Phoenix Numbers

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Abstract

If the digits of number n (in base b) can be split $p|q$ so that $n = q_r p_r$, where p_r and q_r are the numbers formed by reversing the digits of p and q, respectively, then n is called a phoenix number in base b. If q has k digits in base b, then n is called a k-Phoenix number in base b. For example, $1.1.n.n$ is 1-Phoenix in base $b = n^2 - 1$. The existence of infinitely many k-phoenix numbers in any base b is claimed, but not proven.

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This document is as much a scratchpad as a research paper, so it may get a bit "choppy" in places.

1 Introduction

Let n be a positive integer with digits $(d_m \ldots d_1)_b$ in base b. Then n is called a s-Phoenix number in base b if $(d_{s+1} \cdots d_m)_b \times (d_1 \ldots d_s)_b = n$. Briefly, split the number n in base b at the sth digit, reverse the digits in both pieces, regard the two sequences of digits as numbers in base b, and multiply. If the product is equal to n, then n is called a s-Phoenix number in base b. The s-Phoenix property is indeed base-dependent, since, for example, $27 = 123_4$ and $7 \cdot 2 = 14$ but $3 \cdot 21 = 120_4 + 3_4 = 123_4$. In fact, $123_4 = 3^3$ is the first Phoenix number and the first Phoenix power (Section 5).

Phoenix numbers are of interest because of the many patterns they contain. Let us establish some notation. For example, the number $1.1.[\![0]\!].[2.1.0.1].[\![0]\!].2.2$ is 1-phoenix in base 3 and this read as follows: $[2.1.0.1]$ means that the digits 2.1.0.1 can be repeated zero or more times while $[\![0]\!]$ means that 0 must be repeated the same number of times in each position. An instance would be $1.1.0.0.0.2.1.0.1.2.1.0.1.0.0.0.2.2$. The pattern $1.1.[0].2.2$ in base 3 also occurs. Furthermore, the number $1.1.a.a$ is 1-phoenix in base $b = a^2 - 1$ and $1.k.ka.a$ is 1-phoenix in base $b = k(a^2 - 1)$. The aforementioned pattern in base 3 can be generalized to bases $b = k(a^2 - 1)$ (Theorem).

Sequence 1 ([A014575](#)). *Vampire numbers (definition 2): numbers n with an even number of digits which have a factorization $n = ij$ where $length(i) = length(j) = length(n)/2$ and the multiset of the digits of n coincides with the multiset of the digits of i and j.*

Sequence 2 ([A020342](#)). *Vampire numbers (definition 1): n has a nontrivial factorization using n's digits.*

Sequence 3 ([A179482](#)). *A subset of vampire numbers: n has a nontrivial factorization using n's digits in reverse order.*

Sequence 4 ([A009944](#)). *Left-right/right-left numbers: $n = p|q$ such that $n = q_{reversed}p_{reversed}$.*

$$
\begin{aligned}
m &= [10, 9.7.8] \\
n &= [10, 9.8.7] \\
mn &= [10, 7.9.7.8.9.8]
\end{aligned}
$$

3.				(6)	(5)
2.				(5)	(4)
1.				(5)	(4)
				9	8	7
				9	7	8
	(1)				
1.			6	5	8	8
2.		5	8	0	1
3.	7	3	5	3	
	7	9	7	8	9	8

http://mathworld.wolfram.com/VampireNumber.html
https://en.wikipedia.org/wiki/Vampire_number
https://groups.google.com/forum/#!msg/sci.math/eCgYeC3aUbc/2haqpIEie_EJ

2   1-Phoenix numbers

The smallest number of digits a 1-phoenix number can have is 3 digits. Table 1 shows the first 3-digit 1-phoenix numbers in bases from 3 to 12. Note that there are no 1-phoenix numbers in base 2.

Table 1: 1-Phoenix numbers with three digits

b	n
4	1.2.3
6	1.2.4, 1.3.3
8	1.2.5, 1.4.3
9	1.3.4
10	1.2.6, 1.5.3
12	1.2.7, 1.3.5, 1.4.4, 1.6.3

Observe that all the numbers are of the form $1.e.(d+1)$ where d is a proper divisor of the base b and $e = b/d$ is the conjugate divisor of d. The following theorem shows that all three digit 1-phoenix numbers $1.x.y$ are of this form.

Theorem 5. *The number $1.x.y$ is 1-phoenix in base b if and only if $b = x(y-1)$. See Table 1.*

Proof. By assumption, $xy = 1.z$ in base b. The multiplication table is

$x.$	1
	y
1. $z.$	y

Base b.

If $1.x.y$ is 1-phoenix in base b, then $z = x$, $xy = b + x$ and $b = x(y-1)$. If $b = x(y-1)$, then $xy = b + x$ and $1.x.y$ is 1-phoenix in base b. $\qquad\square$

The very first phoenix number is the 1-phoenix number 1.2.3 in base $b = 4$. Numbers of the form $1.e.3$ in even bases $b = 2e$ have many interesting patterns, stated in the following theorem.

Theorem 6 (1-Phoenix numbers with even bases not divisible by 8)**.** *Let $b = 2e$, $e > 1$, and consider 1-phoenix numbers of the form $1.e.3$.*

1. *If $b \equiv 2 \mod 8$, that is, $b = 10 + 8k = 2(5 + 4k)$, then*

$$1.[5 + 4k.7 + 6k.4 + 4k.2 + 2k].5 + 4k.3$$

   is 1-phoenix in base b. Note that the initial term is $1.[5.7.4.2].5.3$ in base 10.

2. *If $b \equiv 4 \mod 8$, that is, $b = 4 + 8k = 2(2 + 4k)$, then*

$$1.[2 + 4k.1 + 3k.[3 + 7k.1 + 4k.2 + 5k].k].2 + 4k.3$$

   is 1-phoenix in base b. Note that the initial term is $1.[2.1.[3.1.2].0].2.3$ in base 4.

3. *If $b \equiv 6 \mod 8$, that is, $b = 6 + 8k = 2(3 + 4k)$, then*

   (a) *$1.[3.4.[0].2.1].3.3$ is 1-phoenix in base 6.*

   (b) *$1.[3 + 4k.4 + 6k.2 + 4k.1 + 2k].3 + 4k.3$ is 1-phoenix in base b. Note that, in base 6, $[3.4.2.1]$ is an instance of $[3.4.[0].2.1]$.*

   (c) *$1.[3 + 4k.4 + 6k.k.2 + 3k.1 + 2k].3 + 4k.3$ is 1-phoenix in base b. Note that, in base 6, $[3.4.0.2.1]$ is an instance of $[3.4.[0].2.1]$.*

   (d) *Let $A_0, A_1, \ldots, A_m$ be instances of $[3+4k.4+6k.2+4k.1+2k]$ and let $Z_0, Z_1, \ldots, Z_m$ be instances of $[3 + 4k.4 + 6k.k.2 + 3k.1 + 2k]$. Then*

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i. The number

$$1.\,[A_m.Z_m \cdots A_1.Z_1.A_0.Z_1.A_1 \cdots Z_m.A_m]\,.3 + 4k.3$$

is 1-phoenix in base b.

ii. The number

$$1.\,[Z_m.A_m \cdots Z_1.A_1.Z_0.A_1.Z_1 \cdots A_m.Z_m]\,.3 + 4k.3$$

is 1-phoenix in base b.

Proof. We will prove 2. The other cases will be left to the reader. Note that the pattern in

$$1.[2 + 4k.1 + 3k.[3 + 7k.1 + 4k.2 + 5k].k].2 + 4k.3$$

can be shifted one place to the right. Thus,

Table 2: Multiplication for Theorem 6.2.

(1)		(1)	(1)	(2)	(1)	(1)		
$[2 + 4k.$	$k.$	$[2 + 5k.$	$1 + 4k.$	$3 + 7k].$	$1 + 3k].$	$2 + 4k.$	1	
								3
1.	$[2 + 4k.$	$1 + 3k.$	$[3 + 7k.$	$1 + 4k.$	$2 + 5k].$	$k].$	$2 + 4k.$	3

Base $b = 4 + 8k$.

$$
\begin{aligned}
3(2 + 4k) - (2 + 4k) &= (1)(4 + 8k), \\
3(1 + 3k) + 1 - k &= (1)(4 + 8k), \\
3(3 + 7k) + 1 - (2 + 5k) &= 9 + 21k + 1 - 2 - 5k \\
&= 8 + 16k = (2)(4 + 8k), \\
3(1 + 4k) + 2 - (1 + 4k) &= (1)(4 + 8k), \\
3(2 + 5k) + 1 - (3 + 7k) &= 6 + 15k + 1 - 3 - 7k \\
&= (1)(4 + 8k), \\
3k + 1 - (1 + 3k) &= 0.
\end{aligned}
$$

□

There is an interesting pattern in bases that are squares.

Theorem 7 (1-Phoenix in base $b = n^2$)**.** *The number*

$$1.[n.n - 1.n^2 - 1.n^2 - n - 1.n^2 - n.0].n.n + 1$$

is 1-phoenix in base $b = n^2$.

Proof. By multiplication.

	⑦	⑥	⑤	④	③	②	①		
(1)		$(n-1)$	$(n-1)$	(n)	(1)	(1)			
	$[n.$	$0.$	$n^2-n.$	$n^2-n-1.$	$n^2-1.$	$n-1].$	$n.$		1
									$n+1$
$1.$	$[n.$	$n-1.$	$n^2-1.$	$n^2-n-1.$	$n^2-n.$	$0].$	$n.$		$n+1$

Base $b = n^2$.

$$① = (n+1)n - n = (1)n^2,$$
$$② = (n+1)(n-1) + (1) = n^2 - 1 + 1 = (1)n^2,$$
$$③ = (n+1)(n^2-1) + (1) - (n^2-n)$$
$$= (n+1)^2(n-1) - n^2 + n + 1$$
$$= n^3 + 2n^2 + n - n^2 - 2n - 1 - n^2 + n + 1$$
$$= (n)n^2,$$
$$④ = (n+1)(n^2-n-1) + n - (n^2-n-1)$$
$$= n(n^2-n-1) + n$$
$$= n^3 - n^2 - n + n$$
$$= (n-1)n^2,$$
$$⑤ = (n+1)(n^2-n) + (n-1) - (n^2-1)$$
$$= (n-1)[(n+1)(n) + 1 - (n+1)]$$
$$= (n-1)[n^2 + n + 1 - n - 1]$$
$$= (n-1)n^2,$$
$$⑦ = ①.$$

$\square$

Theorem 8 (1-Phoenix in base $b = n(n+3)$)**.** *The number*

$$(n+1)^2 - 3.n(n+3) - 2.(n+1)^2 - 1$$

is 1-phoenix in base $b = n(n+3)$, $n \geq 1$. See Table 3.

Remark 9. The following sequences occur in Theorem 8.

A005563 $a(n) = n(n+2)$.

A028552 $a(n) = n(n+3)$. In our case, $(n-1)(n+2)$.

A028872 $a(n) = n^2 - 3$. In our case, $n^2 + 2n - 2$.

(not in OEIS) $a(n) = n^2 + 3n - 2 = \text{A028552}(n) - 2$.

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Table 3: 1-Phoenix numbers of Theorem 8.

n	b	N	N_b
1	4	27	1.2.3
2	10	688	6.8.8
3	18	4515	13.16.15
4	28	18000	22.26.24
5	40	54355	33.38.35
6	54	136992	46.52.48
7	70	303723	61.68.63
8	88	611680	78.86.80
9	108	1142955	97.106.99
10	130	2010960	118.128.120
11	154	3367507	141.152.143
12	180	5410608	166.178.168

Base $b = n(n+3)$.

Proof. By multiplication. It is convenient to write the number in the form $n(n+2) - 2.n(n+3) - 2.n(n+2)$.

	$((n+2)(n-1))$	
	$n(n+3) - 2.$	$n(n+2) - 2$
		$n(n+2)$
$n(n+2) - 2.$	$n(n+3) - 2.$	$n(n+2)$

Base $b = n(n+3)$.

$$
\begin{aligned}
n(n+2)[n(n+2) - 2] - n(n+2) &= n(n+2)[n(n+2) - 2 - 1] \\
&= n(n+2)[n^2 + 2n - 3] \\
&= n(n+2)(n-1)(n+3) \\
&= (n+2)(n-1) \cdot n(n+3), \\
n(n+2)[n(n+3) - 2] + (n+2)(n-1) - (n(n+3) - 2) \\
&= n^2(n+2)(n+3) - 2n(n+2) \\
&\quad + n^2 + n - 2 - n^2 - 3n + 2 \\
&= n^2(n+2)(n+3) - 2n(n+2) - 2n \\
&= n[n(n+2)(n+3) - 2(n+2) - 2] \\
&= n[n(n+2)(n+3) - 2n - 6] \\
&= n[n(n+2)(n+3) - 2(n+3)] \\
&= n(n+3)[n(n+2) - 2]
\end{aligned}
$$

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$$= (n(n+2) - 2) \cdot n(n+3). \qquad \square$$

Theorem 10 (1-Phoenix in base $b = 2n^2 + 6n + 2$)**.** *The number*

$$(1+n)(2+n).(1+n)(3+n).2(1+n)^2$$

is 1-phoenix in base $b = 2n^2 + 6n + 2$, $n \geq 1$.

