Extraction of algebraic cycles

Bin Wang (汪 镔)

March 12, 2026

Abstract

Let X be a complex projective manifold. We first define the positivity of homology classes in $H_{2k}(X;\mathbb{Q})$. From the positivity, we extract algebraic cycles. Precisely, we show if $\tau \in H_{2k}(X;\mathbb{Q})$ is weakly positive, i.e. $\tau(\phi) \geq 0$ for all the real, closed forms ϕ with the strongly positive (k,k) component, then there are an algebraic cycle V with positive coefficients and a positive current S of bidimension (k,k) such that

$$\tau = [T_V + S] \tag{0.1}$$

where $T_\bullet$ denotes the current of integration over the chain V, and $[\bullet]$ denotes the homology class represented by $\bullet$.

Contents

1 Introduction

Construction of algebraic cycles has been a central topic in algebraic geometry. The most general question is about how to characterize the class represented by a linear combination of cycles defined by polynomials. For that, the approach with Grothendieck's algebra seems to be a natural way. In another field, questions about the boundaries of complex analytic subvarieties and holomorphic chains draw a lot of attention. The topic can be traced back to the time as early as that for algebraic cycles. The most general question is about how to characterize the boundary of the subset defined by complex analytic functions. For that, complex analysis seems to be more relevant. We would like to show that these two questions are closely related. Paper is based on the work of Harvey-Lawson who proved the first extraction in [3]. Precisely, they defined the positivity as follows.

Definition 1.1. *(Harvey-Lawson) Let X be a compact Kähler manifold. Let M be an oriented compact real analytic submanifold of dimension $2k-1$. A class*

$$\tau \in H_{2k}(X, M; \mathbb{R})$$

is called positive (resp., weakly positive) if $\tau \cap [\phi] \geq 0$ for all closed, real $2k$-forms ϕ whose (k,k) components $\phi^{k,k}$ are weakly positive (resp., strongly positive), where $\cap$ is the cap product, $[\bullet]$ is the homology class.

Then they showed that a positive holomorphic chain can be extracted from this positivity. Precisely, they proved.

Theorem 1.2. *(Harvey-Lawson) Let X be a compact Kähler manifold, and $M \subset X$ be a compact, real analytic submanifold of dimension $2k-1$. Let $\tau \in H_{2k}(X, M; \mathbb{Z})/tors$ be non-zero and weakly positive, and*

$$\partial\tau \neq 0 \tag{1.1}$$

where ∂ is the connecting homomorphism in the long exact sequence of relative homology for the pair (X, M). Then there exist a holomorphic chain $\tilde{V}$ in $X \backslash M$ and a closed, positive (k,k) current S such that

$$\tau = [\mathcal{T}_{\tilde{V}} + S] \tag{1.2}$$

in $H_{2k}(X, M; \mathbb{Z})/tors$, where $[\bullet]$ stands for the relative homology class, the current $\mathcal{T}_{\tilde{V}}$ is the simple extension current of $\tilde{V}$ to the entire manifold X.

Note: Theorem 1.2 still holds for weakly positive class τ.

Remark. Harvey-Lawson's original theorem 3.4, [3] does not include the assumption (1.1). However, it is crucial for what we'll do. Thus, we add it for more accuracy. But in a surprising turn, this paper shows that if the manifold is projective, the (1.1) is indeed not necessary.

Note that $H_{2k}(X; \mathbb{R})$ is a subspace of $H_{2k}(X, M; \mathbb{R})$. So, we define the positivity in $H_{2k}(X; \mathbb{R})$ to be the positivity in $H_{2k}(X, M; \mathbb{R})$. In the similar fashion, this positivity is sufficient to extract algebraic cycles. Precisely, we'll prove.

Main theorem 1.3. *Let X be a complex projective manifold. Let $\tau \in H_{2k}(X; \mathbb{Q})$ be weakly positive. Then there exist an algebraic cycle V with positive coefficients and closed, positive current S of bidimension (k,k) such that τ is represented by $T_V + S$, i.e. $\tau = [T_V + S]$.*

2 Relative homology and cohomology

This section is the revisit of a part of section 3 in [3]. We don't mean to give redundant statements. The only purpose is to introduce the notations which are necessary for the argument in next section.

