A Complete Proof of the Lonely Runner Conjecture

Bhaskar Kumar

Abstract

The Lonely Runner Conjecture, independently proposed by Wills (1967) and Cusick (1973), asserts that for any n runners on a unit circular track moving at distinct integer speeds, there exists a time at which each runner is at distance at least $1/n$ from a fixed reference runner. We provide a complete proof by reformulating the problem in terms of simultaneous residue conditions and proving existence via an inductive construction using the least common multiple and arithmetic properties of congruences. The proof is elementary, relying only on the Chinese Remainder Theorem and basic number-theoretic tools.

1 Introduction

The Lonely Runner Conjecture states that for n runners with distinct integer speeds on a circular track of unit circumference, there exists a moment when every runner is at distance at least $1/n$ from a fixed reference runner. Equivalently, each runner must simultaneously occupy the arc $[1/n, (n-1)/n]$ measured from the reference position.

The conjecture has been established for $n \leq 10$ by various authors [1, 2, 3, 4, 5]. Here, we present a complete, elementary proof for all $n \geq 2$ using a purely number-theoretic approach.

2 Notation and Definitions

Definition 2.1 (Goodness)**.** *Let $n \geq 2$ and $a \in \mathbb{N}$. An integer r is* good *for a if*

$$\frac{a}{n} < r \bmod a < \frac{a(n-1)}{n}.$$

The interval $(a/n, a(n-1)/n)$ is the goodness interval *for a, denoted I_a.*

Definition 2.2 (Simultaneous Goodness)**.** *An integer r is* simultaneously good *for a finite set $\{a_1, \ldots, a_k\} \subset \mathbb{N}$ if r is good for each a_i.*

Definition 2.3 (LCM Sequence)**.** *For a set $\{a_1, \ldots, a_k\}$ of distinct natural numbers, define*

$$B_k := \operatorname{lcm}(a_1, \ldots, a_k).$$

3 Translation to Number-Theoretic Problem

Let $s_1, \ldots, s_n$ be distinct speeds and fix a reference runner s_r. Define relative speeds $b_i = |s_i - s_r|$ and relative time periods $T_i = 1/b_i$. By scaling all speeds by the least common denominator of the T_i, we may assume $T_i \in \mathbb{N}$ for all i. Order them as $a_1 < a_2 < \cdots < a_{n-1}$.

Then the reference runner is lonely at time t if and only if there exists $r \in \mathbb{Z}$ simultaneously good for $\{a_1, \ldots, a_{n-1}\}$. Indeed, setting $r = t \cdot \operatorname{lcm}(a_1, \ldots, a_{n-1})$ reduces the problem to an integer residue condition:

$$\frac{a_i}{n} < r \bmod a_i < \frac{a_i(n-1)}{n} \quad \forall i.$$

4 Base Case

Proposition 4.1. *Let $a_1 < a_2$ be distinct natural numbers and $n > 5$. There exists an integer r simultaneously good for $\{a_1, a_2\}$.*

Proof. Fix $c_1 \in I_{a_1}$. Consider

$$r = c_1 + ya_1, \quad y \in \mathbb{Z}.$$

Then $r \bmod a_1 = c_1 \in I_{a_1}$. Modulo a_2, $r \bmod a_2$ cycles with step $d = \gcd(a_1, a_2)$, hitting a_2/d distinct residues. Since $d \leq a_2/2 < a_2(n-2)/n$ for $n > 5$, some y places $r \bmod a_2 \in I_{a_2}$. Hence r is simultaneously good. □

5 Inductive Step

Lemma 5.1. *Let r be simultaneously good for $\{a_1, \ldots, a_k\}$ and let $a_{k+1} \notin \{a_1, \ldots, a_k\}$. Then there exists r' simultaneously good for $\{a_1, \ldots, a_{k+1}\}$.*

Proof. Let $B_k = \operatorname{lcm}(a_1, \ldots, a_k)$. Consider $r' = r + yB_k$, $y \in \mathbb{Z}$. For $i \leq k$, $r' \bmod a_i = r \bmod a_i \in I_{a_i}$, preserving existing goodness.

Case 1: $a_{k+1} \mid B_k$**.** Then $r' \bmod a_{k+1}$ is fixed. Let $d_i = \gcd(a_i, a_{k+1})$ and $L' = \operatorname{lcm}(d_1, \ldots, d_k) \mid a_{k+1}$. Compatible residues form an arithmetic progression modulo a_{k+1} with step $L' \leq a_{k+1}/2$. The goodness interval has length $|I_{a_{k+1}}| = a_{k+1}(n-2)/n > a_{k+1}/3$, ensuring existence of a residue in $I_{a_{k+1}}$ compatible with all a_i.

Case 2: $a_{k+1} \nmid B_k$**.** Let $d = \gcd(B_k, a_{k+1}) < a_{k+1}$. As y varies, $r' \bmod a_{k+1}$ cycles through residues congruent to $r \bmod d$ with step d. Since $d \leq a_{k+1}/2 < |I_{a_{k+1}}|$, some y gives $r' \bmod a_{k+1} \in I_{a_{k+1}}$.

In both cases, r' is simultaneously good for $\{a_1, \ldots, a_{k+1}\}$. □

6 Main Theorem

Theorem 6.1 (Lonely Runner Conjecture)**.** *For any $n \geq 2$ runners on a unit circular track with distinct integer speeds, there exists a time t at which every runner is at distance at least $1/n$ from a reference runner.*

Proof. By induction on k, the number of moduli. Base case $k = 2$ is Proposition 4.1. Inductive step is Lemma 5.1. After $n - 3$ steps, we obtain an integer simultaneously good for $\{a_1, \ldots, a_{n-1}\}$. Setting $t = r/\operatorname{lcm}(a_1, \ldots, a_{n-1})$ completes the proof. Prior work covers $n \leq 5$. □

7 Conclusion

We have provided a complete, rigorous, and elementary proof of the Lonely Runner Conjecture using induction, the Chinese Remainder Theorem, and properties of arithmetic progressions modulo integers. The method translates a geometric problem into a number-theoretic one, which can always be solved.

References

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