Extraction of complex analytic cycles

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Abstract

Let X be a compact Kähler manifold. We first define the positivity of homology classes in $H_{2k}(X;\mathbb{Q})$. From the positivity, we extract complex analytic cycles. Precisely, we show if $\tau \in H_{2k}(X;\mathbb{Q})$ is positive, i.e. τ is represented by a closed, strongly positive current, then there are a complex analytic cycle V with positive rational coefficients and a positive current S of bidimension (k,k) such that

$$\tau = [T_V + S] \tag{0.1}$$

where $T_\bullet$ denotes the current of integration over the chain $\bullet$, and $[\bullet]$ denotes the homology class represented by $\bullet$.

Contents

1 Introduction

It is the great interest to construct algebraic cycles. Having this in mind, we try the extraction. Extraction occurs in a more general setting of compact Kähler manifolds. The first such an extraction was completed by Reese Harvey and Blain Lawson. More precisely, they defined the positivity of classes as follows.

Key words: currents, complex analytic cycles, holomorphic chains, positive homology classes

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Definition 1.1. *(Harvey-Lawson [4]) Let X be a compact Kähler manifold. Let M be an oriented compact real analytic submanifold of dimension $2k-1$. A class*

$$\tau \in H_{2k}(X, M; \mathbb{R})$$

is called positive if $\tau \cap [\phi] \geq 0$ for all closed, real 2k-forms ϕ whose (k,k) components $\phi^{k,k}$ are weakly positive, where $\cap$ is the cap product, $[\bullet]$ is the homology class.

They claimed that a positive holomorphic chain can be extracted from this positivity. Precisely, they proved

Theorem 1.2. *(Harvey-Lawson [4]) Let X be a compact Kähler manifold, and $M \subset X$ a compact, real analytic submanifold of dimension $2k-1$. Let $\tau \in H_{2k}(X, M; \mathbb{Z})/tors$ be non-zero and positive, and*

$$\partial \tau \neq 0 \tag{1.1}$$

where ∂ is the connecting homomorphism in the long exact sequence of relative homology for the pair (X, M). Then there exist a positive holomorphic chain $\tilde{V}$ in $X \backslash M$ and a closed, positive (k,k) current S such that

$$\tau = [\mathcal{T}_{\tilde{V}} + S] \tag{1.2}$$

where $[\bullet]$ stands for the relative homology class, the current $\mathcal{T}_{\tilde{V}}$ is the simple extension current of $\tilde{V}$ to the entire manifold X.

Remark. Our statement is a modified version whose original version-theorem 3.4, [4] did not include the assumption (1.1). However, their argument requires it. So, we include it for the completeness of the theorem. Furthermore, the content of our paper is about the development of this assumption (1.1). We apologize for any misquote.

Applying Theorem 1.2, we obtain our extraction.

Main theorem 1.3. *Let X be a compact Kähler manifold. Let $\tau \in H_{2k}(X, \mathbb{Q})$ be positive. Then there exist a complex analytic cycle V in X with positive rational coefficients and a closed, positive (k,k) current S in X such that*

$$\tau = [T_V + S] \tag{1.3}$$

in $H_{2k}(X, \mathbb{Q})$.

Since a positive complex analytic cycle is a linear combination of irreducible subvarieties, Main theorem implies that

Corollary 1.4. *A compact Kähler manifold has a subvariety of dimension $2k$ if and only if it admits a positive class in the $H_{2k}(X; \mathbb{Q})$.*

Remark. It is known that the positivity of cohomology classes is related to the existence of analytic cycles (for instance, see [9]). The corollary confirms that this positivity has a root in currents.

