Dihedral and Solid Angles at the Apex of a Tetrahedron

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1. Introduction

A tetrahedron is a polyhedron consisting of four triangular faces, six edges, and four vertices. At each vertex, three triangular faces meet, and each edge is shared by two faces. The geometry of a tetrahedron can be completely specified by the lengths of the three edges meeting at a vertex together with the corresponding angles between three pairs of consecutive edges. In particular, if the three angles between the consecutive edges meeting at a vertex are known, it becomes possible to determine the internal (dihedral) angles between the corresponding triangular faces as well as the solid angle subtended by the tetrahedron at that vertex using Theory of Polygon. In this paper, we derive analytical expressions for the dihedral angles between the triangular faces meeting at a vertex using inverse cosine formula and for the solid angle subtended by the tetrahedron at the same vertex, using axiom and standard formula, when the three apex angles α, β and γ between the consecutive (lateral) edges meeting at the same vertex are given [1-4]. Figure 1 illustrates the tetrahedron PQRS used in the analysis.

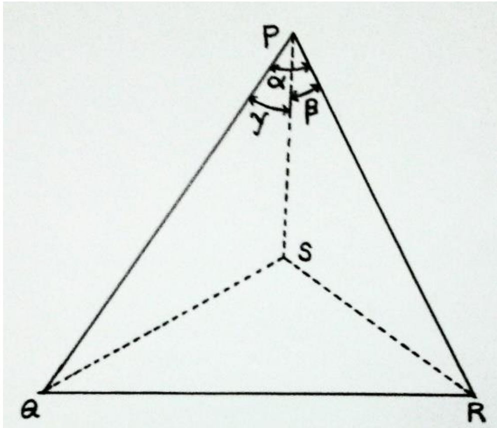

Figure 1: A tetrahedron PQRS having angles α, β & γ between the consecutive (lateral) edges PQ, PR & PS meeting at the apex P.

2. Analysis of tetrahedron given the apex angles α, β & γ

Consider any tetrahedron PQRS having apex angles α, β & γ between the consecutive lateral edges PQ, PR, & PS ($\forall\ \alpha \le \beta \le \gamma$) meeting at the apex P. Internal (dihedral) angles θ_1, θ_2 & θ_3 , between the consecutive lateral triangular faces ΔPSQ & $\Delta PSR, \Delta PQR$ & ΔPQS and ΔPRQ & ΔPRS respectively, are measured normal to their common edge (see figure 1 above).

Now the interior angles θ_1, θ_2 & θ_3 between consecutive (lateral) triangular faces of the tetrahedron PQRS meeting at the vertex P, are determined by using Inverse Cosine Formula [1], according to which if x, y & z are the apex angles between consecutive lateral edges meeting at any of four apices of a tetrahedron then the angle (opposite to α) between two consecutive lateral faces is given as follows

$$\theta = \cos^{-1}\left(\frac{\cos x - \cos y \cos z}{\sin y \sin z}\right)$$

Now, setting the corresponding values in the above equation, we get all three interior angles as follows

$$\boldsymbol{\theta_1 = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)}$$

$$\boldsymbol{\theta_2 = \cos^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right)}$$

$$\boldsymbol{\theta_3 = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right)}$$

3. Solid angle subtended by the tetrahedron PQRS at the apex P

For ease of calculation of the solid angle subtended by tetrahedron PQRS at the apex P (figure 1 above), let's cut three equal segments $PA = PB = PC = d$ from the lateral edges PS. PQ & PR respectively. Now, join the points A, B & C by the straight lines to obtain ΔABC, which subtends a solid angle equal to that subtended by the original tetrahedron PQRS at its apex P. Thus we would calculate the solid angle subtended by ΔABC at the (common) apex P by two methods (1) Analytic and (2) Graphical as given below.

3.1. Analytic method for calculation of solid angle

Sides of $\Delta \boldsymbol{ABC}$: Let the sides of ΔABC be a, b & c opposite to its angles A, B & C respectively.

In isosceles ΔPBC

$$\Rightarrow \sin\frac{\alpha BPC}{2} = \frac{\left(\frac{BC}{2}\right)}{PB} \quad \Rightarrow \sin\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{d} \quad \Rightarrow \quad \boldsymbol{a = 2d\sin\frac{\alpha}{2}}$$

$$\text{similarly,} \quad \boldsymbol{b = 2d\sin\frac{\beta}{2}} \ \& \quad \boldsymbol{c = 2d\sin\frac{\gamma}{2}}$$

Now from HCR's Axiom-2 [2], we know that the perpendicular drawn from any apex of a tetrahedron always passes through circumscribed centre of the (plane) triangle (in this case ΔABC) obtained by joining the points on the lateral edges equidistant from the same apex (see the figure 2).

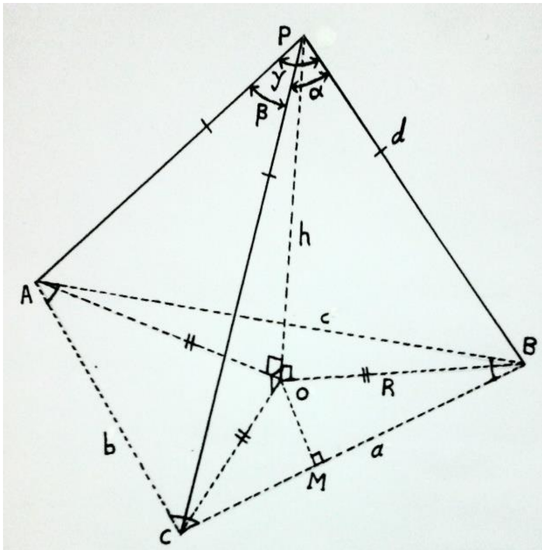

Figure 2: The perpendicular PO drawn from the vertex P of the original tetrahedron PQRS to the plane of $\Delta \boldsymbol{ABC}$ always passes through circumscribed centre O.