Proof. By multiplication.

	$((n+1)^2)$	
	$(n+1)(n+3).$	$(n+1)(n+2)$
		$2(n+1)^2$
$(n+1)(n+2).$	$(n+1)(n+3).$	$2(n+1)^2$

Base $b = 2n^2 + 6n + 2$, $n \geq 1$.

$$
\begin{aligned}
2(n+1)^2 \cdot (n+1)(n+2) - 2(n+1)^2 &= (n+1)^2[2(n+1)(n+2) - 2] \\
&= (n+1)^2(2n^2 + 6n + 2).
\end{aligned}
$$

$$
\begin{aligned}
2(n+1)^2 \cdot (n+1)(n+3) + (n+1)^2 - (n+1)(n+3) &= (n+1)[2(n+1)^2(n+3) - 2] \\
&= 2(n+1)[(n+1)^2(n+3) - 1] \\
&= 2(n+1)[n^3 + 5n^2 + 7n + 2] \\
&= 2(n+1)[(n+2)(n^2 + 3n + 1)] \\
&= (n+1)(n+2)(2n^2 + 6n + 2). \ \square
\end{aligned}
$$

Remark 11. The following sequences occur in Theorem 10.

A000290 The squares: $a(n) = n^2$.

A001105 $a(n) = 2n^2$.

A002378 $a(n) = n(n+1)$. In our case, $(n+1)(n+2)$.

A090288 $a(n) = 2n^2 + 6n + 2$.

A028347 $a(n) = n^2 - 4$. In our case, $n(n+4)$.

A028552 $a(n) = n(n+3)$.

A005563 $a(n) = n(n+2)$. In our case, $n^2 - 1$.

Theorem 12 (1-Phoenix in base $b = n(2n^2 - 3)$)**.** *The number*

$$2(n^2 - 1).2n(n^2 - 2).2n^2$$

is 1-phoenix in base $b = n(2n^2 - 3)$, $n \geq 2$.

Table 4: 1-Phoenix numbers of Theorem 10

n	b	N	N_b
1	10	688	6.8.8
2	22	6156	12.15.18
3	38	29824	20.24.32
4	58	103000	30.35.50
5	82	286416	42.48.72
6	110	684628	56.63.98
7	142	1463296	72.80.128
8	178	2869344	90.99.162
9	218	5254000	110.120.200
10	262	9098716	132.143.242
11	310	15043968	156.168.288
12	362	23920936	182.195.338

Base $b = 2n^2 + 6n + 2$.

Proof. By multiplication.

	$(2n)$	
	$n(2n^2 - 4).$	$2n^2 - 2$
		$2n^2$
$2n^2 - 2.$	$n(2n^2 - 4).$	$2n^2$

Base $b = n(2n^2 - 3)$, $n \geq 2$.

$$
\begin{aligned}
2n^2(2n^2 - 2) - 2n^2 &= 2n^2(2n^2 - 2) - 2n^2 \\
&= 2n^2(2n^2 - 3) \\
&= 2n \cdot n(2n^2 - 3). \\
2n^2 \cdot n(2n^2 - 4) + 2n - n(2n^2 - 4) &= 2n[n^2(2n^2 - 4) + 1 - (n^2 - 2)] \\
&= 2n[2n^4 - 4n^2 + 1 - n^2 + 2] \\
&= 2n(2n^4 - 5n^2 + 3) \\
&= 2n(n^2 - 1)(2n^2 - 3) \\
&= (2n^2 - 2) \cdot n(2n^2 - 3). \qquad \square
\end{aligned}
$$

Remark 13. The following sequences occur in Theorem 12.

(**not in OEIS**) $a(n) = n(2n^3 - 3)$.

[A268581](#) $a(n) = 2n^2 + 8n + 5$. In our case, $2n^2 - 3$, $n \geq 2$.

Table 5: 1-Phoenix numbers for Theorem 12

n	b	N	N_b
2	10	688	6.8.8
3	45	34308	16.42.18
4	116	416704	30.112.32
5	235	2704900	48.230.50
6	414	12166704	70.408.72
7	665	42891268	96.658.98
8	1000	126992128	126.992.128
9	1431	329676804	160.1422.162
10	1970	772279600	198.1960.200
11	2629	1665676804	240.2618.242
12	3420	3356826048	286.3408.288

Base $b = n(2n^2 - 3)$.

Table 6: 1-Phoenix numbers for Theorem 14 (9).

n	b	N_b
2	6	1.[3.4.0.0.2.1].3.3
3	12	1.[4.6.1.4.6.1].4.4
4	20	1.[5.8.2.10.12.1].5.5
5	30	1.[6.10.3.18.20.1].6.6

Base $b = n(n + 1)$.

A054000 $a(n) = 2n^2 - 2$.

(not in OEIS) $a(n) = n(2n^3 - 4)$.

A147973 $a(n) = -2n^2 + 12n - 14$. In our case, $2n^2 - 4$, $n \geq 2$. Technically, not in OEIS.

A005843 $a(n) = 2n$.

3 1-Phoenix numbers in base $b = n(n + 1)$

Numbers of the form $1.n + 1.n + 1$ in base $b = n(n + 1)$ have the following patterns.

Theorem 14 (1-Phoenix of the form $1.n + 1.n + 1$ in base $b = n(n + 1)$)**.** *Suppose $n \geq 2$. Then*

1. *The number* 1.[5].3.3 *is 1-phoenix in base 6.*

2. *The number* 1.[3.4.[0].2.1].3.3 *is 1-phoenix in base 6.*

3. *The number*
$$1.[\![5]\!].[3.4.[0].2.1].[\![5]\!].3.3$$
   is 1-phoenix in base 6.

4. *The number* 1.[4.6.1].4.4 *is 1-phoenix in base 12.*

5. *The number* 1.[7.5.10].4.4 *is 1-phoenix in base 12.*

6. *If* $A_0, \ldots, A_k$ *are instances of* [4.6.1] *and* $B_1, \ldots, B_k$ *are instances of* [7.5.10]*, then*
$$1.[A_k B_k \ldots A_1 B_1 A_0 B_1 A_1 \ldots B_k A_k].4.4$$
   is 1-phoenix in base 12.

7. *If* $A_1, \ldots, A_k$ *are instances of* [4.6.1] *and* $B_0, \ldots, B_k$ *are instances of* [7.5.10]*, then*
$$1.[B_k A_k \ldots B_1 A_1 B_0 A_1 B_1 \ldots A_k B_k].4.4$$
   is 1-phoenix in base 12.

8. *The number* $1.n + 1.n + 1$ *is 1-phoenix in base* $b = n(n+1)$, $n \geq 2$.

9. *The number*
$$1.[n + 1.2n.n - 2.n^2 - n - 2.n^2 - n.1].n + 1.n + 1$$
   is 1-phoenix in base $b = n(n+1)$, $n \geq 2$. *Note that for* $n = 2$ *(b = 6) the pattern is* [3.4.0.0.2.1] *and for* $n = 3$ *(b = 12) the pattern is* [4.6.1.4.6.1].

Remark 15. See Theorem 21 for an arithmetic extension of (9).

Proof. By multiplication. Proof of (3).

	⑩	⑨	⑧	⑦	⑥	⑤	④	③	②	①		
	(2)	(2)		(1)			(2)	(1)	(2)	(2)		
	3.	$[\![5]\!]$	5.	[1.	2.	[0].	0.	4.	3].	$[\![5]\!]$.	5	1
												3
1.	5.	$[\![5]\!]$.	[3.	4.	0.	[0].	2.	1].	5.	$[\![5]\!]$.	3.	3

Base $b = 6$.

Proof of (5).

13

	(3)	(1)	(2)	(3)	(1)	(2)		
	4.	[X.	5.	7.	X.	5.	7]	1
								4
1.	[7.	5.	X.	7.	5.	X]	4.	4

Base $b = 12$.

Proof of (6).

	(2)	(1)	(3)	(1)	(2)		(2)	(1)		
	4.	⟦1.	6.	4⟧.	[X.	5.	7].	⟦1.	6.	4⟧.	1
											4
1.	⟦4.	6.	1⟧.	[7.	5.	X].	⟦4.	6.	1⟧.	4.	4

Base $b = 12$.

Proof of (7).

	(3)	(1)	(2)		(2)	(1)	(3)	(1)	(2)		
	4.	[X.	5.	7].	[1.	6.	4].	⟦X.	5.	7⟧.	1
											4
1.	⟦7.	5.	X⟧.	[4.	6.	1].	⟦7.	5.	X⟧.	4.	4

Base $b = 12$.

Proof of (9).

	⑦		⑥		⑤		④		③		②		①		
			$(n-1)$	$(n-2)$					(2)		(1)				1
	$n+1.$		[1.	$n^2 - n.$	$n^2 - n - 2.$		$n-2.$		$2n.$		$n+1].$		1
															$n+1$
1.	$[n+1.$		$2n.$	$n-2.$	$n^2 - n - 2.$		$n^2 - n.$		1].		$n+1.$		$n+1$

Base $b = n(n+1)$.

① $= (n+1)^2 - (n+1) = n(n+1)$.
② $= (n+1)(2n) + 1 - 1 = 2n(n+1)$.
③ $= (n+1)(n-2) + 2 - (n^2 - n) = n^2 - n - 2 + 2 - n^2 + n = 0$.
④ $= (n+1)(n^2 - n - 2) - (n^2 - n - 2) = n(n+1)(n-2) = (n-2) \cdot n(n+1)$.
⑤ $= (n+1)(n^2 - n) + (n-2) - (n-2) = (n-1) \cdot n(n+1)$.
⑥ $= (n+1)(1) + (n-1) = 2n$,
⑦ $=$ ①. □

Bases that are multiples of 6 show an interesting pattern. Here is a table of the first few examples.

k	b	N	N_b
0	6	57	1.3.3
1	12	522	3.7.6
2	18	1827	5.11.9
3	24	4404	7.15.12
4	30	8685	9.19.15
5	36	15102	11.23.18

Base $b = 6 + 6k$.

Theorem 16 (1-Phoenix in base $b = 6 + 6k$)**.** *The number*

$$1 + 2k.3 + 4k.3 + 3k$$

is 1-phoenix in base $b = 6 + 6k$, $k \geq 0$. Let $\mathrm{B} = b - 1$. Furthermore, the following is true.

1. *$1 + 2k.[\mathrm{B}].3 + 4k.3 + 3k$ is 1-phoenix in base b*

2. *$1 + 2k.[3 + 4k.4 + 4k.[0].2 + 2k.1 + 2k].3 + 4k.3 + 3k$ is 1-phoenix in base b.*

3. *Let $Z_0, Z_1, \ldots, Z_k$ be instances of $[3 + 4k.4 + 4k.[0].2 + 2k.1 + 2k]$, and let $B_1, \ldots, B_k$ be instances of $[\mathrm{B}]$. Then*

$$1 + 2k.\left[Z_k B_k \cdots Z_1 B_1 Z_0 B_1 Z_1 \cdots B_k Z_k\right].3 + 4k.3 + 3k$$

   is 1-phoenix in base b.

4. *Let $B_0, B_1, \ldots, B_k$ be instances of $[\mathrm{B}]$ and let $Z_1, \ldots, Z_k$ be instances of $[3 + 4k.4 + 4k.[0].2 + 2k.1 + 2k]$. Then*

$$1 + 2k.\left[B_k Z_k \cdots B_1 Z_1 B_0 Z_1 B_1 \cdots Z_k B_k\right].3 + 4k.3 + 3k$$

   is 1-phoenix in base b.

Proof. By induction and multiplication.

		(k)
	$3 + 4k.$	$1 + 2k$
		$3 + 3k$
$1 + 2k.$	$3 + 4k.$	$3 + 3k$

Base $b = 6 + 6k$.