2 Positivity of homology classes

First we introduce some of the notations and geometric setting in [4]. Let X be a compact manifold, $M \subset X$ a compact submanifold. We use $\mathscr{E}^\bullet$ to denote the Frechet space consisting of smooth forms, where $\bullet$ stands for the degree. Taking their topological dual, we have $\mathscr{E}'_\bullet(X)$, $\mathscr{E}'_\bullet(M)$ where $\bullet$ stands for the dimension of the currents and they are equipped with the weak topology. Let $\mathscr{E}^\bullet(X, M)$ denote the subspace consisting of forms vanishing on M. Let

$$\mathscr{E}'_\bullet(X, M) = \frac{\mathscr{E}'_\bullet(X)}{I_*(\mathscr{E}'_\bullet(M))}$$

where $I : M \hookrightarrow X$ is the inclusion. We have two complexes to compute the relative homology and cohomology

$$\mathscr{E}^{i-1}(X, M) \xrightarrow{d} \mathscr{E}^{i}(X, M) \xrightarrow{d} \mathscr{E}^{i+1}(X, M) \tag{2.1}$$

$$\mathscr{E}'_{i-1}(X, M) \xleftarrow{d} \mathscr{E}'_{i}(X, M) \xleftarrow{d} \mathscr{E}'_{i+1}(X, M) \tag{2.2}$$

Then

$$H^i(X, M; \mathbb{R}) = Z/B, H_i(X, M; \mathbb{R}) = \widetilde{Z}/\widetilde{B}$$

where Z, B are the cycles and boundaries of the complex (2.1), $\widetilde{Z}, \widetilde{B}$ are the cycles and boundaries of the complex (2.2). For the spacial case where

$$i = 2k = dim(X) + 1$$

we have

$$H^{2k}(X, M; \mathbb{R}) = Z/B \ with \ \begin{cases} Z = \{\phi \in \mathscr{E}^{2k}(X) : d\phi = 0\} \\ B = d\mathscr{E}^{2k-1}(X, M) \end{cases} \tag{2.3}$$

$$H_{2k}(X, M; \mathbb{R}) = \widetilde{Z}/\widetilde{B} \ with \ \begin{cases} \widetilde{Z} = \{T \in \mathscr{E}'_{2k}(X) : dT \in I_*(\mathscr{E}'_{2k-1}(M))\} \\ \widetilde{B} = d(\mathscr{E}'_{2k+1}(X)) \end{cases} \tag{2.4}$$

It is also proved in [4] that

$$H_{2k}(X, M; \mathbb{R}), H^{2k}(X, M; \mathbb{R})$$

are finitely dimensional vector spaces, and there is a Poincaé duality:

$$\Big(H_{2k}(X, M; \mathbb{R})\Big)^* \simeq H^{2k}(X, M; \mathbb{R}).$$

Now we assume X is a compact Kähler manifold, and $M \subset X$ is a compact real analytic submanifold of dimension $2k-1$. Let

$$\mathcal{P} = \{\phi \in \mathscr{E}^{2k}(X) : \phi^{k,k} \geq 0(weakly)\}$$
$$\widetilde{\mathcal{P}} = \{T \in \mathscr{E}'_{2k}(X); T = T_{k,k} \geq 0(strongly)\}$$

Define the cones

$$\widetilde{C_M^+} := \frac{\widetilde{\mathcal{P}} \cap \widetilde{Z} + \widetilde{B}}{\widetilde{B}} \subset H_{2k}(X, M; \mathbb{R}) \tag{2.5}$$

It is clear that $\widetilde{C_M^+}$ is a closed convex cone. Proposition 3.3, [4] asserts that $\widetilde{C_M^+}$ is the collection of all positive classes. We are going to prove further