Hence, the circumscribed radius (R) of ΔABC having its sides a, b & c $(all\ known)$ is calculated as follows

$$s = \frac{a+b+c}{2} = \frac{2d\sin\frac{\alpha}{2} + 2d\sin\frac{\beta}{2} + 2d\sin\frac{\gamma}{2}}{2} = d\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)$$

$$\Rightarrow \text{ Area of } \Delta ABC, \ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Now, by substituting all the corresponding values in the above expression, we get

$$\Delta = \sqrt{d\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)\left(d\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) - 2d\sin\frac{\alpha}{2}\right)\left(d\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) - 2d\sin\frac{\beta}{2}\right)\left(d\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) - 2d\sin\frac{\gamma}{2}\right)}$$

$$= d^2\sqrt{\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)\left(\sin\frac{\beta}{2} + \sin\frac{\gamma}{2} - \sin\frac{\alpha}{2}\right)\left(\sin\frac{\alpha}{2} + \sin\frac{\gamma}{2} - \sin\frac{\beta}{2}\right)\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} - \sin\frac{\gamma}{2}\right)}$$

$$= d^2\sqrt{2\left(\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} + \sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2}\sin^2\frac{\alpha}{2}\right) - \sin^4\frac{\alpha}{2} - \sin^4\frac{\beta}{2} - \sin^4\frac{\gamma}{2}}$$

$$= d^2\sqrt{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}-\sin^2\frac{\gamma}{2}\right)^2}$$

Hence, the circumscribed radius (R) of ΔABC having its sides a, b & c (all known) is given as follows

$$R=\frac{abc}{4\Delta}=\frac{\left(2d\sin\frac{\alpha}{2}\right)\left(2d\sin\frac{\beta}{2}\right)\left(2d\sin\frac{\gamma}{2}\right)}{4\left(d^2\sqrt{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}-\sin^2\frac{\gamma}{2}\right)^2}\right)}$$

$$=\frac{2d\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sqrt{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}-\sin^2\frac{\gamma}{2}\right)^2}}=Kd\ (\text{Let's assume})$$

$$\therefore\ \boldsymbol{R=Kd},\ \text{where, } K=\frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sqrt{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-\left(\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}-\sin^2\frac{\gamma}{2}\right)^2}}=\text{constant}\qquad(0<K<1)$$

$$\forall\ (\alpha+\beta)>\gamma,\ (\beta+\gamma)>\alpha,\ (\gamma+\alpha)>\beta\ \ \&\ \ (\alpha+\beta+\gamma)<360^o$$

Note: For ease of understanding & calculations always follow the sequence $\alpha \le \beta \le \gamma$ for given (known) values of angles $\alpha,\ \beta$ & γ between the consecutive lateral edges meeting at a vertex of any tetrahedron.

Hence, the normal height (h) of ΔABC from the vertex (apex) P of the tetrahedron PQRS is given as follows

In right ΔPOA (Fig. 2)

$$PO=\sqrt{(PA)^2-(OA)^2}=\sqrt{d^2-R^2}=\sqrt{d^2-(Kd)^2}=d\sqrt{1-K^2}$$

$$\therefore\ \boldsymbol{h=d\sqrt{1-K^2}}$$

Now, in right ΔOMB (Fig. 2)

$$\boldsymbol{OM}=\sqrt{(BO)^2-(MB)^2}=\sqrt{R^2-\left(\frac{a}{2}\right)^2}=\sqrt{(Kd)^2-\left(\frac{2d\sin\frac{\alpha}{2}}{2}\right)^2}=d\sqrt{K^2-\sin^2\frac{\alpha}{2}}$$

$$\therefore\ \boldsymbol{OM=d\sqrt{K^2-\sin^2\frac{\alpha}{2}}}$$

Now, the solid angle subtended by the right triangle having its orthogonal sides a & b at any point lying at a height h on the vertical axis passing through the vertex common to the side a & the hypotenuse, is given from standard formula-1 of HCR's Theory of Polygon [3,4] as follows

$$\omega=\sin^{-1}\left(\frac{b}{\sqrt{b^2+a^2}}\right)-\sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+a^2}}\right)\left(\frac{h}{\sqrt{h^2+a^2}}\right)\right\}$$

Hence, the solid angle ($\omega_{\Delta OBC}$) subtended by the isosceles ΔOBC at the vertex P of the tetrahedron

$$=\omega_{\Delta OMB}+\omega_{\Delta OMC}=2(\omega_{\Delta OMB})=2(\text{solid angle subtended by the right }\Delta OMB)$$

$$\Rightarrow \omega_{\Delta OBC} = 2\left[\sin^{-1}\left(\frac{(MB)}{\sqrt{(MB)^2+(OM)^2}}\right) - \sin^{-1}\left\{\left(\frac{(MB)}{\sqrt{(MB)^2+(OM)^2}}\right)\left(\frac{(PO)}{\sqrt{(PO)^2+(OM)^2}}\right)\right\}\right]$$