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$$(3+3k)(1+2k) - (3+3k) = (3+3k)(2k)$$
$$= (k)(6+6k).$$
$$(3+3k)(3+4k) + k - (3+4k) = (3+3k)(3+4k) - (3+3k)$$
$$= (3+3k)(2+4k)$$
$$= (1+2k)(6+6k).$$

Now assume the Theorem is true for n copies of $[5+6k]$. Let us show it is true for $n+1$ copies of $[5+6k]$. Then

		$(2+3k)$	$(2+3k)$	(k)	
		$3+4k.$	$[5+6k].$	$5+6k.$	$1+2k$
					$3+3k$
$1+2k.$	$5+6k.$	$[5+6k].$	$3+4k.$	$3+3k$

Base $b = 6+6k.$

$$(3+3k)(1+2k) - (3+3k) = (k)(6+6k).$$
$$(3+3k)(5+6k) + k - (3+4k) = (3+3k)(5+6k) - (3+3k)$$
$$= (3+3k)(4+6k)$$
$$= (2+3k)(6+6k).$$
$$(3+3k)(3+4k) + (2+3k) - (5+6k) = (3+3k)(3+4k) - (3+3k)$$
$$= (3+3k)(2+4k)$$
$$= (1+2k)(6+6k).$$

The theorem is also true for $n+1$. By induction the theorem is true for all n. $\qquad\square$

Motivated by Theorem 16, we look at base $b = 12$.

Theorem 17 (1-Phoenix in base $b = 12 + 24k$)**.** *The number*

$$1 + 3k.4 + 9k.4 + 8k$$

is 1-phoenix in base $b = 12 + 24k$, $k \geq 0$. Furthermore, the following is true.

1. *The number $1 + 3k.[4 + 9k.6 + 12k.1 + 3k].4 + 9k.4 + 8k$ is 1-phoenix in base b.*

2. *The number $1 + 3k.[7 + 15k.5 + 12k.10 + 21k].4 + 9k.4 + 8k$ is 1-phoenix in base b.*

3. *Let $A_0, A_1, \cdots, A_m$ be instances of $[4+9k.6+12k.1+3k]$ and let $B_1, \cdots B_m$ be instances of $[7 + 15k.5 + 12k.10 + 21k]$. Then*

$$1 + 3k.[A_m.B_m \ldots A_1.B_1.A_0.B_1.A_1...B_m.A_m].4 + 9k.4 + 8k$$

is 1-phoenix in base b.

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4. Let $A_1, \cdots, A_m$ be instances of $[4 + 9k.6 + 12k.1 + 3k]$ and let $B_0, \cdots B_m$ be instances of $[7 + 15k.5 + 12k.10 + 21k]$. Then

$$1 + 3k.[B_m.A_m \ldots B_1.A_1.B_0.A_1.B_1...A_m.B_m].4 + 9k.4 + 8k$$

is 1-phoenix in base b.

Proof. By multiplication.

	(k)	
	$4 + 9k.$	$1 + 3k$
		$4 + 8k$
$1 + 3k.$	$4 + 9k.$	$4 + 8k$

Base $b = 12 + 24k$.

$$(4 + 8k)(1 + 3k) - (4 + 8k) = (k)(12 + 24k).$$
$$(4 + 8k)(4 + 9k) + k - (4 + 9k) = (4 + 8k)(4 + 9k) - (4 + 8k)$$
$$= (4 + 8k)(3 + 9k)$$
$$= (1 + 3k)(12 + 24k).$$

Inductive proofs similar to those of Theorem 16 are left to the reader. □

Theorem 18 (1-Phoenix in base $b = 20 + 60k$). *The number $1 + 4k.5 + 16k.5 + 15k$ is 1-phoenix in base $b = 20 + 60k$.*

Proof. By multiplication.

	(k)	
	$6 + 16k.$	$1 + 4k$
		$5 + 15k$
$1 + 4k.$	$5 + 16k.$	$5 + 15k$

Base $b = 20 + 60k$.

$$(5 + 15k)(1 + 4k) - (5 + 15k) = (k)(20 + 60k).$$
$$(5 + 15k)(5 + 16k) + k - (5 + 16k) = (5 + 15k)(5 + 16k) - (5 + 15k)$$
$$= (5 + 15k)(4 + 16k)$$
$$= (1 + 4k)(20 + 60k). \qquad \square$$

Note that 1.3.3 (base 6), 1.4.4 (base 12), 1.5.5 (base 12) are all initial terms of arithmetic sequences of 1-phoenix numbers. Note that the sequence of bases 6, 12, 20 are numbers of the form $n(n + 1)$. Thus, we have the following.

n	b		.	.
2	$6 + 6k$	$1 + 2k.$	$3 + 4k.$	$3 + 3k$
3	$12 + 24k$	$1 + 3k.$	$4 + 9k.$	$4 + 8k$
4	$20 + 60k$	$1 + 4k.5 + 16k.5 + 15k$		

The following theorem presents itself.

Theorem 19 (1-Phoenix in base $b = n(n + 1) + n(n^2 - 1)k)$). *The number*

$$1 + nk.(n + 1) + n^2k.(n + 1) + (n^2 - 1)k$$

is 1-phoenix in base $b = n(n + 1) + n(n^2 - 1)k$, $n \geq 2$, $k \geq 0$.

Proof. By multiplication. Note that $b = n(n + 1)(1 + (n - 1)k)$.

(k)		
$(n + 1) + n^2k.$		$1 + nk$
		$(n + 1) + (n^2 - 1)k$
$1 + nk.$	$(n + 1) + n^2k.$	$(n + 1) + (n^2 - 1)k$

Base $b = n(n + 1) + n(n^2 - 1)k$, $n \geq 2$, $k \geq 0$.

$$[(n + 1) + (n^2 - 1)k)](1 + nk) - [(n + 1) + (n^2 - 1)k]$$
$$= [(n + 1) + (n^2 - 1)k](nk)$$
$$= (k)[n(n + 1) + n(n^2 - 1)k].$$
$$[(n + 1) + (n^2 - 1)k]((n + 1) + n^2k) + k - [(n + 1) + n^2k]$$
$$= [n + (n^2 - 1)k]((n + 1) + n^2k) + k$$
$$= n(n + 1) + [n^3 + (n^2 - 1)(n + 1) + 1]k + n^2(n^2 - 1)k^2$$
$$= n(n + 1) + [n^3 + n^3 + n^2 - n - 1 + 1]k + n^2(n^2 - 1)k^2$$
$$= n(n + 1) + [2n^3 + n^2 - n]k + n^2(n^2 - 1)k^2$$
$$= n(n + 1) + n(2n - 1)(n + 1)k + n^2(n^2 - 1)k^2$$
$$= n(n + 1)(1 + (2n - 1)k + n(n - 1)k^2)$$
$$= n(n + 1)(1 + nk)(1 + (n - 1)k)$$
$$= (1 + nk)[n(n + 1)(1 + (n - 1)k)]$$
$$= (1 + nk)[n(n + 1) + n(n^2 - 1)k)]. \qquad \square$$

Remark 20. The following sequences occur in Theorem 19.

[A002378](#) $a(n) = n(n + 1)$, the oblong numbers.

[A007531](#) $a(n) = n(n - 1)(n - 2)$. In our case, $n(n^2 - 1)$.

If $1.n + 1.n + 1$ in base $b = n(n + 1)$, then it is natural to consider that Theorem 14 (9) also generalizes to Theorem 19. Indeed, the following is true.

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Theorem 21 (Theorem 14 (9) generalized to Theorem 19)**.** *The number*

$$1 + nk.[(n+1) + n^2k.2n + 2n(n-1)k.(n-2) + n(n-2)k.(n-2)(n+1) + n^2(n-2)k.$$
$$n(n-1) + n(n-1)^2k.1 + nk].(n+1) + n^2k.(n+1) + (n^2-1)k$$

is 1-phoenix in base $b = n(n+1) + n(n^2-1)k$, $n \geq 2$, $k \geq 0$.

Proof. An exercise for the reader. □

Theorem 22. *The number*

$$1.[a_1 + b_1k.a_4 + b_4k.a_3 + b_3k.a_2 + b_2k].a_1 + b_1k.2n - 1,$$

is 1-phoenix in base $b = a_0 + b_0k$, *where*

$$a_0 = 2n^2 - 2 = (n+1)(2n-2), \qquad b_0 = 4n(n-1),$$
$$a_1 = n+1, \qquad b_1 = 2n,$$
$$a_2 = a_1 - 2 = n-1, \qquad b_2 = b_1 - 2 = 2n-2,$$
$$a_3 = a_4 - 2 = 2n^2 - n - 4, \qquad b_3 = b_4 - 2 = 2n(2n-3),$$
$$a_4 = 2n^2 - n - 2, \qquad b_4 = 2(n-1)(2n-1),$$
$$c_1 = 2n-2, \qquad c_2 = 2n-3,$$

and $n \geq 2$, $k \geq 0$.

Proof. By multiplication.

	(c_2)	(c_1)	(1)		
$a_1 + b_1k.$	$a_2 + b_2k.$	$a_3 + b_3k.$	$a_4 + b_4k.$	$a_1 + b_1k.$	1
					$2n-1$
$1.$ $[a_1 + b_1k.$	$a_4 + b_4k.$	$a_3 + b_3k.$	$a_2 + b_2k].$	$a_1 + b_1k.$	$2n-1$

$$\begin{aligned}
(2n-1)(a_1 + b_1k) - (a_1 + b_1k) &= (2n-2)(a_1 + b_1k)\\
&= (2n-2)((n+1) + 2nk)\\
&= (2n^2 - 2) + 4n(n-1)k\\
&= (1)(a_0 + b_0k).\\
(2n-1)(a_4 + b_4k) + 1 - (a_2 + b_2k) &= [(2n-1)a_4 + 1 - a_2] + [(2n-1)b_4 - b_2]k\\
&= 4(n+1)(n-1)^2 + 8n(n-1)^2k\\
&= 2(n-1)(a_0 + b_0k) \quad (c_1 = 2n-2).\\
(2n-1)(a_3 + b_3k) + (2n-2) - (a_3 + b_3k) &= (2n-2)(a_3 + b_3k) + (2n-2)\\
&= (2n-2)((a_3 + 1) + b_3k)\\
&= (2n-2)((2n^2 - n - 3) + 2n(2n-3)k)
\end{aligned}$$

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$$
\begin{aligned}
&= (2n-2)((2n-3)(n+1) + 2n(2n-3)k) \\
&= (2n-3)((2n-2)(n+1) + 2n(2n-2)k) \\
&= (2n-3)(a_0 + b_0 k) \quad (c_2 = 2n-3).
\end{aligned}
$$

$$
\begin{aligned}
(2n-1)(a_2 + b_2 k) + (2n-3) - (a_4 + b_4 k) &= [(2n-1)a_2 + (2n-3) - a_4] + [(2n-1)b_2 - b_4]k \\
&= [(2n-1)(n-1) + (2n-3) - (2n^2 - n - 2)] \\
&\quad + [(2n-1)(2n-2) - 2(n-1)(2n-1)]k \\
&= [2n^2 - 3n + 1 + 2n - 3 - 2n^2 + n + 2] \\
&\quad + [2(n-1)(2n-1) - 2(n-1)(2n-1)]k \\
&= 0. \qquad\qquad\qquad\qquad \square
\end{aligned}
$$

4 1-Phoenix numbers in base $b = n^2 - 1$

An interesting observation is that 1.1.2.2 is 1-phoenix in base 3, 1.1.3.3 is 1-phoenix in base 8, 1.1.4.4 is 1-phoenix in base 15, and so on. Thus, we have the following.

Theorem 23. *The number* 1.1.n.n *is 1-phoenix in base* $b = n^2 - 1$, $n \geq 2$.

Proof. By multiplication. Since $n^2 = (n^2 - 1) + 1$, $n^2 = 1.1$ in base $b = n^2 - 1$.

$n.$	1.	1
		n
1.	1.	$n.$ n

Base $b = n^2 - 1$, $n \geq 2$.

$\square$

Theorem 24 (1-Phoenix in base $b = n^2 - 1$). *The number* 1.1.n.n *is 1-phoenix in base* $b = n^2 - 1$, $n \geq 2$. *Furthermore, if* $c = n(n-1)$ *and* $B = b - 1$, *then the following is true.*

1. *Let* $Z_0, Z_1, \ldots, Z_k$ *be instances of* $[0]$ *and let* $A_1, \ldots, A_k$ *be instances of* $[c, c-1.[B].n-2.n-1]$. *Then*
$$1.1.\,[Z_k A_k \cdots Z_1 A_1 Z_0 A_1 Z_1 \cdots A_k Z_k]\,.n.n$$
is 1-phoenix in base b.

2. *Let* $Z_1, \ldots, Z_k$ *be instances of* $[0]$ *and let* $A_0, A_1, \ldots, A_k$ *be instances of* $[c, c-1.[B].n-2.n-1]$. *Then*
$$1.1.\,[A_k Z_k \cdots A_1 Z_1 A_0 Z_1 A_1 \cdots Z_k A_k]\,.n.n$$
is 1-phoenix in base b.

Proof. An exercise for the reader. $\square$

Another interesting observation is that 1.2.4.2 is 1-phoenix in base 6, 1.3.6.2 is 1-phoenix in base 9. Thus, we have the following.