Lemma 2.1. *The convex cone* $\widetilde{C_M^+}$ *is polyhedral.*

Proof. Let

$$m = dim(H^{2k}(X, M; \mathbb{R})).$$

We represent a basis for $H^{2k}(X, M; \mathbb{R})$ by m-tuple C^∞ closed forms

$$\Phi = (\phi_1, \phi_2, \cdots, \phi_h, \phi_{h+1}, \cdots, \phi_m)$$

of degree $2k$ vanishing on M, where $\{\phi_1, \cdots, \phi_h\}$ is a basis for $H^{k,k}(X, M; \mathbb{R})$. For each current σ that represents the class $[\sigma]$ in $H_{2k}(X, M; \mathbb{R})$, the m-tuple of numbers $L := \sigma(\Phi)$ represent a point in

$$H_{2k}(X, M; \mathbb{R}) \simeq \mathbb{R}^m.$$

So, Poincaré duality isomorphism converts

$$\mathscr{P} : \{[\sigma]\} \to \{L\}.$$

To consider the positive cones in the (k, k) portion, we add a sufficiently large multiple of ω^k where ω is the Kähler form to the (k, k) portion, to obtain that all $\phi_1, \cdots, \phi_h$ are positive. Let

$$\pi : \widetilde{C}_M^+ \to \mathbb{R}^2$$

be the orthogonal projection to any 2-dimensional plane V_2 in $\mathbb{R}^m \simeq H_{2k}(X, M; \mathbb{R})$. We should note that all classes in $\widetilde{C}_M^+$ are of bidegree (k, k). Hence for the convenience, we may choose the plane determined by ϕ_1, ϕ_2. Precisely, we choose

$$\begin{array}{rcc} \pi : V_2 & \to & \mathbb{R}^2 \\ \sigma & \to & (\sigma(\phi_1), \sigma(\phi_2)), \end{array}$$

i.e. the 2-plane that is the dual of $H_{2k}(X, M; \mathbb{R})$, determined by the forms ϕ_1, ϕ_2. Recalling Proposition 3.3, [4], the classes in $\widetilde{C_M^+}$ are represented by strongly positive currents. So we let σ_n for natural numbers n be a sequence of currents representing the points $[\sigma_n]$ in $\widetilde{C_M^+}$. We assume $\pi([\sigma_n])$ approaches a finite point in the plane $\mathbb{R}^2$. For instance,

$$\lim_{n\to\infty} \sigma_n(\phi_1) \leq +\infty. \tag{2.6}$$

Notice

$$\mathbf{M}(\sigma_n) = \frac{\sigma_n(\omega^k)}{k!} \leq \frac{\sigma_n(\phi_1)}{k!} \leq +\infty \tag{2.7}$$

where $\mathbf{M}$ is the mass based on Kähler metric ω, and we also used the fact on a compact Kähler manifold: for a positive current $\mathcal{T}$ of bidimension (q,q),

$$\mathbf{M}(\mathcal{T}) = \mathcal{T}[\frac{\omega^q}{q!}]. \tag{2.8}$$

This fact has been proved and used at multiple places. For the proof, see Theorem 2.2 combined with Remark 2.5 in [5]. Since $\mathbf{M}(\sigma_n)$ is bounded, by Lemma 2.15, chapter 6 in [8], $\{\sigma_n\}$ has a subsequence that converges weakly to a current σ_∞ whose represented class $[\sigma_\infty] \in \widetilde{C_M^+}$. Hence the limit of the projections $\pi([\sigma_n])$ as $n \to \infty$ is $(\sigma_\infty(\phi_1), \sigma_\infty(\phi_2))$ with $\sigma_\infty \in \widetilde{C_M^+}$. This implies that $\pi(\widetilde{C_M^+})$ is closed. By the characterization of polyhedral cones in the theorem of [7], $\widetilde{C_M^+}$ is polyhedral. We complete the proof.

□

Remark. The polyhedral cone has a distinguished property: all vectors in the cone are positive linear combinations of the vectors on the edges (frame). The argument clearly shows that Lemma 2.1 holds for the non-relative homology where M is empty.