Hence, by setting the corresponding values in the above formula, we obtain

$$\omega_{\Delta OBC} = 2\left[\sin^{-1}\left(\frac{\left(\frac{a}{2}\right)}{\sqrt{\left(\frac{a}{2}\right)^2+(OM)^2}}\right) - \sin^{-1}\left\{\left(\frac{\left(\frac{a}{2}\right)}{\sqrt{\left(\frac{a}{2}\right)^2+(OM)^2}}\right)\left(\frac{(h)}{\sqrt{(h)^2+(OM)^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\left(\frac{2d\sin\frac{\alpha}{2}}{2}\right)}{\sqrt{\left(\frac{2d\sin\frac{\alpha}{2}}{2}\right)^2+\left(d\sqrt{K^2-\sin^2\frac{\alpha}{2}}\right)^2}}\right)\right.$$

$$\left. - \sin^{-1}\left\{\left(\frac{\left(\frac{2d\sin\frac{\alpha}{2}}{2}\right)}{\sqrt{\left(\frac{2d\sin\frac{\alpha}{2}}{2}\right)^2+\left(d\sqrt{K^2-\sin^2\frac{\alpha}{2}}\right)^2}}\right)\left(\frac{\left(d\sqrt{1-K^2}\right)}{\sqrt{\left(d\sqrt{1-K^2}\right)^2+\left(d\sqrt{K^2-\sin^2\frac{\alpha}{2}}\right)^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{d\sin\frac{\alpha}{2}}{\sqrt{d^2\sin^2\frac{\alpha}{2}+K^2d^2-d^2\sin^2\frac{\alpha}{2}}}\right)\right.$$

$$\left. - \sin^{-1}\left\{\left(\frac{d\sin\frac{\alpha}{2}}{\sqrt{d^2\sin^2\frac{\alpha}{2}+K^2d^2-d^2\sin^2\frac{\alpha}{2}}}\right)\left(\frac{d\sqrt{1-K^2}}{\sqrt{d^2-K^2d^2+K^2d^2-d^2\sin^2\frac{\alpha}{2}}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{d\sin\frac{\alpha}{2}}{Kd}\right) - \sin^{-1}\left\{\left(\frac{d\sin\frac{\alpha}{2}}{Kd}\right)\left(\frac{d\sqrt{1-K^2}}{d\sqrt{1-\sin^2\frac{\alpha}{2}}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{K}\right) - \sin^{-1}\left\{\left(\frac{\sin\frac{\alpha}{2}}{K}\right)\left(\frac{\sqrt{1-K^2}}{\cos\frac{\alpha}{2}}\right)\right\}\right] = 2\left[\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{K}\right) - \sin^{-1}\left(\tan\frac{\alpha}{2}\sqrt{\left(\frac{1}{K}\right)^2-1}\right)\right]$$

$$\omega_{\Delta OBC} = 2\left[\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{K}\right) - \sin^{-1}\left(\tan\frac{\alpha}{2}\sqrt{\left(\frac{1}{K}\right)^2-1}\right)\right] = \omega_1 \text{ (let)} \qquad \dots\dots\dots (1)$$

Similarly, we can obtain the following,

$$\omega_{\Delta OAC} = 2\left[\sin^{-1}\left(\frac{\sin\frac{\beta}{2}}{K}\right) - \sin^{-1}\left(\tan\frac{\beta}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right] = \omega_2 \ \text{(let)} \qquad \dots\dots\dots(2)$$

$$\omega_{\Delta OAB} = 2\left[\sin^{-1}\left(\frac{\sin\frac{\gamma}{2}}{K}\right) - \sin^{-1}\left(\tan\frac{\gamma}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right] = \omega_3 \ \text{(let)} \qquad \dots\dots\dots(3)$$

Now, we must check out the nature of ΔABC whether it is an acute, a right or an obtuse triangle. Let's assume the largest angle is γ among known values of $\alpha,\ \beta \ \&\ \gamma$ hence we can determine the largest angle C of ΔABC using cosine formula as follows

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{\left(2d\sin\frac{\alpha}{2}\right)^2 + \left(2d\sin\frac{\beta}{2}\right)^2 - \left(2d\sin\frac{\gamma}{2}\right)^2}{2\left(2d\sin\frac{\alpha}{2}\right)\left(2d\sin\frac{\beta}{2}\right)} = \frac{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\therefore \boldsymbol{\cos C = \frac{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}} \quad \boldsymbol{or} \quad \boldsymbol{C = \cos^{-1}\left(\frac{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}\right)} \qquad \forall\, \boldsymbol{\gamma \ge \beta \ge \alpha}$$

By substituting the known values of angles $\alpha,\ \beta \ \&\ \gamma$ in the above expression, we can directly calculate the value of the largest angle C to check out the nature of ΔABC. Thus, there arise two cases to calculate the solid angle subtended by the plane ΔABC at the vertex P of tetrahedron PQRS as follows

Case 1: ΔABC is an acute or a right triangle ($\forall\ \boldsymbol{\gamma \ge \beta \ge \alpha}\ \&\ \boldsymbol{C \le 90^o}$)