Theorem 25 (1-Phoenix in base $b = k(n^2 - 1)$, I)**.** *The number $1.k.kn.n$ is 1-phoenix in base $b = k(n^2 - 1)$, $n \geq 2$ and $k \geq 1$.*

Proof. By multiplication. Since $kn^2 = k(n^2 - 1) + k$, $kn^2 = 1.k$ in base $b = k(n^2 - 1)$.

	$kn.$	$k.$	1
			n
1.	$k.$	$kn.$	n

Base $b = k(n^2 - 1)$, $n \geq 2$.

$\square$

If $n = 2$ in Theorem 25, we obtain the following.

Theorem 26 (1-Phoenix in base $b = 3k$)**.** *The number $1.k.2k.2$ is 1-phoenix in base $b = 3k$, $k \geq 1$. Furthermore, the following numbers are also 1-phoenix in base b.*

1. $1.k.[0].2k.2$

2. $1.k.[2k.2k - 1.k - 1.k].2k.2$

3. $1.k.[2k.2k - 1.[B].k - 1.k].2k.2$

4. $1.k.[\![0]\!].[2k.2k - 1.[B].k - 1.k].[\![0]\!].2k.2$

Proof. Proof of (4). By multiplication.

	⑦	⑥	⑤		④		③		②		①			
					(1)		(1)		(1)					
	$2k.$	$[\![0]\!].$	$[k.$		$k - 1.$		$[B].$		$2k - 1.$	$2k].$	$[\![0]\!].$		$k.$	1
														2
1.	$k.$	$[\![0]\!].$	$[2k.$	$2k - 1.$		$[B].$		$k - 1.$		$k].$	$[\![0]\!].$		$2k.$	2

Base $b = 3k$.

$$① = 2 \cdot 2k - k = 3k.$$
$$② = 2(2k - 1) + 1 - (k - 1) = 4k - 2 + 1 - k + 1 = 3k.$$
$$③ = 2(3k - 1) + 1 - (3k - 1) = 3k - 1 + 1 = 3k.$$
$$⑦ = ①.$$

$\square$

If $n = 3$ in Theorem 25, we obtain the following.

Theorem 27 (1-Phoenix in base $b = 8k$)**.** *The number $1.k.3k.3$ is 1-phoenix in base $b = 8k$. Furthermore, the following numbers are also 1-phoenix in base b.*

1. $1.k.[0].3k.3$

2. $1.k.[6k.6k-1.2k-1.2k].3k.3$

3. $1.k.[6k.6k-1.[B].2k-1.2k].3k.3$

4. $1.k.[\![0]\!].[6k.6k-1.[B].2k-1.2k].[\![0]\!].3k.3$

Proof. An exercise for the reader. □

Theorems 26 and 27 can be generalized to the following.

Theorem 28 (1-Phoenix in base $b = k(n^2 - 1)$, II)**.** *The number*

$$1.k.kn.n$$

is 1-phoenix in base kb, $b = n^2 - 1$, $n \geq 2$, and $k \geq 1$. Furthermore, if $c = n(n-1)$ and $B = kb - 1$, then the following is true.

1. $1.k.[0].kn.n$ *is 1-phoenix in base kb.*

2. $1.k.[kc.kc - 1.k(n-1) - 1.k(n-1)].kn.n$ *is 1-phoenix in base kb.*

3. $1.k.[kc.kc - 1.[B].k(n-1) - 1.k(n-1)].kn.n$ *is 1-phoenix in base kb.*

4. $1.k.[\![0]\!].[kc.kc - 1.[B].k(n-1) - 1.k(n-1)].[\![0]\!].kn.n$ *is 1-phoenix in base kb.*

5. Let $Z_0, Z_1, \ldots, Z_m$ be instances of $[0]$ and let $A_1, \ldots, A_m$ be instances of $[kc.kc-1.[kb-1].k(n-1)-1, k(n-1)]$. Then

$$1.k.\big[Z_m A_m \cdots Z_1 A_1 Z_0 A_1 Z_1 \cdots A_m Z_m\big].kn.n$$

is 1-phoenix in base kb.

6. Let $Z_1, \ldots, Z_m$ be instances of $[0]$ and let $A_0, A_1, \ldots, A_m$ be instances of $[kc.kc-1.[B].k(n-1)-1.k(n-1)]$. Then

$$1.k.\big[A_m Z_m \cdots A_1 Z_1 A_0 Z_1 A_1 \cdots Z_m A_m\big].kn.n$$

is 1-phoenix in base kb.

Proof. An exercise for the reader. □

Theorem 29 (1-Phoenix in base $b = k(n^2-1)$)**.** *Let $b = n^2 - 1$, $n \geq 3$, and let $c = n(n-1)$. Then the number*
$$1.(c+1)k.ck - 1.(n-1)k - 1.(2n-1)k.n$$
is 1-phoenix in base kb, $k \geq 1$. Furthermore,

1. *If* $B = kb - 1$, *then*

$$1.(c+1)k.ck - 1.[B].(n-1)k - 1.(2n-1)k.n$$

   is 1-phoenix in base kb.

2. *The number*

$$1.(c+1)k.ck - 1.\big[(n-1)k - 1.(n-1)k.[0].ck.ck - 1\big].(n-1)k - 1.(2n-1)k.n$$

   is 1-phoenix in base kb.

3. *Let* $Z_0, Z_1, \ldots, Z_m$ *be instances of* $[(n-1)k - 1.(n-1)k.[0].ck.ck - 1]$ *and let* $B_1, \ldots, B_m$ *be instances of* $[B]$, *then*

$$1.(c+1)k.ck - 1.\big[Z_m B_m \cdots Z_1 B_1 Z_0 B_1 Z_1 \cdots B_m Z_m\big].(n-1)k - 1.(2n-1)k.n$$

   is 1-phoenix in base kb.

4. *Let* $B_0, B_1, \ldots, B_m$ *be instances of* $[B]$ *let* $Z_1, \ldots, Z_m$ *be instances of* $[(n-1)k - 1.(n-1)k.[0].ck.ck - 1]$, *then*

$$1.(c+1)k.ck - 1.\big[B_m Z_m \cdots B_1 Z_1 B_0 Z_1 B_1 \cdots Z_m B_m\big].(n-1)k - 1.(2n-1)k.n$$

   is 1-phoenix in base kb.

Proof. By multiplication.

<div align="center">Theorem 29.</div>

			$(n-1)$	$(n-1)$		
$(2n-1)k.$	$(n-1)k - 1.$		$[B].$	$ck - 1.$	$(c+1)k.$	1
						n
1	$(c+1)k.$	$ck - 1.$	$[B].$	$(n-1)k - 1.$	$(2n-1)k.$	n

<div align="center">Base $b = k(n^2 - 1)$, $c = n(n-1)$, $n \geq 3$, $k \geq 1$.</div>

Thus,

$$
\begin{aligned}
n \cdot (c+1)k - (2n-1)k &= k(n(n^2 - n + 1) - 2n + 1) \\
&= k(n^3 - n^2 + n - 2n + 1) \\
&= k(n^3 - n^2 - n + 1) \\
&= (n-1) \cdot k(n^2 - 1). \\
n(ck - 1) + (n-1) - ((n-1)k - 1) &= nck - n + n - 1 - (n-1)k + 1 \\
&= nck - n + n - nk + k \\
&= nck - nk + k \\
&= k(n^2(n-1) - n + 1)
\end{aligned}
$$

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$$= (n-1) \cdot k(n^2 - 1).$$
$$n\text{B} + (n-1) - \text{B} = (n-1)(\text{B}+1) = (n-1) \cdot kb.$$
$$n[(n-1)k - 1] + (n-1) - (ck-1) = ck - n + n - 1 - ck + 1 = 0.$$
$$n(2n-1)k - (c+1)k = k(2n^2 - n - (n^2 - n + 1))$$
$$= k(2n^2 - n - n^2 + n - 1)$$
$$= k(n^2 - 1). \qquad \square$$

Theorem 30 (1-Phoenix in base $b = n^2 - 1$). *The number*

$$n^2 - n - 1.n^2 - 3.n^2 - 2.n^2 - n$$

is 1-phoenix in base $b = n^2 - 1$, $n \geq 2$.

Remark 31. The case $n = 2$ is 1.1.2.2 in base $b = 2^2 - 1 = 3$. See Theorems 24 and 25.

Proof. By multiplication.

<table>
<tr><td colspan="4" align="center">Theorem 30.</td></tr>
<tr><td></td><td>$(n^2 - n - 2)$</td><td>$(n^2 - 2n)$</td><td></td></tr>
<tr><td></td><td>$n^2 - 2.$</td><td>$n^2 - 3.$</td><td>$n^2 - n - 1$</td></tr>
<tr><td></td><td></td><td></td><td>$n^2 - n$</td></tr>
<tr><td>$n^2 - n - 1.$</td><td>$n^2 - 3.$</td><td>$n^2 - 2.$</td><td>$n^2 - n$</td></tr>
</table>

Base $b = n^2 - 1$, $n \geq 2$.

$$(n^2 - n)(n^2 - n - 1) - (n^2 - n) = (n^2 - n)(n^2 - n - 2)$$
$$= (n^2 - 2n)(n^2 - 1).$$
$$(n^2 - n)(n^2 - 3) + (n^2 - 2n) - (n^2 - 2) = (n^2 - n)(n^2 - 3) - (2n - 2)$$
$$= (n-1)(n^3 - 3n - 2)$$
$$= (n-1)(n+1)^2(n-2)$$
$$= (n^2 - n - 2)(n^2 - 1).$$
$$(n^2 - n)(n^2 - 2) + (n^2 - n - 2) - (n^2 - 3) = (n^2 - n)(n^2 - 2) - (n - 1)$$
$$= (n-1)(n^3 - 2n - 1)$$
$$= (n-1)(n+1)(n^2 - n - 1)$$
$$= (n^2 - n - 1)(n^2 - 1). \qquad \square$$

Remark 32. The sequences that occur are in Theorem 30 are

[A005563](#) $n(n+2)$ (or, $(n+1)^2 - 1$). Here: $n^2 - 1$.

[A002378](#) Oblong (or promic, pronic, or heteromecic) numbers: $a(n) = n(n+1)$. In our case, $n^2 - n$.

A165900 Values of Fibonacci polynomial $n^2 - n - 1$.

A008865 $a(n) = n^2 - 2$.

A028872 $a(n) = n^2 - 3$.

A067998 $a(n) = n^2 - 2n$.

A028552 $a(n) = n(n + 3)$. In our case, $n^2 - n - 2$.

Occasionally a sequence is a shift of another sequence. For example, $n(n + 2) \implies (n - 1)(n + 1) = n^2 - 1$. Similarly, $n(n + 3) \implies (n - 2)(n + 1) = n^2 - n - 2$.

5 Phoenix powers

A k-phoenix power is a k-phoenix number that is also a power. k-Phoenix powers with $k > 1$ are much rarer that 1-phoenix powers. Table 8 shows the only known k-phoenix powers with $k > 1$. Most 1-phoenix numbers are squares or cubes, and there are only a few that have

Table 8: k-Phoenix powers with $k > 1$.

b	k	n_b	m^e
7	2	2.4.2.4	30^2
7	2	1.0.2.4.1.4.4	352^2
13	4	5.7.7.10.10.9	1440^2
31	2	6.24.22.8	450^2

powers greater than 3. Table 9 shows the only 1-phoenix powers with powers greater than a cube. The number 1.3.8 in base 21 can be excluded since $1.3.8 = 8^3$ is covered by Theorem

Table 9: 1-Phoenix powers higher than a cube.

b	n_b	m^e
21	1.3.8	2^9
22	1.11.3	3^6
22	1.10.6.5	5^6
120	1.20.7	7^5
9653280	1.338.28561	3107^4

55. The number 1.10.6.5 in base 22 is the only known 1-phoenix power with more than three

digits in its base. From now we consider only 1-phoenix powers of the form $1.e.(d+1)$ in base $b = de$. Table 10 shows the 1-phoenix powers with powers authentically greater than 3, and are discussed in Subsection 5.3. Consult Table 11 for the 1-phoenix powers up to base

Table 10: 1-Phoenix powers authentically higher than a cube.

b	n_b	m^e
22	1.11.3	3^6
120	1.20.7	7^5
9653280	$1.2 \cdot 13^2.13^4$	$(13 \cdot 239)^4$

$b = 120$. Note that base 115 has both a square and a cube and base 120 has both a square and a fifth power. See Theorem 55 for a proof that base 115 is the only semiprime base with both a square and a cube among its factorizations.

Table 11: 1-Phoenix powers up to base $b = 120$

b	n_b	m^e
4	1.2.3	3^3
12	1.4.4	1.2^2
21	1.3.8	8^3
22	1.11.3	3^6
45	1.15.4	1.7^2
48	1.6.9	1.3^2
56	1.4.15	15^3
76	1.19.5	1.9^2
115	1.5.24	24^3
115	1.23.6	1.11^2
120	1.8.16	1.4^2
120	1.20.7	7^5

5.1 1-Phoenix squares

If $1.e.(d+1)$ is a square, it is of the form $(1.x)^2$, with $x^2 = cb + (d+1)$ and $e = 2x + c$. The simplest cases are $c = 0$ (Theorem 33) and $c = 1$ (Theorem 34). See Theorems 37 and 38 for consideration of $c > 1$.