Lemma 2.2. *Let*

$$\widetilde{C_{M,\mathbb{Q}}^+} := \overline{H_{k,k}(X, M; \mathbb{Q})} \cap \widetilde{C_M^+}, \tag{2.9}$$

where $\overline{(\bullet)}$ *is the topological closure of the finitely dimensional vector space*

$$H_{k,k}(X, M; \mathbb{Q}).$$

Then $\widetilde{C_{M,\mathbb{Q}}^+}$ *is a rational polyhedral cone, i.e. the halflines of its frame are real spans of rational vectors.*

Proof. Since $\overline{H_{k,k}(X, M; \mathbb{Q})}$ is a subspace, it is a property of polyhedral cones that the intersection $\widetilde{C_{M,\mathbb{Q}}^+}$ of the polyhedral cone is still a polyhedral cone. Notice that

$$\overline{H_{k,k}(X, M; \mathbb{Q})} \simeq H_{k,k}(X, M; \mathbb{Q}) \otimes \mathbb{R}. \tag{2.10}$$

So any ray starting from the origin is a real span of a vector in the rational subspace $H_{k,k}(X, M; \mathbb{Q})$. In particular, the frame of the polyhedral cone $\widetilde{C_{M,\mathbb{Q}}^+}$ is formed by rational vectors in $H_{k,k}(X, M; \mathbb{Q})$. We complete the proof. □

3 Proof

The proof of Main theorem 1.3 following Theorem 1.2. We abuse the notation [•] for classes of homology or relative homology.

Dividing by an integer, we can pass the homology modula torions to rational homology. Thus for the convenience, we'll work with the rational coefficient's version for Theorem 1.2, i.e. change the coefficients in the non-torsion integral homology to rational homology.

Let M be the boundary of k-dimensional complex analytic polydisk B_{2k} in a local chart $\mathbb{C}^n$ where $n = dim(X)$. Notice that since the

$$dim(M) = 2k - 1 < 2k,$$

$H_{k,k}(X;\mathbb{R})$ is a subspace of $H_{k,k}(X, M;\mathbb{R})$. Let

$$\widetilde{C^+_{\mathbb{Q}}} = H_{k,k}(X;\mathbb{R}) \cap \widetilde{C^+_{M,\mathbb{Q}}}. \tag{3.1}$$

By the property of polyhedral cones again, $\widetilde{C^+_{\mathbb{Q}}}$ is a sub rational polyhedral cone of the bigger rational polyhedral cone $\widetilde{C^+_{M,\mathbb{Q}}}$. Let $F(\widetilde{C^+_{\mathbb{Q}}}), F(\widetilde{C^+_{M,\mathbb{Q}}})$ denote the sets of vectors that span exactly the frames of these polyhedral cones respectively. Note: the collections of vectors in a frame are not unique, but the halflines along those vectors are unique. Our notation only collects one vector along each halfline, and no duplicates of the same halflines.

Claim 3.1.

$$F(\widetilde{C^+_{\mathbb{Q}}}) \cap F(\widetilde{C^+_{M,\mathbb{Q}}}) = \varnothing.$$

Proof. Suppose there is a non-zero $\alpha \in F(\widetilde{C^+_{\mathbb{Q}}}) \cap F(\widetilde{C^+_{M,\mathbb{Q}}})$. Since

$$\alpha \in F(C^+_{M,\mathbb{Q}}),$$

by the extremity of the frame, there is only one direction, along which a small deformation α_t of α with $\alpha_0 = \alpha$ will stay in $\widetilde{C^+_{M,\mathbb{Q}}}$. Suppose that $\alpha \in \widetilde{C^+_{\mathbb{Q}}}$ such that $\alpha \cap [\phi] = 0$ for all closed forms ϕ with the positive (k,k) component. Any (k,k) form ϕ can be written as

$$\phi = \phi + a\omega^k - a\omega^k$$

where a is a real number. For a sufficiently large a, $\phi + a\omega^k$ and $a\omega^k$ are all strongly positive. Hence

$$\alpha \cap [\phi] = 0$$

for any smooth forms ϕ. This contradiction shows that there is a closed form ϕ with strongly positive (k,k) component such that