In this case, the foot point O of the perpendicular drawn from the vertex P to the plane of acute ΔABC lies within or on (in case of right triangle) the boundary of this triangle (See the figure 2 above). All the values of solid angles ω_1 , ω_2 & ω_3 corresponding to the angles $\alpha,\ \beta \ \&\ \gamma$ respectively of a tetrahedron are taken as positive. Hence, the solid angle (ω) subtended by the tetrahedron PQRS at the vertex P is equal to the solid angle ($\omega_{\Delta ABC}$) subtended by the acute/right ΔABC at the vertex P of tetrahedron which is given as the sum of magnitudes of solid angles as follows

$$\omega = \omega_{\Delta ABC} = \omega_{\Delta OBC} + \omega_{\Delta OAC} + \omega_{\Delta OAB} = \omega_1 + \omega_2 + \omega_3$$

$\therefore$ Solid angle subtended by the tetrahedron at the vertex $= \boldsymbol{\omega = \omega_1 + \omega_2 + \omega_3}$

Case 2: ΔABC is an obtuse triangle ($\forall\ \boldsymbol{\gamma > \beta \ge \alpha}\ \&\ \boldsymbol{C > 90^o}$)

In this case, the foot point O of the perpendicular drawn from the vertex P to the plane of obtuse ΔABC lies outside the boundary of this triangle. (See the figure 3). In this case, solid angles ω_1 & ω_2 corresponding to the angles α & β respectively are taken as positive while solid angle ω_3 corresponding to the largest angle γ of a tetrahedron is taken as negative. Hence, the solid angle subtended by the tetrahedron PQRS at the vertex P is equal to the solid angle ($\omega_{\Delta ABC}$) subtended by the obtuse ΔABC at the vertex P of tetrahedron which is given as the algebraic sum of solid angles as follows

$$\omega = \omega_{\Delta ABC} = \omega_{\Delta OBC} + \omega_{\Delta OAC} - \omega_{\Delta OAB} = \omega_1 + \omega_2 - \omega_3$$

$\therefore$ Solid angle subtended by the tetrahedron at the vertex is given as

$$\boldsymbol{\omega = \omega_1 + \omega_2 - \omega_3}$$

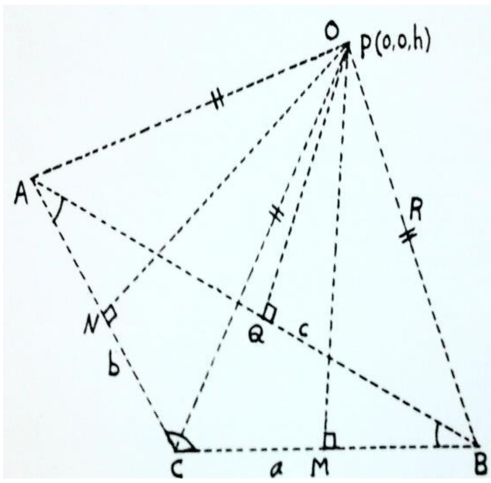

Figure 3: Foot of perpendicular O drawn from the vertex P lies outside the boundary of obtuse ΔABC

3.2. Graphical method for calculation of solid angle

In this method, we first plot the diagram of ΔABC having known sides a, b & c by taking a suitable multiplying factor d (as mentioned above) & then specify the location of foot of perpendicular (F.O.P.) i.e. the circumscribed centre of ΔABC then draw the perpendiculars from circumscribed centre O to all the opposite sides to divide it (i.e. ΔABC) into elementary right triangles. Now, using standard formula-1 [3,4] for a right triangle to find solid angle subtended by each of the elementary right triangles at the centre of sphere, which is given as follows

$$\boldsymbol{\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2+a^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+a^2}}\right)\left(\frac{h}{\sqrt{h^2+a^2}}\right)\right\}}$$

Then find out the algebraic sum (ω) of the solid angles subtended by the elementary right triangles at the vertex of the given tetrahedron depending on the nature of the triangle ABC**.**

$$\therefore \text{Solid angle subtended by the tetrahedron at the vertex} = \omega = \text{algebraic sum of } \omega_1,\ \ \omega_2\ \&\ \ \omega_3$$

4. Important deductions

1. Consider any of eight octants in 3-D co-ordinate system, if three co-ordinates axes X, Y & Z represent the consecutive edges meeting at the origin (vertex) of a tetrahedron then in this case the planes XY, YZ & ZX will represent the consecutive lateral faces meeting/intersecting at the origin (vertex) & we have

$$\alpha = \beta = \gamma = 90^o \text{ (angle between any two orthogonal axes in 3D system)}$$

Now, all the interior angles θ_1, θ_2 & θ_3 opposite to the angles $\alpha,\ \beta$ & γ respectively between the consecutive lateral faces i.e. planes XY, YZ & ZX of the tetrahedron can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow\ \boldsymbol{\theta_1} = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos 90^o - \cos 90^o \cos 90^o}{\sin 90^o \sin 90^o}\right) = \boldsymbol{90^o}$$

$$\boldsymbol{\theta_2} = \cos^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right) = \cos^{-1}\left(\frac{\cos 90^o - \cos 90^o \cos 90^o}{\sin 90^o \sin 90^o}\right) = \boldsymbol{90^o}$$

$$\boldsymbol{\theta_3} = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) = \cos^{-1}\left(\frac{\cos 90^o - \cos 90^o \cos 90^o}{\sin 90^o \sin 90^o}\right) = \boldsymbol{90^o}$$

The above values show that the angle between any two orthogonal planes in 3-D system is 90^o.