Assume $1.e.d + 1 = (1.x)^2$ with carry $c = 0$, so that $(1.x)^2 = (b + x)^2 = b^2 + 2xb + x^2 = 1.2x.x^2$. Consequently, $d = x^2 - 1$, $e = 2x$, and $b = 2x(x^2 - 1)$.

Theorem 33 (1-Phoenix squares, I). *If $x \geq 2$, then the number $1.2x.x^2$ is 1-phoenix in base $b = 2x(x^2 - 1)$, and $1.2x.x^2 = (1.x)^2$. See Table 12.*

The sequence of bases $b = 2x(x^2 - 1)$ is not yet in the OEIS.

Assume $1.e.d + 1 = (1.x)^2$ with carry $c = 1$, so that $(1.x)^2 = (b + x)^2 = b^2 + (2x + 1)b + (d + 1)$. Consequently, $e = 2x + 1$, and

$$x^2 = b + (d + 1)$$
$$x^2 = d(2x + 1) + (d + 1)$$
$$x^2 - 1 = 2(x + 1)d$$
$$x - 1 = 2d \quad \Leftrightarrow \quad x = 2d + 1 \quad \Leftrightarrow \quad d = \frac{x - 1}{2}.$$

Note that x is necessarily odd and $e = 4d + 3$.

Theorem 34 (1-Phoenix squares, II). *1. If $d \geq 2$, then the number $1.4d + 3.d + 1$ is 1-phoenix in base $b = d(4d + 3)$, and*

$$1.4d + 3.d + 1 = (1.2d + 1)^2 = (d + 1)^2(4d + 1)^2.$$

2. Equivalently, given odd $x \geq 5$, let $d = (x - 1)/2$ and $e = 2x + 1$. Then $1.e.(d + 1)$ is 1-phoenix in base $b = de$ and

$$1.e.(d + 1) = (1.x)^2 = \left(\frac{x + 1}{2}\right)^2 (2x - 1)^2.$$

See Table 13.

Unsurprisingly, the sequence of bases in Theorem 34 is in the OEIS.

Sequence 35. *A033954 Second 10-gonal (or decagonal) numbers: $n(4n + 3)$.*

Sequence 36. *A002145 Primes of form $4n + 3$.*

Observe from Table 14 that x_b^2 is of the form $4n - 1.n$ so $d = n - 1$. Furthermore, the sequence $49, 97, 161, \ldots$ is A069129: $a(n) = 8n^2 - 8n + 1$. Therefore, $e = 2a(n) + (4n - 1) = 16n^2 - 12n + 1$ and $b = (n - 1)(16n^2 - 12n + 1)$. It can be shown that

$$1.16n^2 - 12n + 1.n = (n(16n^2 - 20n + 5))^2$$
$$= ((n - 1)(16n^2 - 12n + 1) + (8n^2 - 8n + 1))^2$$
$$= (1.(8n^2 - 8n + 1))^2.$$

Theorem 37. *The number $1.16n^2 - 12n + 1.n$ is 1-phoenix in base $b = (n-1)(16n^2 - 12n + 1)$, $n \geq 3$, and*

$$1.16n^2 - 12n + 1.n = (1.(8n^2 - 8n + 1))^2 = n^2(16n^2 + 20n + 5)^2.$$

See Table 14.

Table 12: 1-Phoenix squares of Theorem 33.

b	N	$1.2x.x^2$	$(1.x)^2$
12	196	1.4.4	1.2^2
48	2601	1.6.9	1.3^2
120	15376	1.8.16	1.4^2
240	60025	1.10.25	1.5^2
420	181476	1.12.36	1.6^2
672	461041	1.14.49	1.7^2
1008	1032256	1.16.64	1.8^2
1440	2099601	1.18.81	1.9^2
1980	3960100	1.20.100	1.10^2
2640	7027801	1.22.121	1.11^2
3432	11861136	1.24.144	1.12^2

Table 13: 1-Phoenix squares of Theorem 34.

$b = de$	N	$1.e.(d+1)$	$(1.2d+1)^2$
22	729	1.11.3	1.5^2
45	2704	1.15.4	1.7^2
76	7225	1.19.5	1.9^2
115	15876	1.23.6	1.11^2
162	30625	1.27.7	1.13^2
217	53824	1.31.8	1.15^2
280	88209	1.35.9	1.17^2
351	136900	1.39.10	1.19^2
430	203401	1.43.11	1.21^2
517	291600	1.47.12	1.23^2
612	405769	1.51.13	1.25^2

The remaining squares that do not conform to Theorems 34, 34, and 37 nevertheless always occur as solutions to a generalized Pell equation.

Theorem 38. *Given $d \geq 2$, there exist infinitely many e such that $1.e.d + 1$ is 1-phoenix and a square in base $b = de$. See Table 15.*

Proof. Clearly, $1.e.d + 1 = (de)^2 + e \cdot de + (d+1) = d(d+1)e^2 + (d+1)$ in base $b = de$ so we are led to consider solutions to the generalized Pell equation

$$x^2 - d(d+1)y^2 = d+1. \tag{1}$$

Table 14: 1-Phoenix squares of Theorem 37.

d	e	b	$1.e.(d+1)$	$(1.x)^2$	x_b^2
2	109	218	1.109.3	1.49^2	11.3
3	209	627	1.209.4	1.97^2	15.4
4	341	1364	1.341.5	1.161^2	19.5
5	505	2525	1.505.6	1.241^2	23.6
6	701	4206	1.701.7	1.337^2	27.7
7	929	6503	1.929.8	1.449^2	31.8
8	1189	9512	1.1189.9	1.577^2	35.9
9	1481	13329	1.1481.10	1.721^2	39.10
10	1805	18050	1.1805.11	1.881^2	43.11
11	2161	23771	1.2161.12	1.1057^2	47.12
12	2549	30588	1.2549.13	1.1249^2	51.13

It is necessary that D is not a square if 1 is to have a solution, and any number of the form $d(d+1)$ is never a power [5]. Furthermore, if it has one solution, it has infinitely many. Observe that $(x_0, y_0) = (d+1, 1)$ is in general a fundamental solution to (1). However, if $d + 1 = c^2$, then $(x_0, y_0) = (c, 0)$, $c > 0$, is also a fundamental solution. Further observe that $r = 2d + 1$ and $s = 2$ is a fundamental solution to $r^2 - d(d+1)s^2 = 1$. Thus, the solutions to (1) are recursively generated by

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} 2d+1 & 2d(d+1) \\ 2 & 2d+1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix}, \quad (k \geq 1). \tag{2a}$$

where

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} d+1 \\ 1 \end{bmatrix} \quad \text{(all } d\text{)}, \tag{2b}$$

and, additionally,

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} \quad \text{(if } d+1 = c^2\text{)}. \tag{2c}$$

See Table 17 for the first nontrivial solutions to (1) using (2). □

Table 16: Solutions to $x^2 - d(d+1)y^2 = d+1$, $2 \leq d \leq 15$.

▶ Solutions to $x^2 - 6y^2 = 3$.

$x:$	27, 267, 2643, 26163, 258987, 2563707
$y:$	11, 109, 1079, 10681, 105731, 1046629

► Solutions to $x^2 - 12y^2 = 4$.

$x:$	14, 52, 194, 724, 2702, 10084
$y:$	4, 15, 56, 209, 780, 2911

► Solutions to $x^2 - 20y^2 = 5$.

$x:$	85, 1525, 27365, 491045, 8811445, 158114965
$y:$	19, 341, 6119, 109801, 1970299, 35355581

► Solutions to $x^2 - 30y^2 = 6$.

$x:$	126, 2766, 60726, 1333206, 29269806, 642602526
$y:$	23, 505, 11087, 243409, 5343911, 117322633

► Solutions to $x^2 - 42y^2 = 7$.

$x:$	175, 4543, 117943, 3061975, 79493407, 2063766607
$y:$	27, 701, 18199, 472473, 12266099, 318446101

► Solutions to $x^2 - 56y^2 = 8$.

$x:$	232, 6952, 208328, 6242888, 187078312, 5606106472
$y:$	31, 929, 27839, 834241, 24999391, 749147489

► Solutions to $x^2 - 72y^2 = 9$.

$x:$	51, 297, 1731, 10089, 58803, 342729
$y:$	6, 35, 204, 1189, 6930, 40391

► Solutions to $x^2 - 90y^2 = 10$.

$x:$	370, 14050, 533530, 20260090, 769349890, 29215035730
$y:$	39, 1481, 56239, 2135601, 81096599, 3079535161

► Solutions to $x^2 - 110y^2 = 11$.

$x:$	451, 18931, 794651, 33356411, 1400174611, 58773977251
$y:$	43, 1805, 75767, 3180409, 133501411, 5603878853

► Solutions to $x^2 - 132y^2 = 12$.

$x:$	540, 24828, 1141548, 52486380, 2413231932, 110956182492
$y:$	47, 2161, 99359, 4568353, 210044879, 9657496081

► Solutions to $x^2 - 156y^2 = 13$.

$x:$	637, 31837, 1591213, 79528813, 3974849437, 198662943037
$y:$	51, 2549, 127399, 6367401, 318242651, 15905765149

► Solutions to $x^2 - 182y^2 = 14$.

$x:$	742, 40054, 2162174, 116717342, 6300574294, 340114294534
$y:$	55, 2969, 160271, 8651665, 467029639, 25210948841

► Solutions to $x^2 - 210y^2 = 15$.

$x:$	855, 49575, 2874495, 166671135, 9664051335, 560348306295
$y:$	59, 3421, 198359, 11501401, 666882899, 38667706741

► Solutions to $x^2 - 240y^2 = 16$.

$x:$	124, 976, 7684, 60496, 476284, 3749776
$y:$	8, 63, 496, 3905, 30744, 242047

End of Table 16.

Sequence 39 (not in OEIS). *First nontrivial x solution to $x^2 - n(n+1)y^2 = n+1$, $n \geq 2$.*

$$27, 14, 85, 126, 175, 232, 51, 370, 451, 540, 637, \ldots$$

Sequence 40 (not in OEIS). *First nontrivial y solution to $x^2 - n(n+1)y^2 = n+1$, $n \geq 2$.*

$$11, 4, 19, 23, 27, 31, 6, 39, 43, 47, 51, \ldots$$

Here's the empirical data of squares of the form $1.e.(d+1) = (1.x)^2$ with $2 \leq d \leq 16$. The first four terms of each sequence $1.e_i.(d+1)$ are given when $d+1$ is not a square, and the first six terms when $d+1$ is a square.

b	n	n_b	$(1,x)^2$	$(x^2)_b$
218	71289	1.109.3	$(1.49)^2$	11.3
2158	6985449	1.1079.3	$(1.485)^2$	109.3
21362	684502569	1.10681.3	$(1.4801)^2$	1079.3
211462	67074266169	1.105731.3	$(1.47525)^2$	10681.3
12	196	1.4.4	$(1.2)^2$	4
45	2704	1.15.4	$(1.7)^2$	1.4
168	37636	1.56.4	$(1.26)^2$	4.4
627	524176	1.209.4	$(1.97)^2$	15.4
2340	7300804	1.780.4	$(1.362)^2$	56.4
8733	101687056	1.2911.4	$(1.1351)^2$	209.4
76	7225	1.19.5	$(1.9)^2$	1.5
1364	2325625	1.341.5	$(1.161)^2$	19.5
24476	748843225	1.6119.5	$(1.2889)^2$	341.5
439204	241125192025	1.109801.5	$(1.51841)^2$	6119.5
115	15876	1.23.6	$(1.11)^2$	1.6
2525	7650756	1.505.6	$(1.241)^2$	23.6
55435	3687647076	1.11087.6	$(1.5291)^2$	505.6
1217045	1777438238436	1.243409.6	$(1.116161)^2$	11087.6
162	30625	1.27.7	$(1.13)^2$	1.7
4206	20638849	1.701.7	$(1.337)^2$	27.7
109194	13910551249	1.18199.7	$(1.8749)^2$	701.7
2834838	9375690900625	1.472473.7	$(1.227137)^2$	18199.7
217	53824	1.31.8	$(1.15)^2$	1.8
6503	48330304	1.929.8	$(1.449)^2$	31.8

