Infinite cycle classes

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March 25, 2026

Abstract

We define and study infinite cycle classes - particular types of cohomology classes in the convex-cone-approximation of currents. Furthermore, this convex cone descends to a polyhedral cone in homology. This phenomenon implies that on a compact Kähler manifold, infinite cycle classes are linear combinations of analytic subvarieties with real coefficients, and on a complex projective manifold, infinite cycle classes are algebraic cycle classes with rational coefficients.

Contents

1 Statement

It is the great interest in algebraic geometry to characterize those cohomology classes that can be represented by a linear combination of subvarieties. We explore this topic with currents. More specifically, we study a type of cohomology class that can be represented by infinitely series of subvarieties with the strong convergence. Let's begin with the definition.

Definition 1.1. *(Infinite cycle class) Let X be a compact complex manifold. Let $T_\bullet$ denote the integration current over a chain $\bullet$. Let $\mathbf{M}$ denote a mass of*

Key words: currents, algebraic cycles.
2020 Mathematics subject classification: 49Q15, 32J25

currents, based on a Hermitian metric. A class $u \in H^{2p}(X;\mathbb{R})$ *is an infinite cycle class if it is represented by a closed current of an absolutely mass-convergent series of currents*

$$\sum_{i=1}^{\infty} r_i T_{V_i} \tag{1.1}$$

where V_i *are irreducible subvarieties coupled with real coefficients* r_i*. The absolute mass-convergence is the absolute norm convergence, i.e.*

$$\sum_{i=1}^{\infty} \mathbf{M}(r_i T_{V_i}) < +\infty$$

or equivalently

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{1.2}$$

for any $N' \geq N$*. The closed current in (1.1) is called an infinitely complex analytic cycle. Such a class (or cycle) has positivity if* $r_i > 0$ *for all* i*.*

So, infinite cycle classes form a subspace and those with positivity form a convex cone.

Remark The definition is independent of Hermitian metric. The approximation (1.1) is a particular type of convex-cone-approximation ([2]). This type approximation is closely related holomorphic chains. However, since an infinite cycle class uses the convergence of currents, it is not a holomorphic chain. Following is such an example. It is based on a counter-example in [6].

Example 1.2. *Let* $\mathbb{C}P^1$ *be the projective space over* $\mathbb{C}$*. Let* $z_i \in \mathbb{C}P^1$ *for the positive integer* i *be a sequence of points that converges to* $\mathbf{o} \in \mathbb{C}P^1$*. Let* r_i *be a sequence of positive numbers such that* $\sum_{i=1}^{\infty} r_i = \lambda$ *is a finite number. The following is the behavior of such sequences.*

(1) $\sum_{i=1}^{\infty} r_i T_{\{z_i\}}$ *is a closed current whose cohomology class is the infinitely algebraic class* $\lambda[z_1]$ *where* $[z_1] \in H^2(\mathbb{C}P^1;\mathbb{Z})$ *is represented by* z_1*.*

(2) the current $\sum_{i=1}^{\infty} r_i T_{\{z_i\}}$ *is not a holomorphic chain with real coefficients because* $\cup_{i=1}^{\infty}\{z_i\}$ *is not a subvariety of* $\mathbb{C}P^1$*.*

(3) Let $T^o_{\{z_i\}}$ *be the restriction of* $T_{\{z_i\}}$ *to the affine open set* $\mathbb{C}P^1 \backslash \{\mathbf{o}\}$*, in which,* $\sum_{i=1}^{\infty} r_i T^o_{\{z_i\}}$ *is still a closed current that represents the cohomology of the point* z_1*. But in* $\mathbb{C}P^1 \backslash \{\mathbf{o}\}$*, it is also a holomorphic chain with real coefficients because* $\cup_{i=1}^{\infty}\{z_i\}$ *is a subvariety.*

Main theorem 1.3.

(1) If X is a compact Kähler manifold and $u \in H^{2p}(X;\mathbb{R})$ is an infinite cycle class, then u is represented by a linear combination of subvarieties with real coefficient.

(2) If X is a complex projective manifold and $u \in H^{2p}(X;\mathbb{Q})$ is an infinite cycle class, then u is represented by an algebraic cycle with rational coefficients.

