Infinite cycle classes

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Abstract

Let X be a complex projective manifold X. If a rational cohomology class can be approximated by irreducible subvarieties in a particular way, then it is represented by an algebraic cycle with rational coefficients.

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1 Statement

It is the great interest in algebraic geometry to characterize those cohomology classes that can be represented by a linear combination of subvarieties. In this paper, we explore the topic with currents. First we define the particular current's approximation mentioned in the abstract.

Definition 1.1. *(Infinite complexity) Let X be a compact complex manifold. Let $T_\bullet$ denote the integration current over a chain $\bullet$. Let $\mathbf{M}$ denote a mass of currents, based on a Hermitian metric. A class $u \in H^{2p}(X;\mathbb{R})$ is an infinite cycle class if it is represented by a closed current of an absolutely mass-convergent series of currents*

$$\sum_{i=1}^{\infty} r_i T_{V_i}. \tag{1.1}$$

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The absolute mass-convergence is the absolute norm convergence, i.e.

$$\sum_{i=1}^{\infty}\mathbf{M}(r_i T_{V_i}) < +\infty$$

or equivalently

$$\lim_{N\to\infty}\sum_{i=N}^{N'}|r_i|\mathbf{M}(T_{V_i}) = 0. \tag{1.2}$$

for any $N' \geq N$. The closed current in (1.1) is called an infinitely complex cycle. Such a class (or cycle) has positivity if $r_i > 0$ for all i.

So, infinite cycle classes form a subspace and those with positivity form a convex cone.

Remark The definition is independent of Hermitian metric. The approximation (1.1) is a particular type of convex-cone-approximation ([2]). If X is complex projective, we call infinite complexity infinite algebraicity which is closely related to algebraicity in a projective manifold. For instance, if X is a complex projective manifold and $\cup_i V_i$ is a subvariety, i.e. the set $\{V_i\}_i$ is locally finite, then this type of infinitely algebraic classes are represented by holomorphic chains with real coefficients, which are algebraic cycles with real coefficients. However, infinite algebraic cycles as in (1.1) are not holomorphic chains. Following is such an example. It is based on a counter-example in [5].

Example 1.2. *Let $\mathbb{C}P^1$ be the projective space over $\mathbb{C}$. Let $z_i \in \mathbb{C}P^1$ for the positive integer i be a sequence of points that converges to $\mathbf{o} \in \mathbb{C}P^1$. Let r_i be a sequence of positive numbers such that $\sum_{i=1}^{\infty} r_i = \lambda$ is a finite number. The following is the behavior of such sequences.*

(1) $\sum_{i=1}^{\infty} r_i T_{\{z_i\}}$ is a closed current whose cohomology class is the infinitely algebraic class $\lambda[z_1]$ where $[z_1] \in H^2(\mathbb{C}P^1;\mathbb{Z})$ is represented by z_1.

(2) the current $\sum_{i=1}^{\infty} r_i T_{\{z_i\}}$ is not a holomorphic chain with real coefficients because $\cup_{i=1}^{\infty}\{z_i\}$ is not a subvariety of $\mathbb{C}P^1$.

(3) Let $T^o_{\{z_i\}}$ be the restriction of $T_{\{z_i\}}$ to the affine open set $\mathbb{C}P^1\backslash\{\mathbf{o}\}$, in which, $\sum_{i=1}^{\infty} r_i T^o_{\{z_i\}}$ is still a closed current that represents the cohomology of the point z_1. But in $\mathbb{C}P^1\backslash\{\mathbf{o}\}$, it is also a holomorphic chain with real coefficients because $\cup_{i=1}^{\infty}\{z_i\}$ is a subvariety.

Main theorem 1.3. *Let X be a complex projective manifold. If $u \in H^{2p}(X;\mathbb{Q})$ is an infinite cycle class, u is represented by an algebraic cycle with rational coefficients.*

2 Positivity of homology classes

Definition 2.1. *(Harvey-Lawson) Let X be a compact Kähler manifold. A class*

$$\tau \in H_{2k}(X;\mathbb{R})$$

is called positive if $\tau \cap [\phi] \geq 0$ for all closed, real 2k-forms ϕ whose (k,k) component $\phi^{k,k}$ is weakly positive where $\cap$ is the cap product, and $[\bullet]$ denotes the cohomology class of $\bullet$. We call the Poincaré dual of a positive homology class also positive class.

Remark. The definition is just a special case of the relative version of positive classes defined by Harvey-Lawson [3].