194873	43400555584	1.27839.8	$(1.13455)^2$	929.8
5839687	38973650580544	1.834241.8	$(1.403201)^2$	27839.8
48	2601	1.6.9	$(1.3)^2$	9
280	88209	1.35.9	$(1.17)^2$	1.9
1632	2996361	1.204.9	$(1.99)^2$	6.9
9512	101787921	1.1189.9	$(1.577)^2$	35.9
55440	3457792809	1.6930.9	$(1.3363)^2$	204.9
323128	117463167441	1.40391.9	$(1.19601)^2$	1189.9
351	136900	1.39.10	$(1.19)^2$	1.10
13329	197402500	1.1481.10	$(1.721)^2$	39.10
506151	284654260900	1.56239.10	$(1.27379)^2$	1481.10
19220409	410471246808100	1.2135601.10	$(1.1039681)^2$	56239.10
430	203401	1.43.11	$(1.21)^2$	1.11
18050	358382761	1.1805.11	$(1.881)^2$	43.11
757670	631470211801	1.75767.11	$(1.36981)^2$	1805.11
31804090	1112650154800921	1.3180409.11	$(1.1552321)^2$	75767.11
517	291600	1.47.12	$(1.23)^2$	1.12
23771	616429584	1.2161.12	$(1.1057)^2$	47.12
1092949	1303131836304	1.99359.12	$(1.48599)^2$	2161.12
50251883	2754820085504400	1.4568353.12	$(1.2234497)^2$	99359.12
612	405769	1.51.13	$(1.25)^2$	1.13
30588	1013594569	1.2549.13	$(1.1249)^2$	51.13
1528788	2531958811369	1.127399.13	$(1.62425)^2$	2549.13
76408812	6324832097188969	1.6367401.13	$(1.3120001)^2$	127399.13
715	550564	1.55.14	$(1.27)^2$	1.14
38597	1604322916	1.2969.14	$(1.1457)^2$	55.14
2083523	4674996406276	1.160271.14	$(1.78651)^2$	2969.14
112471645	13622937923544964	1.8651665.14	$(1.4245697)^2$	160271.14
826	731025	1.59.15	$(1.29)^2$	1.15
47894	2457680625	1.3421.15	$(1.1681)^2$	59.15
2777026	8262721505025	1.198359.15	$(1.97469)^2$	3421.15
161019614	27779267242188225	1.11501401.15	$(1.5651521)^2$	198359.15
120	15376	1.8.16	$(1.4)^2$	16
945	952576	1.63.16	$(1.31)^2$	1.16
7440	59043856	1.496.16	$(1.244)^2$	8.16
58575	3659766016	1.3905.16	$(1.1921)^2$	63.16
461160	226846448656	1.30744.16	$(1.15124)^2$	496.16
3630705	14060820050176	1.242047.16	$(1.119071)^2$	3905.16
1072	1221025	1.67.17	$(1.33)^2$	1.17
70736	5316305569	1.4421.17	$(1.2177)^2$	67.17
4667504	23147193189409	1.291719.17	$(1.143649)^2$	4421.17

307984528	100782873830344225	1.19249033.17	$(1.9478657)^2$	291719.17

5.2   1-Phoenix cubes

It is interesting to note that the most 1-phoenix cubes are of the form $1.e.e^2 - 1 = (e^2 - 1)^3$ for $e > 1$. In fact, we have

Theorem 41. *If $e \geq 2$, then $1.e.(e^2 - 1)$ is 1-phoenix in base $b = e(e^2 - 2)$ and*

$$1.e.(e^2 - 1) = (e^2 - 1)^3.$$

Proof. Since $e(e^2 - 1) = e(e^2 - 2) + e = 1.e$ in base $b = e(e^2 - 2)$, $1.e.(e^2 - 1)$ is 1-phoenix in base b. It is left as an exercise for the reader to verify that $1.e.(e^2 - 1) = (e^2 - 1)^3$. $\square$

The only exceptions to Theorem 41 out to base $b = 10^6$ are in Table 20.

Since $e^2 - 1 = (e - 1)(e + 1)$, it is interesting to consider when $e \pm 1$ have special properties.

Corollary 42 (Twin cubes). *Let e be a number such that $e - 1$ and $e + 1$ are prime powers, that is, let $e - 1 = p^u$ and $e + 1 = q^v$. then, with $d = e^2 - 2$, the number $1.e.(e^2 - 1)$ is 1-phoenix in base $b = de$ and $1.e.(e^2 - 1) = p^{3u}q^{3v}$. In particular, if p and q are twin primes, then $1.(p + q)/2.(pq) = (pq)^3$. See Tables 21, 22, and 23.*

1.*A000027*.*A005563* $= $ *A005563*^{3}, *A005563* $= $ *A000027*$^2 - 1$.
1.*A014574*.(*A001359* $\cdot$ *A006512*) $= $ (*A001359* $\cdot$ *A006512*)^3

Sequence 43 (*A074852*). *Composite n (*A002808*) such that n and $n + 2$ are prime powers (*A000961*).*

Sequence 44 (*A120432*). *Numbers n such that $n - 1$ and $n + 1$ are prime powers.*

Sequence 45 (*A076445*). *The smaller of a pair of powerful numbers (*A001694*) that differ by 2.*

Sequence 46 (not in OEIS). *Numbers n such that $n - 1$ and $n + 1$ are powerful numbers (*A001694*).*

$$26, 70226, 130576328, \ldots$$

Sequence 47 (not in OEIS). *Numbers n such that $n^2 - 1$ is powerful.*

Sequence 48 (not in OEIS). *Numbers n such that $n(n + 2)$ is powerful.*

Table 15: 1-Phoenix squares of Theorem 38.

$b = de$	$1.e.(d+1)$	squares	e	OEIS
$2e$	$1.e.3$	$6e^2 + 3$	$11, 109, 1079, \ldots$	A054320
$3e$	$1.e.4$	$12e^2 + 4$	$15, 209, 2911, \ldots$	A028230 (submit correction?)
$4e$	$1.e.5$	$20e^2 + 5$	$19, 341, 6119, \ldots$	A049629
$5e$	$1.e.6$	$30e^2 + 6$	$23, 505, 11087, \ldots$	A133283
$6e$	$1.e.7$	$42e^2 + 7$	$27, 701, 18199, \ldots$	A157461
$7e$	$1.e.8$	$56e^2 + 8$	$31, 929, 27839, \ldots$	A157878
$8e$	$1.e.9$	$72e^2 + 9$	$6, 35, 204, \ldots$	A001109
$9e$	$1.e.10$	$90e^2 + 10$	$39, 1481, 56239, \ldots$	A097314
$10e$	$1.e.11$	$110e^2 + 11$	$43, 1805, 75767, \ldots$	(not in OEIS)
$11e$	$1.e.12$	$132e^2 + 12$	$47, 2161, 99359, \ldots$	A189173
$12e$	$1.e.13$	$156e^2 + 13$	$51, 2549, 127399, \ldots$	(not in OEIS)

It is interesting to note that A189173, $d = 11$, is the sequence of integers m such that m^3 is the sum of squares of m consecutive integers.

Table 17: First nontrivial solutions to $x^2 - d(d+1)y^2 = d+1$.

d	y	x	x_b
2	11	27	1.5
3	4	14	1.2
4	19	85	1.9
5	23	126	1.11
6	27	175	1.13
7	31	232	1.15
8	6	51	1.3
9	39	370	1.19
10	43	451	1.21
11	47	540	1.23
12	51	637	1.25
13	55	742	1.27
14	59	855	1.29
15	8	124	1.4

Note that, in base $b = dy$, $x = 1.2d + 1$ unless $d + 1 = c^2$, and then $x = 1.c$.

Table 19: 1-Phoenix cubes of Theorem 41.

$b = de$	$1.e.(e^2 - 1)$	$(e^2 - 1)^3$
4	1.2.3	3^3
21	1.3.8	8^3
56	1.4.15	15^3
115	1.5.24	24^3
204	1.6.35	35^3
329	1.7.48	48^3
496	1.8.63	63^3
711	1.9.80	80^3
980	1.10.99	99^3
1309	1.11.120	120^3
1704	1.12.143	143^3

Table 20: Exceptions to Theorem 41.

b	N_b	N
273	1.39.8	44^3
819	1.13.64	88^3
28236	1.1086.27	939^3

5.3 Higher 1-phoenix powers

Sequence 49 (A001597). *Perfect powers: m^k where $m > 0$ and $k \geq 2$.*

It is interesting to note that $1.20.7 = 7^5$ in base $b = 120$. Let us consider $1.e.(d + 1) = (d + 1)^n$ for $n \geq 3$. Thus,

$$(ed)^2 + e(ed) + (d + 1) = (d + 1)^n$$
$$e^2 d(d + 1) = (d + 1)^n - (d + 1)$$
$$e^2 = \frac{(d + 1)^{n-1} - 1}{d}$$

Define

$$S_n(d) = \frac{(d + 1)^{n-1} - 1}{d}, \quad d \geq 2, n \geq 3. \tag{3}$$

Theorem 50. *If $S_n(d) = e^2$, then $1.e.(d + 1) = (d + 1)^n$ in base $b = de$. In particular, if $n = 3$, then $S_3(x^2 - 2) = x^2$ so $1.x.(x^2 - 1) = (x^2 - 1)^3$ in base $b = x(x^2 - 2)$.*

Table 21: 1-Phoenix cubes of Corollary 42 with $e \pm 1$ twin primes.

b	N	N_b	$((e-1) \cdot (e+1))^3$
56	3375	1.4.15	$(3 \cdot 5)^3$
204	42875	1.6.35	$(5 \cdot 7)^3$
1704	2924207	1.12.143	$(11 \cdot 13)^3$
5796	33698267	1.18.323	$(17 \cdot 19)^3$
26940	726572699	1.30.899	$(29 \cdot 31)^3$
74004	5479701947	1.42.1763	$(41 \cdot 43)^3$
215880	46617130799	1.60.3599	$(59 \cdot 61)^3$
373104	139233463487	1.72.5183	$(71 \cdot 73)^3$
1061004	1125837720827	1.102.10403	$(101 \cdot 103)^3$
1259496	1586466211247	1.108.11663	$(107 \cdot 109)^3$
2627796	6905674472507	1.138.19043	$(137 \cdot 139)^3$
3374700	11389106317499	1.150.22499	$(149 \cdot 151)^3$

Table 22: 1-Phoenix cubes of Corollary 42 with $u > 1$ or $v > 1$.

b	N	N_b	$(p^u \cdot q^v)^3$
21	512	1.3.8	$(2 \cdot 2^2)^3$
496	250047	1.8.63	$(7 \cdot 3^2)^3$
980	970299	1.10.99	$(3^2 \cdot 11)^3$
13776	190109375	1.24.575	$(23 \cdot 5^2)^3$
17524	307546875	1.26.675	$(5^2 \cdot 3^3)^3$
21896	480048687	1.28.783	$(3^3 \cdot 29)^3$
110496	12214672127	1.48.2303	$(47 \cdot 7^2)^3$
511840	262021139199	1.80.6399	$(79 \cdot 3^4)^3$
551204	303871055067	1.82.6723	$(3^4 \cdot 83)^3$
2000124	4000748046875	1.126.15875	$(5^3 \cdot 127)^3$
4741296	22480684325567	1.168.28223	$(167 \cdot 13^2)^3$
14172004	200849127059547	1.242.58563	$(241 \cdot 3^5)^3$

See Table 25 for exceptions to Theorem 50.

Sequence 51 ([A008865](#)). $n^2 - 2$.

Sequence 52 ([A242135](#)). $n^3 - 2n$.

Sequence 53 ([A028871](#)). *Primes of the form $n^2 - 2$.*

		$((e-1)\cdot(e+1))^3$
b	17524	
N	307546875	
N_b	1.26.675	$(3^3 \cdot 5^2)^3$
b	346332937362724	
N	1199465035266140824765110 46875	
N_b	1.70226.4931691075	$(5^2\, 53^2 \cdot 3^5\, 17^2)^3$
b	2226349561075427023286896	
N	4956632368100746852078875138788242734414329476287	
N_b	1.130576328.17050177433963583	$(7^3\, 617^2 \cdot 3^2\, 13^2\, 293^2)^3$

A computer search out to $b = 10^7$ reveals that the only powers higher than a cube and not considered a lower power are given in Table 24. See Theorem 54 for why these are the only powers.

Table 24: 1-Phoenix powers higher than a cube.

b	N_b	N
22	1.11.3	3^6
120	1.20.7	7^5
9653280	1.338.28561	3107^4

Theorem 54. *1. The only 1-phoenix number of the form $1.2x.x^2 = (1.x)^2$ in base $b = 2x(x^2 - 1)$ that is a higher power than a square is when $x = 13^2$, so that*

$$1.2(13^2).(13^4) = (1.13^2)^2 = (13 \cdot 239)^4$$

in base $b = 2 \cdot 13^2(13^4 - 1) = 9653280$.

2. The only powers higher than a cube that arise from Theorem 50 are $1.11.3 = 3^6$ in base $b = 22$ and $1.20.7 = 7^5$ in base $b = 120$.

See Table 24.