2 Positivity of homology classes

In [3], Harvey-Lawson introduced a positivity of homology classes. In this section, we further show that the cone of these positive classes is a polyhedral cone. We first recall some of the notations in currents. Let X be a compact manifold, M a compact submanifold. We use $\mathscr{E}^\bullet$ to denote the Frechet space consisting of smooth forms, where $\bullet$ stands for the degree. Taking the topological dual, we have $\mathscr{E}'_\bullet(X)$ equipped with the weak topology. We have two complexes to compute the homology and cohomology

$$\mathscr{E}^{i-1}(X) \quad \stackrel{d}{\rightarrow} \quad \mathscr{E}^{i}(X) \quad \stackrel{d}{\rightarrow} \quad \mathscr{E}^{i+1}(X) \tag{2.1}$$

$$\mathscr{E}'_{i-1}(X) \quad \stackrel{d}{\leftarrow} \quad \mathscr{E}'_{i}(X) \quad \stackrel{d}{\leftarrow} \quad \mathscr{E}'_{i+1}(X) \tag{2.2}$$

Then

$$H^i(X;\mathbb{R}) = Z/B, H_i(X;\mathbb{R}) = \widetilde{Z}/\widetilde{B}$$

where Z, B are the cycles and boundaries of the complex (2.1), $\widetilde{Z}, \widetilde{B}$ are the cycles and boundaries of the complex (2.2).

So,

$$H_{2k}(X;\mathbb{R}), H^{2k}(X;\mathbb{R})$$

are finitely dimensional vector spaces, and there is a Poincaé duality:

$$\left(H_{2k}(X;\mathbb{R})\right)^* \simeq H^{2k}(X;\mathbb{R}).$$

Definition 2.1. *Let X be a compact Kähler manifold. A class*

$$\tau \in H_{2k}(X;\mathbb{R})$$

is called positive if τ is represented by a strongly positive current of bidimension (k,k).

Remark. The definition is consistent with the positivity of relative classes defined by Harvey-Lawson [3].

Let $\mathcal{Z}^p(X)$ be the free Abelian group generated by complex analytic subvarieties of codimension p. Let

$$[\bullet] : \{\textit{closed currents}\} \quad \rightarrow \quad \textit{homology group} \tag{2.3}$$

be the reduction homomorphism. Let

$$C^p_{\mathbb{Z}} \subset H_{2k}(X;\mathbb{Z})$$

be the image $[\mathcal{Z}^p(X)]$, where k is the dimension of the subvarieties, and a subvariety is regarded as the integration currents. Let

$$C^p_{\mathbb{R}} := C^p_{\mathbb{Z}} \otimes \mathbb{R} \ \subset H_{2k}(X;\mathbb{R}).$$

Let

$$E^p_{\mathbb{R}} \subset C^p_{\mathbb{R}}$$

be the cone that consists of weakly positive classes as in Definition 2.1.

Now we assume X is a compact Kähler manifold. Let

$$\widetilde{\mathcal{P}} = \{T \in \mathscr{E}'_{2k}(X); T = T_{k,k} \geq 0(weakly)\}$$

Define the cones

$$\widetilde{C^+} := \frac{\widetilde{\mathcal{P}} \cap \widetilde{Z} + \widetilde{B}}{\widetilde{B}} \subset H_{2k}(X;\mathbb{R}) \tag{2.4}$$

$$\tag{2.5}$$

It is clear that $\widetilde{C^+}$ is a closed convex cone consists of all weakly positive classes. We are going to prove further

Lemma 2.2. *The convex cones $\widetilde{C^+}$ is polyhedral. In particular,*

$$E^p_{\mathbb{R}} = \widetilde{C^+} \cap C^p_{\mathbb{R}} \tag{2.6}$$

is polyhedral.

Proof. Let

$$m = dim(H^{2k}(X;\mathbb{R})).$$

We represent a basis for $H^{2k}(X;\mathbb{R})$ by m-tuple C^∞ closed forms

$$\Phi = (\phi_1, \phi_2, \cdots, \phi_h, \phi_{h+1}, \cdots, \phi_m)$$

of degree $2k$, where $\{\phi_1, \cdots, \phi_h\}$ is a basis for $H^{k,k}(X;\mathbb{R})$. For each current σ that represents the class $[\sigma]$ in $H_{2k}(X;\mathbb{R})$, the m-tuple of numbers $L := \sigma(\Phi)$ represent a point in