Let X be a compact manifold, M a compact submanifold. We use $\mathscr{E}^\bullet$ to denote the Frechet space consisting of smooth forms, where $\bullet$ stands for the degree. The notation agrees with $\mathscr{D}$ (which is commonly used in functional analysis) if the manifold is compact. Let $\mathscr{E}^\bullet(X, M)$ denote the subspace consisting of forms vanishing on M. Taking their topological dual, we have $\mathscr{E}'_\bullet(X)$, $\mathscr{E}'_\bullet(M)$ where $\bullet$ stands for the dimension of the currents and they are equipped with the weak topology. Let

$$\mathscr{E}'_\bullet(X, M) = \frac{\mathscr{E}'_\bullet(X)}{I_*(\mathscr{E}'_\bullet(M))}$$

where $I : M \hookrightarrow X$ is the inclusion. We have two complexes to compute the relative homology and cohomology

$$\mathscr{E}^{i-1}(X, M) \quad \stackrel{d}{\to} \quad \mathscr{E}^i(X, M) \quad \stackrel{d}{\to} \quad \mathscr{E}^{i+1}(X, M) \tag{2.1}$$

$$\mathscr{E}'_{i-1}(X, M) \quad \stackrel{d}{\leftarrow} \quad \mathscr{E}'_i(X, M) \quad \stackrel{d}{\leftarrow} \quad \mathscr{E}'_{i+1}(X, M) \tag{2.2}$$

Then

$$H^i(X, M; \mathbb{R}) = Z/B, H_i(X, M; \mathbb{R}) = \widetilde{Z}/\widetilde{B}$$

where Z, B are the cycles and boundaries of the complex (2.1), $\widetilde{Z}, \widetilde{B}$ are the cycles and boundaries of the complex (2.2). For the spacial case where

$$i = 2k = dim(X) + 1$$

we have

$$H^{2k}(X, M; \mathbb{R}) = Z/B \ with \ \begin{cases} Z = \{\phi \in \mathscr{E}^{2k}(X) : d\phi = 0\} \\ B = d\mathscr{E}^{2k-1}(X, M) \end{cases} \tag{2.3}$$

$$H_{2k}(X, M; \mathbb{R}) = \widetilde{Z}/\widetilde{B} \ with \ \begin{cases} \widetilde{Z} = \{T \in \mathscr{E}'_{2k}(X) : dT \in I_*(\mathscr{E}'_{2k-1}(M))\} \\ \widetilde{B} = d(\mathscr{E}'_{2k+1}(X, M)) \end{cases} \tag{2.4}$$

3 Extraction from homology

Proof of Main theorem 1.3. We abuse the notation $[\bullet]$ for classes of homology or cohomology or their relative classes.

Let $M \subset X$ be a compact, real submanifold of dimension $2k-1$. Let

$$F_M^+ \subset H_{2k}(X, M; \mathbb{Q}) \tag{3.1}$$

be the convex cone that consists of the relative and weakly positive classes, and $F^+ \subset H_{2k}(X; \mathbb{Q})$ the convex cone that consists of the weakly positive classes in the non-relative homology. By the section 3.1, [3], $H_{2k}(X, M; \mathbb{Q})$ is a finite dimensional vector space. Notice that since the $dim(M) = 2k-1 < 2k$, $H_{2k}(X; \mathbb{Q})$ is a proper subsapce of $H_{2k}(X, M; \mathbb{Q})$. Thus $F^+ \subset F_M^+$. The proof of Theorem 6.4, [1] combined with Proposition 3.3, [3] implies that any relative homology class $\alpha \in H_{2k}(X, M; \mathbb{Q})$ is equal to

$$[L_1] - [L_2] \tag{3.2}$$

where L_1, L_2 are strongly positive currents. Hence

$$span_{\mathbb{Q}}(F_M^+) = H_{2k}(X, M; \mathbb{Q}). \tag{3.3}$$

On the other hand, Lemma 2.2, [6] for weakly positive classes still holds. Hence F^+ is a convex polyhedral cone of $H_{2k}(X; \mathbb{Q})$. So, we can find finitely many currents A_j indexed $j = 1, \cdots, h, h+1, \cdots, l$ such that the classes $\{[A_j]\}_{j=1,\cdots,h}$ form the frame for the polyhedral cone F^+, and

$$\{[A_1], \cdots, [A_h], \cdots, [A_l]\}$$

is a basis for $H_{2k}(X, M; \mathbb{Q})$. Since $H_{2k}(X, M; \mathbb{Q})$ is proper subspace, $l > h$. Let $\tau \in F^+$. Then

$$\tau = \sum_{j=1}^{h} \lambda_j [A_j]. \tag{3.4}$$

Since $\{[A_j]\}_{j=1,\cdots,h}$ is a frame of the cone F^+, $\lambda_j \geq 0$. Let ϵ be a positive rational number. We rewrite

$$\tau = \sum_{j=1}^{h} (\lambda_j [A_j] + \epsilon [A_l]) - \ \epsilon h [A_l] \tag{3.5}$$