Proof. Proof of 1. If $1.2x.x^2 = (1.x)^2$ is to be a power higher than a square, then $1.x$ in base $b = 2x(x^2 - 1)$ must itself be a power. Thus,

$$1.x = 2x(x^2 - 1) + x$$

$$= 2x^3 - x$$
$$= x(2x^2 - 1).$$

Suppose $x(2x^2 - 1)$ is a power. Since x and $2x^2 - 1$ are relatively prime, assume that $x = y^p$ and $2x^2 - 1 = z^p$ so that we have the Diophantine equation $z^p = 2y^{2p} - 1$. In [9] it was shown that the equation $z^n = 2y^m - 1$ with $m, n > 1$ has solutions only for $m = n = 2$ and $n = 2$, $m = 4$, and that $239^2 = 2 \cdot 13^4 - 1$ is the only solution in the latter case. Thus, $1.338.28561 = (13 \cdot 239)^4$ is the only higher power of the form $1.2x.x^2$ in base $b = 2x(x^2 - 1)$.

Proof of 2. Observe that $S_n(d) = e^2$ of Theorem 50 can be written in the form

$$\frac{(d+1)^{n-1} - 1}{(d+1) - 1} = e^2,$$

so one is naturally lead to consider the equation

$$\frac{x^n - 1}{x - 1} = y^q, \quad |x| > 1, |y| > 1, n > 2, q \geq 2, \tag{4}$$

called the *Nagell-Ljunggren equation.* It is conjectured that

$$\frac{3^5 - 1}{3 - 1} = 11^2, \quad \frac{7^4 - 1}{7 - 1} = 20^2, \quad \frac{18^3 - 1}{18 - 1} = 7^3, \quad \frac{(-19)^3 - 1}{(-19) - 1} = 7^3. \tag{5}$$

are the only solutions. However, Ljunggren [10] shows that (4) with $q = 2$, that is,

$$\frac{x^n - 1}{x - 1} = y^2, \quad |x| > 1, |y| > 1, n > 2, \tag{6}$$

has only the solutions

$$\frac{3^5 - 1}{3 - 1} = 11^2, \quad \frac{7^4 - 1}{7 - 1} = 20^2. \tag{7}$$

Thus, $1.11.3 = 3^6$ in base $b = 22$ and $1.20.7 = 7^5$ in base $b = 120$. □

Recall from Table 11 that base $b = 115$ has $1.23.6 = 126^2$ and $1.5.24 = 24^3$ as 1-phoenix powers. In fact, it is the only semiprime with this property.

Theorem 55. *The only semiprime base $b = de$ where $1.e.(d+1)$ is a square and $1.d.(e+1)$ is a cube is $b = 5 \cdot 23 = 115$.*

Proof. The condition for a square is $e = 4d + 3$ and the condition for a cube is $e = d^2 - 2$. Thus,

$$d^2 - 2 = 4d + 3,$$
$$d^2 - 4d - 5 = 0,$$
$$(d - 1)(d - 5) = 0,$$

and so $d = 5$ and $e = 23$. Thus, $1.23.6 = 1.11^2 = 126^2$ and $1.5.24 = 24^3$ in base $b = 115$. □

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Table 25: Exceptions to Theorem 50.

b	x	$1.e.(d+1)$	m^n	$S_n(d)$
273	85184	1.39.8	44^3	$S_3(7) = 7 + 2 = 9 \implies e = 3$
819	681472	1.13.64	88^3	$S_3(63) = 65 = 5 \cdot 13$
28236	827936019	1.1086.27	939^3	$S_3(26) = 28 = 2^2 \cdot 7$
1488136	2214792817875	1.164.9075	13035^3	$S_3(9074) = 9076 = 2^2 \cdot 2269$
2342950	5489965305721	1.235.9971	17641^3	$S_3(9970) = 9972 = 2^2 \cdot 3^2 \cdot 277$
9653280	93189077595601	1.338.28561	3107^4	$S_4(28560) = \dfrac{28561^3 - 1}{28560}$
				$= 815759283 = 3 \cdot 61 \cdot 157 \cdot 28393$

This proof only works because $65 = 5 \cdot 13$ is semiprime. The same proof will not work for 120 since the square $1.8.16 = 1.4^2$ and fifth power $1.20.7 = 7^5$ come from distinct factorizations. See Table 26 for all bases out to $b = 10^7$ with more than one 1-phoenix power.

Table 26: Bases with more than one 1-phoenix power.

b	N	N_b	m_b^p
115	13824	1.5.24	24^3
115	15876	1.23.6	1.11^2
120	15376	1.8.16	1.4^2
120	16807	1.20.7	7^5
332640	110685963025	1.110.3025	1.55^2
332640	111033569089	1.1155.289	1.577^2

6 Arithmetic progressions of 3-digit 1-phoenix numbers

An arithmetic progression of 1-phoenix numbers of the first kind has the form $1.m + b_1k.n$ in base $b = m(n-1) + b_0k$. An arithmetic progression of 1-phoenix numbers of the second kind has the form $1 + b_3k.m + b_2k.n + b_1k$ in base $b = m(n-1) + b_0k$.

Theorem 56 (Arithmetic extension of the first kind)**.** *The number* $1.m + b_1k.n$ *is 1-phoenix in base* $b = m(n-1) + b_0k$ *if and only if* $b_0 = n^2 - 1$ *and* $b_1 = n+1$*. Summarizing,*

$$1.m + (n+1)k.n$$

is 1-phoenix in base $b = m(n-1) + (n^2-1)k$*.*

6.1 Arithmetic progressions of the first kind

Theorem 57. *The 1-phoenix number* $1.x.y$ *in base* $b = x(y-1)$ *can be extended to an arithmetic progression of the form* $1.x + (y+1)k.y$ *with* $b = x(y-1) + (y^2-1)k$*.*

Proof. An exercise for the reader. $\qquad\square$

An interesting phenomenon is that there may exist a finite sequence P of arithmetic progressions such that $1.[P].x + (y+1)k.y$ is 1-phoenix in base $b = x(y-1) + (y^2-1)k$.

Let us first consider the case $1.[p].x.y$, where p is a digit. Then the multiplication is

(c)			
	x	p	1
			y
1	p	x	y

The equations are

$$yp - x = cb,$$
$$yp + c - p = cb,$$
$$yx + c - p = b.$$

and have the solution

$$c = \frac{x(y-1)}{xy - x - y}, \quad p = \frac{x(xy - x - 1)}{xy - x - y}, \quad b = x(y-1). \tag{8}$$

It is true that $x = p - c$. If $xy - x - y = 1$, then $c = b$ so this case is excluded. If $xy - x - y = x$, then $y = 2x/(x-1)$ so and either $x = 2$ or $x = 3$. Then $y = 4$ and $y = 3$, respectively. In either case, $b = 2(4-1) = 6$ and $b = 3(3-1) = 6$. Thus, $1.[5].3.3$ and $1.[5].2.4$ in base 6.

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6.1.1 Arithmetic extension of the first kind with 3-digit repetitions

The arithmetic extension of $1.m + (n+1)k, n$ in base $b = m(n-1) + (n^2-1)k$ can itself have an arithmetic extension of the form

	(c)	(1)			
$A_1.$	$[A_3.$	$A_2.$	$A_1].$	1	
					n
1	$[A_1.$	A_3	$A_2].$	A_1	n

Table 27: Arithmetic extensions of the first kind with 3-digit repetitions.

b	N_b	c
$4 + 8k$	$1.[2 + 4k.1 + 3k.0 + 1k].2 + 4k.3$	1
$12 + 15k$	$1.[4 + 5k.6 + 8k.1 + 2k].4 + 5k.4$	2
$15 + 35k$	$1.[3 + 7k.10 + 24k.1 + 4k].3 + 7k.6$	3
$32 + 80k$	$1.[4 + 10k.25 + 63k.2 + 7k].4 + 10k.9$	4
$36 + 48k$	$1.[6 + 8k.26 + 35k.3 + 5k].6 + 8k.7$	
$55 + 143k$	$1.[5 + 13k.46 + 120k.3 + 10k].5 + 13k.12$	
$72 + 99k$	$1.[8 + 11k.58 + 80k.5 + 8k].8 + 11k.10$	
$84 + 224k$	$1.[6 + 16k.73 + 195k.4 + 13k].6 + 16k.15$	
$119 + 323k$	$1.[7 + 19k.106 + 288k.5 + 16k].7 + 19k.18$	
$120 + 168k$	$1.[10 + 14k.102 + 143k.7 + 11k].10 + 14k.13$	
$160 + 440k$	$1.[8 + 22k.145 + 399k.6 + 19k].8 + 22k.21$	
$180 + 255k$	$1.[12 + 17k.158 + 224k.9 + 14k].12 + 17k.16$	
$207 + 575k$	$1.[9 + 25k.190 + 528k.7 + 22k].9 + 25k.24$	
$252 + 360k$	$1.[14 + 20k.226 + 323k.11 + 17k].14 + 20k.19$	
$260 + 728k$	$1.[10 + 28k.241 + 675k.8 + 25k].10 + 28k.27$	

It appears that $n = 5 + 3j$ does not occur. Thus, let's consider the cases $n = 3 + 3j$ and $n = 4 + 3j$ separately.

Case $n = 3 + 3j$.

b	N_b	c
$4 + 8k$	$1.[2 + 4k.1 + 3k.0 + 1k].2 + 4k.3$	1
$15 + 35k$	$1.[3 + 7k.10 + 24k.1 + 4k].3 + 7k.6$	4
$32 + 80k$	$1.[4 + 10k.25 + 63k.2 + 7k].4 + 10k.9$	7

The sequences are adjusted to start at $n = 1$.
The sequence of a_0 is $4, 15, 32, \ldots$ is $a(n) = 3n^2 + 2n - 1 = (3n - 1)(n + 1)$.
The sequence of b_0 is $8, 35, 80, \ldots$ is $a(n) = 9n^2 - 1$.

The sequence of a_1 is $2, 3, 4, \ldots$ is $a(n) = n + 1$.
The sequence of b_1 is $4, 7, 10, \ldots$ is $a(n) = 3n + 1$.
The sequence of a_2 is $0, 1, 2, \ldots$ is $a(n) = n - 1$.
The sequence of b_2 is $1, 4, 7, \ldots$ is $a(n) = 3n - 2$.
The sequence of a_3 is $1, 10, 25, \ldots$ is $a(n) = 3n^2 - 2$.
The sequence of b_3 is $3, 24, 63, \ldots$ is $a(n) = 9n^2 - 6n$.

Theorem 58. *If $n \geq 1$, the number $1.[A_1.A_3.A_2].A_1.3n$ is 1-phoenix in base b, where*

$$
\begin{aligned}
b &= (3n^2 + 2n - 1) + (9n^2 - 1)k, \\
A_1 &= (n + 1) + (3n + 1)k, \\
A_2 &= (n - 1) + (3n - 2)k, \\
A_3 &= (3n^2 - 2) + (9n^2 - 6n)k, \\
c &= 3n - 2.
\end{aligned}
$$

It appears that Theorem $6(2)$ does not generalize to 58.
Case $n = 4 + 3j$.

$$
\begin{array}{lll}
12 + 15k & 1.[4 + 5k.6 + 8k.1 + 2k].4 + 5k.4 & 2 \\
36 + 48k & 1.[6 + 8k.26 + 35k.3 + 5k].6 + 8k.7 & 5 \\
72 + 99k & 1.[8 + 11k.58 + 80k.5 + 8k].8 + 11k.10 & 8
\end{array}
$$

The sequence of a_0 is $12, 36, 72, \ldots$ is $a(n) = 6n^2 + 6n$.
The sequence of b_0 is $15, 48, 99, \ldots$ is $a(n) = 9n^2 + 6n$.
The sequence of a_1 is $4, 6, 8, \ldots$ is $a(n) = 2n + 2$.
The sequence of b_1 is $5, 8, 11, \ldots$ is $a(n) = 3n + 2$.
The sequence of a_2 is $1, 3, 5, \ldots$ is $a(n) = 2n - 1$.
The sequence of b_2 is $2, 5, 8, \ldots$ is $a(n) = 3n - 1$.
The sequence of a_3 is $6, 26, 58, \ldots$ is $a(n) = 6n^2 + 2n - 2$.
The sequence of b_3 is $8, 35, 80, \ldots$ is $a(n) = 9n^2 - 1$.

