u}(\mathbf{x}, 0)$  and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$ , where  $x'_i = x_i + \pi$  and/or  $x'_i = -x_i$  ( $i = 1, 2, 3$ ), we have according to Lemma C (setting  $t_0 = 0$ ) the spatial symmetry

$$\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t), \quad t \in [0, \delta_0],$$

where  $\delta_0 < \rho_0$ , and  $\rho_0 > 0$  is the non-zero radius of convergence of its Taylor series at  $t = 0$ .

Similarly, since there exists the spatial symmetry  $\mathbf{u}'(\mathbf{x}', \delta_0) = \mathbf{u}(\mathbf{x}, \delta_0)$  at  $t = \delta_0$  and  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$ , we have according to Lemma C (setting  $t_0 = \delta_0$ ) the spatial symmetry

$$\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t), \quad t \in [\delta_0, \delta_1],$$

where  $\delta_1 < \rho_1$ , and  $\rho_1 > 0$  is the non-zero radius of convergence of its Taylor series at  $t = \delta_0$ .

Note that

$$[0, \delta_0] \cup [\delta_0, \delta_1] = [0, \delta_1],$$

thus in this way there exists the spatial symmetry

$$\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t), \quad t \in [0, \delta_1],$$

in a larger interval of time. The same process can keep going and the time interval becomes larger and larger, since the temporal Taylor series of  $\mathbf{u}(\mathbf{x}, t)$  always exists with a non-zero radius of convergence. Thus, it holds the spatial symmetry

$$\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t), \quad t \in [0, +\infty).$$

This ends the proof of the Theorem of Spatial Symmetry Invariance.

### 3 Remarks and discussions

**Remark A** The Theorem of Spatial Symmetry Invariance for Navier-Stokes equations can be used as a criterion to verify the correctness and accuracy of numerical simulation of Navier-Stokes equations in the corresponding conditions: numerical simulations of Navier-Stokes equations must depart from exact solution greatly if they distinctly violate this theorem.

**Remark B** Let  $O$  denote one observer in the coordinate system  $O - x_1x_2x_3$ , and  $O'$  denote another observer in the coordinate system  $O' - x'_1x'_2x'_3$ , respectively, where  $x'_i = x_i + \pi$  and/or  $x'_i = -x_i$ . When there exist the spatial symmetry  $\mathbf{f}'(\mathbf{x}') = \mathbf{f}(\mathbf{x})$  for external force and  $\mathbf{u}'(\mathbf{x}', 0) = \mathbf{u}(\mathbf{x}, 0)$  for initial condition, the governing equations and the initial condition are the *same* for both of the observer  $O$  in  $O - x_1x_2x_3$  and the observer  $O'$  in the coordinate system  $O' - x'_1x'_2x'_3$ . Physically speaking, for the two observers at the two different coordinate systems, they should observe the same flow if the initial condition and the external force are the same, say,  $\mathbf{u}'(\mathbf{x}', t) = \mathbf{u}(\mathbf{x}, t)$ , for  $t \in [0, +\infty)$ . From the viewpoint of the third observer knowing their results simultaneously, the flow governed by NS equations should have a kind of spatial symmetry. This physical insight is indeed true as mathematically proved above in this paper. So, the Theorem of Spatial Symmetry Invariance is easy to understand from physical viewpoint.

Note that Liao [2] proved a similar theorem about the two-dimensional Kolmogorov flow, and used it to check the validity of numerical simulations given by direct numerical simulation (DNS) and clean numerical simulation (CNS) [3–9]. It was found [7–9] that CNS results agree with this theorem in the whole time-interval of simulation, but DNS results violate it: this clearly indicates that spatio-temporal trajectories given by DNS are indeed quickly polluted by numerical noises badly. For detailed discussions and comparisons about CNS and DNS, please refer to Liao [2].

It should be emphasized that, although CNS results are numerical, they can provide us enlightenments to approach some mathematical truths such as the theorem of spatial symmetry invariance proved above. Hopefully, CNS [3–9] could be used as a new, powerful tool to study chaotic systems and turbulence.

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## Data and materials availability

All data are available by sending requirement to the corresponding author.

## Competing interests

The authors declare no competing interests.

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