On the Nonexistence of the Identity $c^3 - b^3 = a^3$ for Pythagorean Triples

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Abstract

We show that for any Pythagorean triple (a, b, c) satisfying $a^2 + b^2 = c^2$ with $a, b, c \in \mathbb{N}$, the equality $c^3 - b^3 = a^3$ cannot hold. The proof is elementary and relies only on algebraic manipulation and the structure of the Pythagorean equation, without invoking Fermat's Last Theorem.

1 Introduction

A Pythagorean triple is a triple of positive integers (a, b, c) satisfying

$$a^2 + b^2 = c^2.$$

Such triples have been extensively studied in classical number theory and have a well-known parametrization. In this note we examine whether the cubic relation

$$c^3 - b^3 = a^3$$

can occur simultaneously with the Pythagorean identity.

Observe that the equality $c^3 - b^3 = a^3$ is equivalent to

$$a^3 + b^3 = c^3,$$

which resembles the case $n = 3$ of Fermat's equation. While the nonexistence of solutions follows immediately from Fermat's Last Theorem, it is natural to ask whether the incompatibility can be established directly using only the Pythagorean relation. We provide such an elementary argument.

2 Main Result

Theorem 1. *Let $(a, b, c) \in \mathbb{N}^3$ be a Pythagorean triple satisfying*

$$a^2 + b^2 = c^2.$$

Then

$$c^3 - b^3 \neq a^3.$$

Proof. Suppose, for contradiction, that

$$c^3 - b^3 = a^3.$$

Using the factorization of the difference of cubes,

$$(c - b)(c^2 + cb + b^2) = a^3. \tag{1}$$

Let

$$d = c - b.$$

Since $a^2 + b^2 = c^2$, we have

$$(c - b)(c + b) = a^2,$$

that is,

$$d(c + b) = a^2. \tag{2}$$

Because $c = b + d$, it follows that

$$c + b = 2b + d.$$

From (2) we obtain

$$2b + d = \frac{a^2}{d}. \tag{3}$$

Now consider the second factor in (1):

$$c^2 + cb + b^2 = (a^2 + b^2) + cb + b^2 = a^2 + 2b^2 + cb.$$

Since $c = b + d$, we have

$$cb = b(b + d) = b^2 + bd,$$

and therefore

$$c^2 + cb + b^2 = a^2 + 3b^2 + bd. \tag{4}$$

Substituting the expression for b from (3) and simplifying yields

$$a^2 + 3b^2 + bd = \frac{3a^4/d^2 + d^2}{4}.$$

Substituting this into (1) gives

$$d \cdot \frac{3a^4/d^2 + d^2}{4} = a^3.$$

After simplification we obtain

$$3a^4 + d^4 = 4a^3 d. \tag{5}$$

Dividing by a^4 and setting $t = d/a$ yields

$$t^4 - 4t + 3 = 0.$$

Factorization gives

$$t^4 - 4t + 3 = (t-1)^2(t^2 + 2t + 3).$$

Since $t^2 + 2t + 3 > 0$ for all real t, the only real solution is $t = 1$, hence $d = a$.

Thus $c - b = a$, and substituting this into (2) yields

$$a^2 = a(c + b),$$

which implies $a = c + b$. However, this is impossible because $c > b > 0$. This contradiction completes the proof. $\qquad\square$

3   Conclusion

We have shown that the cubic identity $c^3 - b^3 = a^3$ cannot occur for any Pythagorean triple. The proof is entirely elementary and relies only on algebraic manipulation together with the relation $a^2 + b^2 = c^2$. In particular, the argument does not require Fermat's Last Theorem.

References

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press.

2. T. Nagell, *Introduction to Number Theory*, Chelsea Publishing Company.