$$H_{2k}(X;\mathbb{R}) \simeq \mathbb{R}^m.$$

So, Poincaré duality isomorphism converts

$$\mathscr{P} : \{[\sigma]\} \to \{L\}.$$

To consider the positive cones in the (k,k) portion, we add a sufficiently large multiple of ω^k where ω is the Kähler form to the (k,k) portion, to obtain that all $\phi_1, \cdots, \phi_h$ are positive. Let

$$\pi : \widetilde{C}^+ \to \mathbb{R}^2$$

be the orthogonal projection to any 2-dimensional plane V_2 in $\mathbb{R}^m \simeq H_{2k}(X;\mathbb{R})$. We should note that all classes in $\widetilde{C}^+$ are of bidegree (k,k). Hence for the convenience, we may choose the plane determined by ϕ_1, ϕ_2. Precisely, we choose

$$\begin{array}{ccc} \pi : V_2 & \to & \mathbb{R}^2 \\ \sigma & \to & (\sigma(\phi_1), \sigma(\phi_2)), \end{array}$$

i.e. the 2-plane that is the dual of $H_{2k}(X;\mathbb{R})$, determined by the forms ϕ_1, ϕ_2. So we let σ_n for natural numbers n be a sequence of closed currents representing the points $[\sigma_n]$ in $\widetilde{C^+}$. We assume $\pi([\sigma_n])$ approaches a finite point in the plane $\mathbb{R}^2$. For instance,

$$\lim_{n\to\infty} \sigma_n(\phi_1) \le +\infty. \tag{2.7}$$

Notice

$$\mathbf{M}(\sigma_n) = \frac{\sigma_n(\omega^k)}{k!} \le \frac{\sigma_n(\phi_1)}{k!} \le +\infty \tag{2.8}$$

where $\mathbf{M}$ is the mass based on Kähler metric ω, and we also used the fact on a compact Kähler manifold: for a positive current $\mathcal{T}$ of bidimension (q,q),

$$\mathbf{M}(\mathcal{T}) = \mathcal{T}[\frac{\omega^q}{q!}]. \tag{2.9}$$

This fact has been proved and used at multiple places. For the proof, see Theorem 2.2 combined with Remark 2.5 in [4]. Since $\mathbf{M}(\sigma_n)$ is bounded, by Lemma 2.15, chapter 6 in [8], $\{\sigma_n\}$ has a subsequence that converges weakly to a current σ_∞ whose represented class $[\sigma_\infty] \in \widetilde{C^+}$. Hence the limit of the projections $\pi([\sigma_n])$ as $n \to \infty$ is $(\sigma_\infty(\phi_1), \sigma_\infty(\phi_2))$ with $\sigma_\infty \in \widetilde{C_M^+}$. This implies that $\pi(\widetilde{C^+})$ is closed. By the characterization of polyhedral cones in the theorem of [5], $\widetilde{C^+}$ is polyhedral. We complete the proof.

□

Remark. Through Poincaré duality, we are addressing the positivity for cohomology also. In complex geometry, the positivity for cohomology classes plays a significant role. Positivity first appeared in the bidegree $(1,1)$ classes. In 1977, Steve Zucker produced a compact Kähler manifold that does not admit complex analytic subvarieties of middle dimension $2n$ for $n \ge 2$ ([7]). The example indicated the importance of positivity in subvarieties. However, two years earlier in [4] Lawson showed that his version of positivity does agree with the positivity of the subvarieties. In 2009, Harvey-Lawson introduced the positivity similar to that in Definition 2.1, whose origin is the postivity of currents. In the same paper [4], they also showed that this positivity leads to some of their fundamental results on the boundaries of holomorphic chains.

3 Proof of Main theorem 1.3

(1): Let u be a rational, infinitely complex class. So we write a representation

$$\sum_{i=1}^{\infty} r_i T_{V_i} \tag{3.1}$$

where V_i are irreducible subvarieties of dimension $k = dim(X) - p$. Let

$$[V_i] = \sum_{finite\ j} \lambda_i^j [A^j]$$

where λ_i^j are real numbers, and A_j are those from the frame of the polyhedral cone $E_{\mathbb{R}}^p$ as in Lemma 2.2. By Lemma 2.2, $E_{\mathbb{R}}^p$ is polyhedral. Since $[V_i]$ is in the cone $E_{\mathbb{R}}^p$, all coefficients λ_i^j must be non-negative. By the mass formula (2.9), the evaluation with the power of Kähler form ω

$$T_{V_i}[\frac{\omega^k}{k!}] = \sum_{finite\ j} \lambda_i^j T_{A^j}[\frac{\omega^k}{k!}]. \tag{3.2}$$

implies

$$\mathbf{M}(T_{V_i}) = \sum_{finite\ j} \lambda_i^j \mathbf{M}(T_{A^j}), \tag{3.3}$$

where A_j by Proposition 3.3, [3] can be chosen to be strongly positive current. On the other hand, by the absolute mass-convergence of (1.1), we have