Now, calculate the constant K by using the formula as follows

$$K = \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sqrt{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} - \left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}\right)^2}}$$

$$\boldsymbol{K} = \frac{2\sin\frac{90^o}{2}\sin\frac{90^o}{2}\sin\frac{90^o}{2}}{\sqrt{4\sin^2\frac{90^o}{2}\sin^2\frac{90^o}{2} - \left(\sin^2\frac{90^o}{2} + \sin^2\frac{90^o}{2} - \sin^2\frac{90^o}{2}\right)^2}} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\sqrt{2}}{\sqrt{3}} = \boldsymbol{\sqrt{\frac{2}{3}}}$$

Now, by substituting all the corresponding values, we get

$$\Rightarrow \boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{K}\right) - \sin^{-1}\left(\tan\frac{\alpha}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\sin\frac{90^o}{2}}{\left(\sqrt{\frac{2}{3}}\right)}\right) - \sin^{-1}\left(\tan\frac{90^o}{2}\sqrt{\left(\frac{1}{\left(\sqrt{\frac{2}{3}}\right)}\right)^2 - 1}\right)\right]$$

$$\boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \mathbf{2}\left[\frac{\boldsymbol{\pi}}{\mathbf{3}} - \frac{\boldsymbol{\pi}}{\mathbf{4}}\right] = \frac{\boldsymbol{\pi}}{\mathbf{6}}$$

Similarly, we can find out the following values

$$\Rightarrow \boldsymbol{\omega_2} = \frac{\boldsymbol{\pi}}{\mathbf{6}} \quad \& \ \boldsymbol{\omega_3} = \frac{\boldsymbol{\pi}}{\mathbf{6}}$$

The largest angle of ΔABC is C which is calculated by using cosine formula as follows

$$C = \cos^{-1}\left(\frac{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}\right) = \cos^{-1}\left(\frac{\sin^2\frac{90^o}{2} + \sin^2\frac{90^o}{2} - \sin^2\frac{90^o}{2}}{2\sin\frac{90^o}{2}\sin\frac{90^o}{2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^o$$

$$\Rightarrow C = 60^o < 90^o$$

This implies that the plane ΔABC is an acute angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex (origin) to the plane of ΔABC will lie within the boundary of ΔABC (See the figure 2 above) hence, the solid angle subtended by the tetrahedron at the vertex is the sum of the magnitudes of solid angles as follows

$$\boldsymbol{\omega} = \omega_1 + \omega_2 + \omega_3 = \frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{6} = \frac{\boldsymbol{\pi}}{\mathbf{2}}\ \boldsymbol{sr} \qquad \text{(solid angle subtended by each octant at the origin)}$$

Above value shows that the solid angle subtended by each of eight octants in 3-D co-ordinate system at the origin is ${}^{\pi}/{}_{2}\ sr$. It can also be calculated by the following expression

$$\text{Solid angle subtended by each octant at the origin} = \boldsymbol{\omega} = \frac{\text{Total solid angle}}{\text{No. of octants}} = \frac{4\pi}{8} = \frac{\boldsymbol{\pi}}{\mathbf{2}}\ \boldsymbol{sr}$$

2. Consider a regular tetrahedron which has four congruent equilateral triangular faces each three meeting at each of four identical vertices. Hence, for any vertex of a regular tetrahedron, we have

$$\alpha = \beta = \gamma = 60^o \text{ (angle between any two consecutive edges of a regular tetrahedron)}$$

Now, all the interior angles θ_1, θ_2 & θ_3 opposite to the angles α, β & γ respectively between the consecutive equilateral triangular faces of the regular tetrahedron can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow \boldsymbol{\theta_1} = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos60^o - \cos60^o\cos60^o}{\sin60^o\sin60^o}\right) = \cos^{-1}\left(\frac{1}{3}\right) \approx \mathbf{70.52877937^o}$$

$$\boldsymbol{\theta_2} = \cos^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right) = \cos^{-1}\left(\frac{\cos60^o - \cos60^o\cos60^o}{\sin60^o\sin60^o}\right) = \cos^{-1}\left(\frac{1}{3}\right) \approx \mathbf{70.52877937^o}$$

$$\boldsymbol{\theta_3} = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) = \cos^{-1}\left(\frac{\cos60^{\circ} - \cos60^{\circ}\cos60^{\circ}}{\sin60^{\circ}\sin60^{\circ}}\right) = \cos^{-1}\left(\frac{1}{3}\right) \approx \mathbf{70.52877937^{o}}$$

The above values show that the dihedral angle between any two consecutive equilateral triangular faces of a regular tetrahedron is $\cos^{-1}(1/3) \approx 70.52877937^{o}$ [5,6].