X continues to be a compact Kähler manifold. Let $\mathcal{Z}^p(X)$ be the free Abelian group generated by complex analytic subvarieties of codimension p. Let

$$[\bullet] : \{closed\ currents\} \quad \rightarrow \quad homology\ group \tag{2.1}$$

be the reduction homomorphism. Let

$$C^p_{\mathbb{Z}} \subset H_{2k}(X;\mathbb{Z})$$

be the image $[\mathcal{Z}^p(X)]$, where k is the dimension of the subvarieties, and a subvariety is regarded as the integration currents. Let

$$C^p_{\mathbb{R}} := C^p_{\mathbb{Z}} \otimes \mathbb{R} \ \subset H_{2k}(X;\mathbb{R}).$$

Let $E^p_{\mathbb{R}} \subset C^p_{\mathbb{R}}$ be the cone that consists of positive classes.

Lemma 2.2. *Let X be a compact Kähler manifold. Then there exist finitely many real coefficient complex analytic cycles A_j with $[A_j]$ in $E^p_{\mathbb{R}}$ such that for any irreducible subvariety V of codimension p,*

$$[V] = \sum_{finite\ j} \lambda_j [A_j] \tag{2.2}$$

where λ_j are non-negative real numbers.

Proof. Notice $E^p_{\mathbb{R}}$ is a convex cone in the finite dimensional vector space $C^p_{\mathbb{R}}$. Definition 2.1 shows that $E^p_{\mathbb{R}}$ is defined by finitely many homogeneous linear inequalities (≥ 0). Hence $E^p_{\mathbb{R}}$ is a polyhedral cone. Thus there exist finitely many cycles A_j such that $\{[A_j]\}_j$ is a frame of the polyhedral cone. So, any element $\tau \in E^p_{\mathbb{R}}$ is a linear combination of $\{[A_j]\}_j$ with non-negative coefficients. Since $[V]$ lies in $E^p_{\mathbb{R}}$, we complete the proof. □

Remark. By Poincaré duality, we are addressing the poisitivity for cohomology also. In complex geometry, the positivity for cohomology classes plays a significant role. Historically, positivity first appeared in the bidegree $(1,1)$ classes. It led to the proof of Kodaira embedding theorem and many others. There was also a large smoke screen surround this positivity. So, it was not clear that which direction is more relevant. Later in 1977, Steve Zucker produced a compact Kähler manifold that does not admit complex analytic subvarieties of middle dimension $2n$ for $n \geq 2$ ([6]). Thus even the Lefschetz $(1,1)$ theorem is false for compact Kähler manifolds. This example showed the importance of the positivity. So, Zucker proposed the study of the positivity in general. However, he restricted the attention to subvarieties only. Precisely, Zucker only proposed to study those classes that can be represented by positive linear combinations of subvarieties, i.e. to study the positivity of coefficients. Actually, in 1975 Lawson in [4] already showed that the positivity does not come from the positivity of the subvarieties. Precisely, Lawson proved that there are complex projective manifolds whose class positivity does not coincide with Zucker's positivity. He also showed that there are many other manifolds whose class positivity does coincide with Zucker's. So, the situation is rather subtle. But it raises the awareness of the truth that the positivity provided by subvarieties is not sufficient for the Hodge conjecture. For instance, it was proved in [4] that even if $E^p_{\mathbb{R}}$ is defined with subvarieties, $[A_j]$ may not be a linear combination of subvarieites with positive coefficients. In 2009, Harvey-Lawson introduced the positivity as that in Definition 2.1, whose focus is the positivity of the currents. It was a surprise as they put it. But it was not the end of the surprise.

3 Proof of Main theorem 1.3

The proof has two steps: 1) reduce the infinite series (1.1) in currents to a finite sum in cohomology; 2) convert $\mathbb{R}$-coefficients to $\mathbb{Q}$-coefficients.

Step 1: First we state a fact on a compact Kähler manifold: for a positive current $\mathcal{T}$ of bidimension (q,q),

$$\mathbf{M}(\mathcal{T}) = \mathcal{T}[\frac{\omega^q}{q!}] \tag{3.1}$$

where ω is the Kähler form. This fact has been proved and used at multiple places. For the proof, see Theorem 2.2 combined with Remark 2.5 in [4].