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{3.4}$$

where $N' \geq N$. Plugging (3.3) into (3.4), we obtain

$$\sum_{finite\ j} \mathbf{M}(T_{A^j}) \left(\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \lambda_i^j \right) = 0, \tag{3.5}$$

Since $\mathbf{M}(T_{A^j})$ is positive and λ_i^j are all non-negative, for each j

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \lambda_i^j = 0. \tag{3.6}$$

Then (3.6), for each j, implies the absolute convergence for the series

$$\alpha_j := \sum_{i=1}^{\infty} r_i \lambda_i^j. \tag{3.7}$$

Now we work with the convergence in cohomology. Due to the finiteness of Betti number, cohomological convergence is determined by the convergence of the real numbers on each axis. Precisely, we see that the convergence (3.6) implies that u, which is

$$\left[\sum_{i=1}^{\infty} r_i V_i\right], \tag{3.8}$$

is approached by the cycle classes with real coefficients,

$$\left[\sum_{i=1}^{N} r_i V_i\right] = \sum_{finite\ j} \left(\sum_{i=1}^{N} r_i \lambda_i^j\right)[A^j],\ as\ N \to \infty. \tag{3.9}$$

($[\bullet]$ is continuous). Notice that the cohomology class (3.9), by the convergence (3.6), also converges to a cycle class with real coefficients written as

$$u = \sum_{finite\ j} \alpha_j [A^j]; \tag{3.10}$$

$$\alpha_j = \sum_{i=1}^{\infty} r_i \lambda_i^j. \tag{3.11}$$

So, (3.10) proves Part (1).

(2): In this case, X is a complex projetive manifold. We convert (3.10) to that with $\mathbb{Q}$-coefficients. For any closed subset $W \subset X$, the subgroup

$$ker\left(H^i(X;\mathbb{Q}) \to H^i(X\backslash W;\mathbb{Q})\right) \tag{3.12}$$

will be denoted by $H^i_{(W)}(X;\mathbb{Q})$ where ker stands for the kernel of the restriction map. A class $\gamma \in H^i(X;\mathbb{Q})$ is said to be class-supported on W if $\gamma \in H^i_{(W)}(X;\mathbb{Q})$. On the other hand, we say a class is current-supported on W if it is represented by a closed current supported on W. De Rahms's homology of currents implies that a class current-supported on a W is a class class-supported on W. Recall (3.10)

$$u = \sum_{finite\ j} \alpha_j [A^j] \tag{3.13}$$

where α_j are real and A^j are algebraic cycles with real coefficients. If we let $V = \bigcup_{finite\ j} |A^j|$ be the algebraic set, u is current-supported on V, then u is also class-supported on V. Let

$$\widetilde{V} \stackrel{J}{\to} V \stackrel{I}{\hookrightarrow} X$$

be the composite such that J is a smooth resolution and I is the inclusion. Since the codimension condition

$$deg(u) - 2cod(V) \geq 0$$

is satisfied, we apply Deligne's corollary 8.2.8, [1] which addresses the class-support. Precisely it states that the Gysin map

$$(I \circ J)_! : H^0(\widetilde{V}; \mathbb{Q}) \to H^{2p}_{(V)}(X; \mathbb{Q}) \tag{3.14}$$

is surjective. Then a preimage $\tilde{u}$ of u is a cohomological class of degree 0 on the complex manifold $\widetilde{V}$. So, $\tilde{u}$ must be represented by a rational linear combinations of irreducible components of $\widetilde{V}$. Since J is a complex analytic map from $\widetilde{V}$ onto V, $u = (I \circ J)_!(\tilde{u})$ is represented by a rational, linear combination of irreducible components of V. The proof is completed.

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