Now, calculate the constant K by using the formula as follows

$$K = \frac{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sqrt{4\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} - \left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}\right)^2}}$$

$$\boldsymbol{K} = \frac{2\sin\frac{60^{o}}{2}\sin\frac{60^{o}}{2}\sin\frac{60^{o}}{2}}{\sqrt{4\sin^2\frac{60^{o}}{2}\sin^2\frac{60^{o}}{2} - \left(\sin^2\frac{60^{o}}{2} + \sin^2\frac{60^{o}}{2} - \sin^2\frac{60^{o}}{2}\right)^2}} = \frac{\left(\frac{1}{4}\right)}{\sqrt{\frac{1}{4} - \left(\frac{1}{4}\right)^2}} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}$$

Now, by substituting all the corresponding values, we get

$$\Rightarrow \boldsymbol{\omega_1} = \mathbf{2}\left[\mathbf{sin^{-1}}\left(\frac{\mathbf{sin}\frac{\boldsymbol{\alpha}}{\mathbf{2}}}{\boldsymbol{K}}\right) - \mathbf{sin^{-1}}\left(\mathbf{tan}\frac{\boldsymbol{\alpha}}{\mathbf{2}}\sqrt{\left(\frac{\mathbf{1}}{\boldsymbol{K}}\right)^{\mathbf{2}} - \mathbf{1}}\right)\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\sin\frac{60^{o}}{2}}{\left(\frac{1}{\sqrt{3}}\right)}\right) - \sin^{-1}\left(\tan\frac{60^{o}}{2}\sqrt{\left(\frac{1}{\left(\frac{1}{\sqrt{3}}\right)}\right)^2 - 1}\right)\right]$$

$$\boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right] = \mathbf{2}\left[\frac{\boldsymbol{\pi}}{\mathbf{3}} - \mathbf{sin^{-1}}\left(\sqrt{\frac{\mathbf{2}}{\mathbf{3}}}\right)\right]$$

Similarly, we can find out the following values

$$\Rightarrow \boldsymbol{\omega_2} = \mathbf{2}\left[\frac{\boldsymbol{\pi}}{\mathbf{3}} - \mathbf{sin^{-1}}\left(\sqrt{\frac{\mathbf{2}}{\mathbf{3}}}\right)\right] \quad \& \ \boldsymbol{\omega_3} = \mathbf{2}\left[\frac{\boldsymbol{\pi}}{\mathbf{3}} - \mathbf{sin^{-1}}\left(\sqrt{\frac{\mathbf{2}}{\mathbf{3}}}\right)\right]$$

The largest angle of ΔABC is C which is calculated by using cosine formula as follows

$$C = \cos^{-1}\left(\frac{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}\right) = \cos^{-1}\left(\frac{\sin^2\frac{60^{o}}{2} + \sin^2\frac{60^{o}}{2} - \sin^2\frac{60^{o}}{2}}{2\sin\frac{60^{o}}{2}\sin\frac{60^{o}}{2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^{o}$$

$$\Rightarrow C = 60^{o} < 90^{o}$$

Hence, the plane ΔABC is an acute angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex of tetrahedron to the plane of ΔABC will lie within the boundary of ΔABC (See the figure 2 above) hence, the solid angle subtended by the tetrahedron at the vertex is the sum of the magnitudes of solid angles as follows

$$\boldsymbol{\omega} = \omega_1 + \omega_2 + \omega_3 = 2\left[\frac{\pi}{3} - \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right] + 2\left[\frac{\pi}{3} - \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right] + 2\left[\frac{\pi}{3} - \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right]$$

$$= \boldsymbol{2\pi - 6\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \approx 0.551285598\ sr}$$

The above value shows that the solid angle subtended by a regular tetrahedron at any of its four vertices is $0.551285598\ sr$ [7]. It can also be computed using HCR's standard formula of solid angle [8] as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)}$$

For a regular tetrahedron, we have

$$n = \text{no. of sides in each face} = 3 \quad \& \quad \alpha = \text{angle between adjacent lateral edges} = 60^o$$

$$\therefore\ \boldsymbol{\omega} = 2\pi - 2(3)\sin^{-1}\left(\cos\frac{\pi}{3}\sqrt{\tan^2\frac{\pi}{3} - \tan^2\frac{60^o}{2}}\right) = 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\left(\sqrt{3}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{8}{3}}\right) = \boldsymbol{2\pi - 6\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)} \approx \boldsymbol{0.551285598\ sr}$$

Both the above results are equal hence the generalized formula of a tetrahedron is verified.

5. Application and validation of formula for tetrahedron

These examples are based on all above articles which are very practical and directly & simply applicable to calculate the dihedral angles between the consecutive faces & the solid angle subtended by the tetrahedron at the vertex. For ease of understanding & calculations, value of angle γ of ΔABC is taken as the largest one).

Example 1: Calculate the interior angles between the consecutive faces and the solid angle subtended by a tetrahedron at the vertex such that the angles between the consecutive edges meeting at the same vertex are $30^o, 40^o$ and 50^o.

Solution. Here, we have

$$\alpha = 30^o\ ,\ \ \beta = 40^o\ \ \&\ \ \gamma = 50^o\ \Rightarrow\ \theta_1, \theta_2\ \&\ \theta_3 = ?\ \&\ \text{ solid angle, }\ \omega = ?$$

Now, all the interior angles θ_1, θ_2 & θ_3 opposite to the angles α, β & γ respectively of the tetrahedron can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow\ \boldsymbol{\theta_1} = cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) = cos^{-1}\left(\frac{cos30^o - cos40^o cos50^o}{sin40^o sin50^o}\right) \approx \boldsymbol{40.64407403^o} \approx \boldsymbol{40^o 38' 38.67''}$$

$$\boldsymbol{\theta_2} = cos^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right) = cos^{-1}\left(\frac{cos40^o - cos50^o cos30^o}{sin50^o sin30^o}\right) \approx \boldsymbol{56.86341165^o} \approx \boldsymbol{56^o 51' 48.28''}$$

$$\boldsymbol{\theta_3} = cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right) = cos^{-1}\left(\frac{cos50^o - cos30^o cos40^o}{sin30^o sin40^o}\right) \approx \boldsymbol{93.6796444^o} \approx \boldsymbol{93^o 40' 46.72''}$$