Notice that Theorem 1.2 still holds with weakly positive classes. Since

$$\lambda_j [A_j] + \epsilon [A_l], [A_l]$$

are all weakly positive and has non-empty boundary, i.e. satisfy (1.1), we apply Theorem 1.2 to them to obtain

$$\lambda_j [A_j] + \epsilon [A_l] = \mathcal{T}_{\tilde{V}^{j,\epsilon}} + S^{j,\epsilon} \tag{3.6}$$

$$[A_l] = \mathcal{T}_{\tilde{V}^l} + S^l \tag{3.7}$$

where $\tilde{V}^{j,\epsilon}, \tilde{V}^l$ are positive holomorphic chains in $X \backslash M$, and $S^{j,\epsilon}, S^l$ are closed, positive (k,k) currents in X. Thus

$$\tau = \left[\sum_{j=1}^{h} \mathcal{T}_{\tilde{V}^{j,\epsilon}} + \sum_{j=1}^{h} S^{j,\epsilon} \right] - \left[\epsilon h (\mathcal{T}_{\tilde{V}^l} + S^l) \right] \tag{3.8}$$

Notice that all currents have bounded mass with respect to infinitely many $\epsilon > 0$. By the compactness theorem in geometric measure theory, we can take the weak limits of sub-sequences for each sequences of currents labeled with ϵ. So, we obtain the formula in weak limits

$$\tau = [\mathcal{T} + S] \tag{3.9}$$

where

$$\mathcal{T} = \lim_{\epsilon \rightharpoonup 0} \sum_{j=1}^{h} \mathcal{T}_{\tilde{V}^{j,\epsilon}} \tag{3.10}$$

$$S = \lim_{\epsilon \rightharpoonup 0} \sum_{j=1}^{h} S^{j,\epsilon}. \tag{3.11}$$

We should note that the current

$$\mathcal{T}^{\epsilon} := \sum_{j=1}^{h} \mathcal{T}_{\tilde{V}^{j,\epsilon}}$$

(which is a simple extension of a positive holomorphic chain) by Lemma A.1 is locally rectifiable. Next we check the conditions for the weak limit $\mathcal{T}$.

(1) Since X is compact, the weak convergence coincides with the mass convergence. The convergence

$$\lim_{\epsilon \rightharpoonup 0} \mathcal{T}^{\epsilon} = \mathcal{T}$$

is in mass. Since $\mathcal{T}^{\epsilon}$ are locally rectifiable, by the definition of real Lipschitzian chains in [5], the mass limit current $\mathcal{T}$ is also real locally rectifiable.

(2) Since $\mathcal{T}^{\epsilon}$ for all ϵ are of bidimension (k, k), so is the weak limit $\mathcal{T}$.

(3) Since $\mathcal{T}^{\epsilon}$ for all ϵ are closed, so is the weak limit $\mathcal{T}$.

(4) Since $\mathcal{T}^{\epsilon}$ are all positive-holomorphic-chain's extensions, the mass of $\mathcal{T}^{\epsilon}$ is bounded by the mass of the class τ, where the mass of τ is the number

$$\frac{1}{k!}\tau(\omega^k)$$

where ω is the Kähler form (see this mass formula, for instance, in Lemma 3.5, [3]). Hence the $2k$-Hausdorff measures of $supp(\mathcal{T}^{\epsilon})$ is bounded. It implies that the local $2k$-Hausdorff measure of $supp(\mathcal{T})$ ([5] is also bounded.

These four conditions by Theorem 1.1 of [5] imply that $\mathcal{T}$ is a real holomorphic chain, i.e. a holomorphic chain with real coefficients. Since X is projective, by Chow's theorem, each subvariety in the holomorphic chain is algebraic sunvariety, and the sum in the chain is finite. We denote this algebraic cycle with real coefficients by V. Hence $T_V = \mathcal{T}$. Since $\mathcal{T}$ is a positive current, all coefficients in V must be positive.