Let u be a rational, infinitely complex class. So we write a representation

$$\sum_{i=1}^{\infty} r_i T_{V_i} \tag{3.2}$$

where V_i are irreducible subvarieties of dimension $k = dim(X) - p$. Let

$$[V_i] = \sum_{finite\ j} \lambda_i^j [A^j]$$

where λ_i^j are real numbers, and A_j are those in $E_{\mathbb{R}}^p$ as in Lemma 2.2. Since $[V_i]$ is in the cone $E_{\mathbb{R}}^p$, by Lemma 2.2 (the positivity), all coefficients λ_i^j must be non-negative. The evaluation

$$T_{V_i}[\frac{\omega^k}{k!}] = \sum_{finite\ j} \lambda_i^j T_{A^j}[\frac{\omega^k}{k!}]. \tag{3.3}$$

implies

$$\mathbf{M}(T_{V_i}) = \sum_{finite\ j} \lambda_i^j \mathbb{M}(T_{A^j}), \tag{3.4}$$

where $\mathbb{M}(T_{A^j})$ is the evaluation $T_{A^j}[\frac{\omega^k}{k!}]$ which is non-negative because the class $[A^j]$ is positive and the form $\frac{\omega^k}{k!}$ is strongly positive. On the other hand, by the absolute mass-convergence of (1.1), we have

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{3.5}$$

where $N' \geq N$. Plugging (3.4) into (3.5), we obtain

$$\sum_{finite\ j} \mathbb{M}(T_{A^j}) \left(\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \lambda_i^j \right) = 0, \tag{3.6}$$

Since $\mathbb{M}(T_{A^j})$ and λ_i^j are all non-negative, for each j

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \lambda_i^j = 0. \tag{3.7}$$

Then (3.7), for each j, implies the absolute convergence for the series

$$\alpha_j := \sum_{i=1}^{\infty} r_i \lambda_i^j. \tag{3.8}$$

Now we work with the convergence in cohomology. Due to the finiteness of Betti number, cohomological convergence is determined by the convergence of the real numbers on each axis. Precisely, we see that the convergence (3.7) implies that u, which is

$$\left[\sum_{i=1}^{\infty} r_i V_i \right], \tag{3.9}$$

is approached by the cycle classes with real coefficients,

$$\left[\sum_{i=1}^{N} r_i V_i\right] = \sum_{finite\ j} \left(\sum_{i=1}^{N} r_i \lambda_i^j\right)[A^j],\ as\ N \to \infty. \tag{3.10}$$

([•] is continuous). Notice that the cohomology class (3.10), by the convergence (3.7), also converges to a cycle class with real coefficients written as

$$u = \sum_{finite\ j} \alpha_j [A^j]; \tag{3.11}$$

$$\alpha_j = \sum_{i=1}^{\infty} r_i \lambda_i^j. \tag{3.12}$$

So, u is a cycle class with real coefficients.

Step 2: Next we convert it to $\mathbb{Q}$-coefficients. For any closed subset $W \subset X$, the subgroup

$$ker\bigg(H^i(X;\mathbb{Q}) \to H^i(X\backslash W;\mathbb{Q})\bigg) \tag{3.13}$$

will be denoted by $H^i_{(W)}(X;\mathbb{Q})$ where ker stands for the kernel of the restriction map. A class $\gamma \in H^i(X;\mathbb{Q})$ is said to be class-supported on W if $\gamma \in H^i_{(W)}(X;\mathbb{Q})$. In another direction, we say a class is current-supported on W if it is represented by a closed current supported on W. The homology of currents implies that a class current-supported on a W is a class class-supported on W (but the converse is false). Recall (3.11)

$$u = \sum_{finite\ j} \alpha_j [A^j] \tag{3.14}$$

where α_j are real and A^j are algebraic cycles with real coefficients. If we let $V = \bigcup_{finite\ j} |A^j|$ be the algebraic set, u is current-supported on V, then u is also class-supported on V. Let

$$\widetilde{V} \stackrel{J}{\to} V \stackrel{I}{\hookrightarrow} X$$

be the composite such that J is a smooth resolution and I is the inclusion. Since the codimension condition

$$deg(u) - 2cod(V) \geq 0$$

is satisfied, we apply Deligne's corollary 8.2.8, [1] which addresses the class-support. Precisely it states that the Gysin map

$$(I \circ J)_! : H^0(\widetilde{V};\mathbb{Q}) \to H^{2p}_{(V)}(X;\mathbb{Q}) \tag{3.15}$$

is surjective. Then a pre-image $\tilde{u}$ of u is a cohomological class of degree 0 on the complex manifold $\widetilde{V}$. So, $\tilde{u}$ must be represented by a rational linear combinations of irreducible components of $\widetilde{V}$. Since J is a complex analytic map from $\widetilde{V}$ onto V, $u = (I \circ J)_!(\tilde{u})$ is represented by a rational, linear combination of irreducible components of V. The proof is completed.

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