Now, calculate the constant K by using the formula as follows

$$K = \frac{2sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}}{\sqrt{4sin^2\frac{\alpha}{2}sin^2\frac{\beta}{2} - \left(sin^2\frac{\alpha}{2} + sin^2\frac{\beta}{2} - sin^2\frac{\gamma}{2}\right)^2}}$$

Now, by substituting all the corresponding values, we get

$$K = \frac{2sin\frac{30^o}{2}sin\frac{40^o}{2}sin\frac{50^o}{2}}{\sqrt{4sin^2\frac{30^o}{2}sin^2\frac{40^o}{2} - \left(sin^2\frac{30^o}{2} + sin^2\frac{40^o}{2} - sin^2\frac{50^o}{2}\right)^2}} \approx 0.422811997$$

$$\Rightarrow \omega_1 = 2\left[\sin^{-1}\left(\frac{sin\frac{\alpha}{2}}{K}\right) - \sin^{-1}\left(tan\frac{\alpha}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore \omega_1 = 2\left[\sin^{-1}\left(\frac{sin\frac{30^o}{2}}{0.422811997}\right) - \sin^{-1}\left(tan\frac{30^o}{2}\sqrt{\left(\frac{1}{0.422811997}\right)^2 - 1}\right)\right] \approx 0.094028018\ sr$$

$$\Rightarrow \omega_2 = 2\left[\sin^{-1}\left(\frac{sin\frac{\beta}{2}}{K}\right) - \sin^{-1}\left(tan\frac{\beta}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore \omega_2 = 2\left[\sin^{-1}\left(\frac{sin\frac{40^o}{2}}{0.422811997}\right) - \sin^{-1}\left(tan\frac{40^o}{2}\sqrt{\left(\frac{1}{0.422811997}\right)^2 - 1}\right)\right] \approx 0.094962792\ sr$$

$$\Rightarrow \omega_3 = 2\left[\sin^{-1}\left(\frac{sin\frac{\gamma}{2}}{K}\right) - \sin^{-1}\left(tan\frac{\gamma}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore \omega_3 = 2\left[\sin^{-1}\left(\frac{sin\frac{50^o}{2}}{0.422811997}\right) - \sin^{-1}\left(tan\frac{50^o}{2}\sqrt{\left(\frac{1}{0.422811997}\right)^2 - 1}\right)\right] \approx 0.00626143758\ sr$$

The largest angle of ΔABC is C which is calculated by using cosine formula as follows

$$C = \cos^{-1}\left(\frac{sin^2\frac{\alpha}{2} + sin^2\frac{\beta}{2} - sin^2\frac{\gamma}{2}}{2sin\frac{\alpha}{2}sin\frac{\beta}{2}}\right) = \cos^{-1}\left(\frac{sin^2\frac{30^o}{2} + sin^2\frac{40^o}{2} - sin^2\frac{50^o}{2}}{2sin\frac{30^o}{2}sin\frac{40^o}{2}}\right) \approx 88.26545646^o < 90^o$$

Hence, the plane ΔABC is an acute angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex of tetrahedron to the plane of ΔABC will lie within the boundary of ΔABC (See the figure 2 above) hence, the solid angle subtended by the tetrahedron at the vertex is the sum of the magnitudes of solid angles as follows

$$\boldsymbol{\omega} = \omega_1 + \omega_2 + \omega_3 \approx 0.094028018 + 0.094962792 + 0.00626143758 \approx \mathbf{0.195252247\ sr} \qquad \boldsymbol{Ans.}$$

The above value of area implies that the given tetrahedron subtends a solid angle $\approx 0.195252247\ sr$ at the vertex irrespective of its geometrical dimensions.

Example 2: Calculate the interior angles between the consecutive faces & the solid angle subtended by a tetrahedron at the vertex such that the angles between the consecutive edges meeting at the same vertex are $40^o, 70^o$ & 85^o.

Solution. Here, we have

$$\alpha = 40^o\ ,\ \ \beta = 70^o\ \ \&\ \ \gamma = 85^o\ \Rightarrow\ \theta_1, \theta_2\ \&\ \theta_3 = ?\ \&\ \text{ solid angle, }\ \omega = ?$$

Now, all the interior angles θ_1, θ_2 & θ_3 opposite to the angles $\alpha,\ \beta$ & γ respectively of the tetrahedron can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow\ \boldsymbol{\theta_1} = cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) = cos^{-1}\left(\frac{cos40^o - cos70^o cos85^o}{sin70^o sin85^o}\right) \approx \mathbf{38.1424026^o} \approx \mathbf{38^o 8' 32.65''}$$

$$\boldsymbol{\theta_2} = cos^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right) = cos^{-1}\left(\frac{cos70^o - cos85^o cos40^o}{sin85^o sin40^o}\right) \approx \mathbf{64.54154954^o} \approx \mathbf{64^o 32' 29.58''}$$

$$\boldsymbol{\theta_3} = cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right) = cos^{-1}\left(\frac{cos85^o - cos40^o cos70^o}{sin40^o sin70^o}\right) \approx \mathbf{106.8262695^o}$$
$$\approx \mathbf{106^o 49' 34.57''}$$