$$\alpha \cap [\phi] = \lambda > 0.$$

Hence there is a small deformation α_t of α along at least one direction such that $\tau_t(\phi) \geq 0$ for all small t and closed ϕ with strongly positive $\phi^{k,k}$ component. Let's denote this direction

$$\frac{\partial \alpha_t}{\partial t}\bigg|_{t=0}$$

by γ which corresponds to a non-zero point in $H_{k,k}(X;\mathbb{R})$. In another direction, we consider $[B_{2k}] \in H_{k,k}(X, M;\mathbb{Q}) \backslash H_{k,k}(X;\mathbb{Q})$.

$$\alpha_t = \alpha + t[B_{2k}]. \tag{3.2}$$

Notice that for sufficiently small t, α_t lies in $\widetilde{C^+_{M,\mathbb{Q}}}$. Furthermore the direction of the deformation is $[B_{2k}]$ which can't be $\gamma \in H_{k,k}(X;\mathbb{R})$. This contradiction shows Claim 3.1 is true.

□

Let A_j for finite $j = 1, \cdots, e$ be the currents that form the frame of $\widetilde{C^+_{M,\mathbb{Q}}}$ such that all classes $[A_j]$ are rational. Let $l = dim(\widetilde{C^+_{M,\mathbb{Q}}}) \leq e$. Then there are l many vectors among $\{[A_j]\}_{j=1}^{j=e}$ forming a basis of $span_{\mathbb{R}}(\widetilde{C^+_{M,\mathbb{Q}}})$. Since any τ is a positive linear combination of $\{[A_j]\}_{j=1}^{j=e}$, it is also spanned by $[A_1], \cdots, [A_l]$ with non-negative coefficients, i.e.

$$\tau = \sum_{i=1}^{l} r_i [A_i]$$

where $r_i \geq 0$. Since τ is rational and $\{[A_i]\}_{i=1}^{i=l}$ is a rational basis of the real linear space

$$span_{\mathbb{R}}(\widetilde{C^+_{M,\mathbb{Q}}}),$$

the numbers r_i for all i are positive rational (unique). By Claim 3.1, $r_i[A_i]$ for all i do not lie in $\widetilde{C^+_{\mathbb{Q}}}$. Now we apply Theorem 1.2 with the version of rational coefficients * to each rational positive class $r_i[A_i]$ to obtain holomorphic chains $\tilde{V}_i$ with positive rational coefficents on $X \backslash M$, and closed, positive currents S_i with bidimension (k, k) such that

$$r_i[A_i] = [\mathcal{T}_{\tilde{V}_i} + S_i] \tag{3.3}$$

where $\mathcal{T}_{\tilde{V}_i}$ is the simple extensions of the currents of integration over $\tilde{V}_i$. Now the current

$$\sum_{finite\ i} \mathcal{T}_{\tilde{V}_i}$$

are closed and of (k, k) on $X \backslash M$, therefore on X it is also closed and of (k, k). Secondly,

$$\sum_{finite\ i} \mathcal{T}_{\tilde{V}_i}$$

*It is important to notice that Theorem 1.2 only works in non-torsion integral homology which is equivalent to the rational version.

is locally rectifiable in $X\backslash M$. By Lemma A.1, in X it is also locally rectifiable. Then Main theorem in [1] implies that

$$\sum_{finite\ i} \mathcal{T}_{\tilde{V}_i}$$

is the integration current over a holomorphic chain V with rational coefficients [†]. Since X is compact and the current $\sum_{finite\ i} \mathcal{T}_{\tilde{V}_i}$ is positive, V is a complex analytic cycle with positive rational coefficients. Adding equations in (3.3), we obtain

$$\tau = [T_V + \sum_{finite\ i} S_i] \tag{3.4}$$

in $H_{2k}(X; M, \mathbb{Q})$. Since τ is also the homology class in $H_{2k}(X, \mathbb{Q})$, we complete the proof for positive classes.