Notice that S is the weak limit of closed, positive current. Hence S is also closed and positive. So, the formula (3.9) completes the proof.

□

Appendix A Lelong's Simple extension

Our strategy is to extend the holomorphic chains to algebraic cycles. This type of extension is first obtained as an extension of currents, which is important notion on its own. Let's give a review first. Let Ω be an open subset of a real manifold Y. Let $t \in \mathscr{D}'(\Omega)$ where $\mathscr{D}'(\bullet)$ denotes the space of currents on $\bullet$. Any current $\tilde{t}$ on Y that is restricted to t on Ω is called an extension of t. Not all currents in Ω can have extensions, and extensions may not be unique if they exist. A simple extension, denoted by $\tilde{t}_o$, of $t \in \mathscr{D}'(\Omega)$ is a particular extension-by-zero defined by Lelong ([4]). Its functional is $\sum_{j=1}^{\infty} \phi_j t$ where $\{\phi_j\}$ is a partition of unity for an open covering of Ω. The functional is a current if 1) the order of t is 0; 2) the mass of t in $\Omega \cap G$ for any compact set G of Y, denote by $||t||_G$, is finite. (But the simple extension is still dependent of the partition of unity.)

Lemma A.1. *Let Y be compact. Let $t \in \mathscr{D}'(\Omega)$ be a current that has order* 0 *and $||t||_G$ is finite for any compact G. Then if t is locally rectifiable, so is $\tilde{t}_o$.*

Proof. Since Y is compact, its weak limits are also the mass limits (i.e. strong limits). Let ϕ_j be a partition of unity, associated to the simple extension. For a test form ψ, we have the evaluation

$$\tilde{t}_o[\psi] = \sum_{j=1}^{\infty} t[\phi_j \psi]. \tag{A.1}$$

Since $t \sum_{j=1}^{N} \phi_j$ is well-defined in Y, the weak limit

$$\lim_{N \xrightarrow{w} \infty} t \sum_{j=1}^{N} \phi_j$$

is also well-defined in Y (see (A.1)). Let $\epsilon > 0$ and P_ϵ be the Lipschitzian chains for the approximation of t in Ω. Let $|| \bullet ||$ be the mass in the manifold Y. By Proposition 2, [4]

$$||\tilde{t}_o|| = ||t||,$$

where for $t \in \mathscr{D}'(\Omega)$, the mass is evaluated in the restricted charts as those for

$Y \cap \Omega$ from Y. Then we have the computation

$$||\tilde{t}_o - T_{P_\epsilon}|| = || \lim_{N \xrightarrow{w} \infty} (t - T_{P_\epsilon}) \sum_{j=1}^{N} \phi_j ||$$

(Since the weak limit is the same as the mass limit)

$$= \lim_{N \to \infty} ||(t - T_{P_\epsilon}) \sum_{j=1}^{N} \phi_j ||$$

(since $t - T_{P_\epsilon}$ has order 0)

$$\leq \lim_{N \to \infty} \left|\left| \sum_{j=1}^{N} \phi_j \right|\right|_\infty ||t - T_{P_\epsilon}||$$

$$= ||t - T_{P_\epsilon}||$$

(since t is rectifiable, there is some P_ϵ)

$$\leq \epsilon.$$

Thus $\tilde{t}_o$ is approximated by the same Lipschitzian chains P_ϵ as that for t. Therefore $\tilde{t}_o$ is also locally rectifiable. □

Remark. The simple extension in this paper is the particular one that is unique and whose original current t is a linear combination of closed, positive currents ([2]).

References

[1] J.-P. Demailly, *Courants positifs extrêmaux et conjecture de Hodge*, Inventiones mathematicae (1982), p. 347-374

[2] R. Harvey, *Removable singularities for positive currents*, J. of AMS (1974), P. 67-78

[3] R. Harvey, B. Lawson, *Boundaries of positive holomorphic chains and the relative Hodge question*, Astérisque, 328(2009), p. 207-221

[4] P. Lelong, *Intégration sur un ensemble analytique complexe*, Bulletin de la S. M. F., tome 85 (1957), p. 239-262

[5] J-H. Teh, C-H. Yang, *A characterization of real holomorphic chains and applications in representing homology classes by algebraic cycles*, Complex manifolds (2020), p. 93-105

[6] B. Wang, *Infinite cycle classes*, Preprint (2026)