Now, calculate the constant K by using the formula as follows

$$K = \frac{2sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}}{\sqrt{4sin^2\frac{\alpha}{2}sin^2\frac{\beta}{2} - \left(sin^2\frac{\alpha}{2} + sin^2\frac{\beta}{2} - sin^2\frac{\gamma}{2}\right)^2}}$$

Now, by substituting all the corresponding values, we get

$$K = \frac{2sin\frac{40^o}{2}sin\frac{70^o}{2}sin\frac{85^o}{2}}{\sqrt{4sin^2\frac{40^o}{2}sin^2\frac{70^o}{2} - \left(sin^2\frac{40^o}{2} + sin^2\frac{70^o}{2} - sin^2\frac{85^o}{2}\right)^2}} \approx 0.675830167$$

$$\Rightarrow\ \omega_1 = 2\left[\sin^{-1}\left(\frac{sin\frac{\alpha}{2}}{K}\right) - \sin^{-1}\left(tan\frac{\alpha}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore \omega_1 = 2\left[\sin^{-1}\left(\frac{sin\frac{40^o}{2}}{0.675830167}\right) - \sin^{-1}\left(tan\frac{40^o}{2}\sqrt{\left(\frac{1}{0.675830167}\right)^2 - 1}\right)\right] \approx 0.244883226\ sr$$

$$\Rightarrow\ \omega_2 = 2\left[\sin^{-1}\left(\frac{sin\frac{\beta}{2}}{K}\right) - \sin^{-1}\left(tan\frac{\beta}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore\ \omega_2 = 2\left[\sin^{-1}\left(\frac{sin\frac{70^o}{2}}{0.675830167}\right) - \sin^{-1}\left(tan\frac{70^o}{2}\sqrt{\left(\frac{1}{0.675830167}\right)^2 - 1}\right)\right] \approx 0.289166683\ sr$$

$$\Rightarrow\ \omega_3 = 2\left[\sin^{-1}\left(\frac{sin\frac{\gamma}{2}}{K}\right) - \sin^{-1}\left(tan\frac{\gamma}{2}\sqrt{\left(\frac{1}{K}\right)^2 - 1}\right)\right]$$

$$\therefore\ \omega_3 = 2\left[\sin^{-1}\left(\frac{sin\frac{85^o}{2}}{0.675830167}\right) - \sin^{-1}\left(tan\frac{85^o}{2}\sqrt{\left(\frac{1}{0.675830167}\right)^2 - 1}\right)\right] \approx 0.018999357\ sr$$

The largest angle of ΔABC is C which is calculated by using cosine formula as follows

$$C = \cos^{-1}\left(\frac{sin^2\frac{\alpha}{2} + sin^2\frac{\beta}{2} - sin^2\frac{\gamma}{2}}{2sin\frac{\alpha}{2}sin\frac{\beta}{2}}\right) = \cos^{-1}\left(\frac{sin^2\frac{40^o}{2} + sin^2\frac{70^o}{2} - sin^2\frac{85^o}{2}}{2sin\frac{40^o}{2}sin\frac{70^o}{2}}\right) \approx 91.52686653^o > 90^o$$

Hence, the plane ΔABC is an obtuse angled triangle.

Hence the foot of perpendicular (F.O.P.) drawn from the vertex of tetrahedron to the plane of ΔABC will lie outside the boundary of plane ΔABC (See the figure 3 above) hence, the solid angle subtended by the tetrahedron at the vertex is the algebraic sum of the solid angles as follows

$$\boldsymbol{\omega} = \omega_1 + \omega_2 - \omega_3 \approx 0.244883226 + 0.289166683 - 0.018999357 \approx \mathbf{0.515050552\ sr} \qquad \boldsymbol{Ans.}$$

The above value of area implies that the given tetrahedron subtends a solid angle $\approx 0.515050552\ sr$ at the vertex irrespective of its geometrical dimensions.

Conclusions

All the formulae presented in this work have been derived using elementary principles of geometry and trigonometry. The resulting analytical expressions provide a simple and systematic method for determining the internal (dihedral) angles between consecutive lateral faces of an arbitrary tetrahedron at any of its four vertices, as well as the solid angle subtended by the tetrahedron at a vertex when the angles between the corresponding edges are known. Owing to their generalized form, these relations can also be applied to configurations in which three faces meet at a vertex of various regular and uniform polyhedra, thereby facilitating the evaluation of vertex solid angles in such solids. Beyond their theoretical significance in polyhedral geometry, the derived formulae have potential applications in fields such as computational geometry, structural and architectural design, crystallography, and geometric modeling, where accurate determination of angular relationships and spatial configurations is essential. In addition, the results may be useful in problems involving solid-angle calculations in radiative transfer, illumination analysis, and physical simulations of polyhedral assemblies. Future work may extend the present approach to more complex polyhedral structures, including irregular polyhedra and higher-dimensional analogues, as well as to the development of unified analytical frameworks for evaluating dihedral and solid angles in general spatial networks. Such extensions could further enhance the applicability of the derived relations in both theoretical investigations and practical geometric design problems.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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