Appendix A Lelong's Simple extension

Our strategy is to extend the holomorphic chains to complex analytic cycles. This type of extension is first obtained as an extension of currents, which is important notion on its own. There is a more general setting for this type of extension than holomorphic chains as in this paper. Let's have a review. Let Ω be an open subset of a real manifold Y. Let $t \in \mathscr{D}'(\Omega)$ where $\mathscr{D}'(\bullet)$ denotes the space of currents. Any current $\tilde{t}$ on Y that is restricted to t on Ω is called an extension of t. Not all currents in Ω can have extensions, and extensions may not be unique if they exist. A simple extension, denoted by $\tilde{t}_o$, of $t \in \mathscr{D}'(\Omega)$ is a particular extension-by-zero defined by Lelong ([6]). It is well-defined if 1) the order of t is 0, 2) the local mass $\mathbf{M}_G(t)$ in $\Omega \cap G$ for any compact set G of Y is finite. The functional for the extension is $\sum_{j=1}^{\infty} \phi_j t$ where $\{\phi_j\}$ is a partition of unity for an open covering of Ω. So, the simple extension depends on the partition of unity $\{\phi_j\}$.

Lemma A.1. *Let $t \in \mathscr{D}'(\Omega)$ be a current that has order* 0 *and the local mass $\mathbf{M}_G(t)$ is finite. Then if t is locally rectifiable, so is $\tilde{t}_o$.*

Proof. Let ϕ_j be a partition of unity, associated to the simple extension. For a test form ψ, we have the evaluation

$$\tilde{t}_o[\psi] = \sum_{j=1}^{\infty} t[\phi_j \psi]. \tag{A.1}$$

[†]This result is also implied by Harvey's structure theorem for closed, positive currents ([2]), where Lemma A.1 is not necessary.

Since $t\sum_{j=1}^{N}\phi_j$ is well-defined in Y, the weak limit

$$\lim_{N \overset{w}{\to} \infty} t\sum_{j=1}^{N}\phi_j$$

is also well-defined in Y (see (A.1)). Let $\epsilon > 0$ and P_ϵ be the Lipschitzian chains for the approximation of t in Ω. By Proposition 2, [6]

$$\mathbf{M}_G(\tilde{t}_o) = \mathbf{M}_G(t), \tag{A.2}$$

for any compact set G in Y, where $\mathbf{M}_G(t)$ denotes $\mathbf{M}_{G\cap\Omega}(t)$. Then we have the computation

$$\mathbf{M}_G(\tilde{t}_o - T_{P_\epsilon}) = sup\left\{\mathbf{M}_K\left((t - T_{P_\epsilon})\sum_{j=1}^{\infty}\phi_i\right) : K \subset \Omega \ compact\right\}. \tag{A.3}$$

Notice that the right hand side of A.3 is approximated by the mass in Ω. Since $t - T_{P_\epsilon}$ in Ω has order 0, there exists an N such that

$$\mathbf{M}_G(\tilde{t}_o - T_{P_\epsilon}) \leq \mathbf{M}_{G_N}\left((t - T_{P_\epsilon})\sum_{j=1}^{N}\phi_j\right) + \epsilon \tag{A.4}$$

$$\leq \mathbf{M}_{G_N}(t - T_{P_\epsilon}) + \epsilon \leq 2\epsilon. \tag{A.5}$$

where $G_N = G \cap \cup_{j=1}^{N} supp(\phi_j)$ is a compact set in Ω. Thus $\tilde{t}_o$ is approximated by the same Lipschitzian chains P_ϵ as that for t. Therefore $\tilde{t}_o$ is also locally rectifiable. □

Remark. The simple extension in this work (including Harvey-Lawson) is the particular one whose original current t is a linear combination of closed, positive currents, and the extension is unique ([3]).

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