Theorem 59. *If $n \geq 1$, the number $1.[A_1.A_3.A_2].A_1.3n + 1$ is 1-phoenix in base b, where*

$$
\begin{aligned}
b &= (6n^2 + 6n) + (9n^2 + 6n)k, \\
A_1 &= (2n + 2) + (3n + 2)k, \\
A_2 &= (2n - 1) + (3n - 1)k, \\
A_3 &= (6n^2 + 2n - 2) + (9n^2 - 1)k, \\
c &= 3n - 1.
\end{aligned}
$$

6.1.2 Arithmetic extensions of the first kind with 4-digit repetitions

$$a_3 = \frac{a_1{}^2 c_2\, n^2 - 2\, a_1{}^2 c_2\, n + a_1{}^2 c_2 - a_2\, n^2 - c_2\, n + a_2 - 1}{a_1\, (n-1)^2},$$

$$a_4 = na_2 + c_2,$$

$$b_2 = \frac{a_2\, n^2 + c_2\, n - a_2 + 1}{a_1\, (n-1)}$$

$$b_3 = c_2\, n + c_2,$$

$$b_4 = \frac{n\, (a_2\, n^2 + c_2\, n - a_2 + 1)}{a_1\, (n-1)},$$

$$c_1 = \frac{a_2\, n^2 + c_2\, n - a_2 + 1}{a_1\, (n-1)},$$

$$c_3 = 0,$$

$$d_1 = 0,$$

$$d_2 = 0,$$

$$d_3 = 0.$$

Table 28: Arithmetic extensions of the first kind with 4-digit repetitions.

$6 + 8k$	$1.[3 + 4k.4 + 6k.2 + 4k.1 + 2k].3 + 4k.3$	2	1
$10 + 8k$	$1.[5 + 4k.7 + 6k.4 + 4k.2 + 2k].5 + 4k.3$	2	1
$14 + 8k$	$1.[7 + 4k.10 + 6k.6 + 4k.3 + 2k].7 + 4k.3$	2	1
$18 + 8k$	$1.[9 + 4k.13 + 6k.8 + 4k.4 + 2k].9 + 4k.3$	2	1
$18 + 15k$	$1.[6 + 5k.14 + 12k.11 + 10k.3 + 3k].6 + 5k.4$	3	2
$33 + 15k$	$1.[11 + 5k.26 + 12k.21 + 10k.6 + 3k].11 + 5k.4$	3	2
$48 + 15k$	$1.[16 + 5k.38 + 12k.31 + 10k.9 + 3k].16 + 5k.4$	3	2
$63 + 15k$	$1.[21 + 5k.50 + 12k.41 + 10k.12 + 3k].21 + 5k.4$	3	2
$16 + 24k$	$1.[4 + 6k.13 + 20k.11 + 18k.2 + 4k].4 + 6k.5$	4	3
$28 + 24k$	$1.[7 + 6k.23 + 20k.20 + 18k.4 + 4k].7 + 6k.5$	4	3
$40 + 24k$	$1.[10 + 6k.33 + 20k.29 + 18k.6 + 4k].10 + 6k.5$	4	3
$52 + 24k$	$1.[13 + 6k.43 + 20k.38 + 18k.8 + 4k].13 + 6k.5$	4	3
$40 + 35k$	$1.[8 + 7k.34 + 30k.31 + 28k.5 + 5k].8 + 7k.6$	5	4
$75 + 35k$	$1.[15 + 7k.64 + 30k.59 + 28k.10 + 5k].15 + 7k.6$	5	4
$110 + 35k$	$1.[22 + 7k.94 + 30k.87 + 28k.15 + 5k].22 + 7k.6$	5	4
$145 + 35k$	$1.[29 + 7k.124 + 30k.115 + 28k.20 + 5k].29 + 7k.6$	5	4
$30 + 48k$	$1.[5 + 8k.26 + 42k.24 + 40k.3 + 6k].5 + 8k.7$	6	5
$54 + 48k$	$1.[9 + 8k.47 + 42k.44 + 40k.6 + 6k].9 + 8k.7$	6	5
$78 + 48k$	$1.[13 + 8k.68 + 42k.64 + 40k.9 + 6k].13 + 8k.7$	6	5

b	N_b	c_1	c_2
$102 + 48k$	$1.[17 + 8k.89 + 42k.84 + 40k.12 + 6k].17 + 8k.7$	6	5
$70 + 63k$	$1.[10 + 9k.62 + 56k.59 + 54k.7 + 7k].10 + 9k.8$	7	6
$133 + 63k$	$1.[19 + 9k.118 + 56k.113 + 54k.14 + 7k].19 + 9k.8$	7	6
$196 + 63k$	$1.[28 + 9k.174 + 56k.167 + 54k.21 + 7k].28 + 9k.8$	7	6
$259 + 63k$	$1.[37 + 9k.230 + 56k.221 + 54k.28 + 7k].37 + 9k.8$	7	6
$48 + 80k$	$1.[6 + 10k.43 + 72k.41 + 70k.4 + 8k].6 + 10k.9$	8	7
$88 + 80k$	$1.[11 + 10k.79 + 72k.76 + 70k.8 + 8k].11 + 10k.9$	8	7
$128 + 80k$	$1.[16 + 10k.115 + 72k.111 + 70k.12 + 8k].16 + 10k.9$	8	7
$168 + 80k$	$1.[21 + 10k.151 + 72k.146 + 70k.16 + 8k].21 + 10k.9$	8	7
$108 + 99k$	$1.[12 + 11k.98 + 90k.95 + 88k.9 + 9k].12 + 11k.10$	9	8
$207 + 99k$	$1.[23 + 11k.188 + 90k.183 + 88k.18 + 9k].23 + 11k.10$	9	8
$306 + 99k$	$1.[34 + 11k.278 + 90k.271 + 88k.27 + 9k].34 + 11k.10$	9	8
$405 + 99k$	$1.[45 + 11k.368 + 90k.359 + 88k.36 + 9k].45 + 11k.10$	9	8
$70 + 120k$	$1.[7 + 12k.64 + 110k.62 + 108k.5 + 10k].7 + 12k.11$	10	9
$130 + 120k$	$1.[13 + 12k.119 + 110k.116 + 108k.10 + 10k].13 + 12k.11$	10	9
$190 + 120k$	$1.[19 + 12k.174 + 110k.170 + 108k.15 + 10k].19 + 12k.11$	10	9
$250 + 120k$	$1.[25 + 12k.229 + 110k.224 + 108k.20 + 10k].25 + 12k.11$	10	9
$154 + 143k$	$1.[14 + 13k.142 + 132k.139 + 130k.11 + 11k].14 + 13k.12$	11	10
$297 + 143k$	$1.[27 + 13k.274 + 132k.269 + 130k.22 + 11k].27 + 13k.12$	11	10
$440 + 143k$	$1.[40 + 13k.406 + 132k.399 + 130k.33 + 11k].40 + 13k.12$	11	10
$583 + 143k$	$1.[53 + 13k.538 + 132k.529 + 130k.44 + 11k].53 + 13k.12$	11	10

Recall that we are considering 1-phoenix numbers of the form

$$1.[a_1 + b_1k.a_4 + b_4k.a_3 + b_3k.a_2 + b_2k].a_1 + b_1k.n, \quad \text{in base } b = a_1(n - 1) + (n^2 - 1)k.$$

An interesting observation is that when $b_0 = n^2 - 1$ is even, then all b_i are even and the common difference between "constant terms" is $b_i/2$, but if $b_0 = n^2 - 1$ is odd, then the b_i can be of mixed parity, and the common difference between constant terms is b_i. Thus, if b_0 is odd and a_0 is minimal, then

$$1.[a_1 + b_1k.a_4 + b_4k.a_3 + b_3k.a_2 + b_2k].a_1 + b_1k.n, \quad \text{in base } b = a_1(n - 1) + (n^2 - 1)k.$$

represents all arithmetic extensions with four digit repetitions. However, if b_0 is even, then

$$1.[a_1 + (b_1/2)j + b_1k.a_4 + (b_4/2)j + b_4k.a_3 + (b_3/2)j + b_3k.a_2 + (b_2/2)j + b_2k].a_1 + (b_1/2)j + b_1k.n,$$

in base $b = (a_1 + (b_1/2)j)(n - 1) + (n^2 - 1)k$.

The first instances when $n^2 - 1$ is even are

b	N_b	c_1	c_2
$6 + 8k$	$1.[3 + 4k.4 + 6k.2 + 4k.1 + 2k].3 + 4k.3$	2	1
$16 + 24k$	$1.[4 + 6k.13 + 20k.11 + 18k.2 + 4k].4 + 6k.5$	4	3
$30 + 48k$	$1.[5 + 8k.26 + 42k.24 + 40k.3 + 6k].5 + 8k.7$	6	5
$48 + 80k$	$1.[6 + 10k.43 + 72k.41 + 70k.4 + 8k].6 + 10k.9$	8	7
$70 + 120k$	$1.[7 + 12k.64 + 110k.62 + 108k.5 + 10k].7 + 12k.11$	10	9

NOTE: Adjust all sequences to start with $n = 1$.
The sequence of a_0 is $6, 16, 30, 48, \ldots$ is [A054000](#) $a(n) = 2n^2 + 4n$.
The sequence of b_0 is $8, 24, 48, 80, \ldots$ is [A033996](#) $a(n) = 4n^2 + 4n$.
The sequence of a_1 is $n + 2$.
The sequence of b_1 is $2n + 2$.
The sequence of a_2 is n.
The sequence b_2 is $2n$.
The sequence of a_3 is $2, 11, 24, 41, \ldots$ is (not in OEIS) $a(n) = 2n^2 - n - 4$.
The sequence of b_3 is $4, 18, 40, 70, \ldots$ is [A187297](#) $a(n) = 4n^2 - 6n$.
The sequence of a_4 is $4, 13, 26, 43, \ldots$ is [A091823](#) $a(n) = 2n^2 + 3n - 1$.
The sequence of b_4 is $6, 20, 42, 72, \ldots$ is [A002943](#) $a(n) = 2n(2n + 1)$.
The carry digits are $c_1 = 2n$ and $c_2 = 2n - 1$, respectively.

Theorem 60. *If $n \geq 1$, then $1.[A_1.A_4.A_3.A_2].A_1.2n + 1$ is 1-phoenix in base b, where*

$$b = 2n(n + 2) + 2n(n + 1)j + 4n(4n + 1)k,$$
$$A_1 = (n + 2) + (n + 1)j + 2(n + 1)k,$$
$$A_2 = n + nj + 2nk,$$
$$A_3 = (2n^2 + 3n - 3) + (2n^2 + n - 1)j + 2(2n^2 + n - 1)k,$$
$$A_4 = (2n^2 + 3n - 1) + (2n^2 + n)j + 2(2n^2 + n)k,$$
$$c_1 = 2n,$$
$$c_2 = 2n - 1.$$

and $j, k \geq 0$.

The first instances of $n^2 - 1$ odd are
The first instances when $n^2 - 1$ is even are

b	N_b	c_1	c_2
$18 + 15k$	$1.[6 + 5k.14 + 12k.11 + 10k.3 + 3k].6 + 5k.4$	3	2
$40 + 35k$	$1.[8 + 7k.34 + 30k.31 + 28k.5 + 5k].8 + 7k.6$	5	4
$70 + 63k$	$1.[10 + 9k.62 + 56k.59 + 54k.7 + 7k].10 + 9k.8$	7	6
$108 + 99k$	$1.[12 + 11k.98 + 90k.95 + 88k.9 + 9k].12 + 11k.10$	9	8
$154 + 143k$	$1.[14 + 13k.142 + 132k.139 + 130k.11 + 11k].14 + 13k.12$	11	10

NOTE: All sequences adjusted to $n = 2$ as first term.
The sequence of a_0 is $18, 40, 70, 108, \ldots$ is $a(n) = 4n^2 + 2n - 2$.
The sequence of b_0 is $15, 35, 63, 99, \ldots$ is $a(n) = 4n^2 - 1$.
The sequence of a_1 is $6, 8, 10, 12, \ldots$ is $a(n) = 2n + 2$.
The sequence of b_1 is $5, 7, 9, 77, \ldots$ is $a(n) = 2n + 1$.
The sequence of a_2 is $3, 5, 7, 9, \ldots$ is $a(n) = 2n - 1$.
The sequence of b_2 is $3, 5, 7, 9, \ldots$ is $a(n) = 2n - 1$.

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The sequence of a_3 is $10, 28, 54, 88, \ldots$ is $a(n) = 4n^2 - 2n - 2$.
The sequence of b_3 is $11, 31, 59, 95, \ldots$ is $a(n) = 4n^2 - 5$.
The sequence of a_4 is $14, 34, 62, 98, \ldots$ is $a(n) = 4n^2 - 2$.
The sequence of b_4 is $12, 30, 56, 90, \ldots$ is $a(n) = 4n^2 - 2n$.

Theorem 61. *If $n \geq 2$, then $1.[A_1.A_4.A_3.A_2].A_1.2n$ is 1-phoenix in base b, where*

$$b = (4n^2 + 2n - 2) + (4n^2 - 1)k,$$
$$A_1 = (2n + 2) + (2n + 1)k,$$
$$A_2 = (2n - 1) + (2n - 1)k,$$
$$A_3 = (4n^2 - 2n - 2) + (4n^2 - 5)k,$$
$$A_4 = (4n^2 - 2) + (4n^2 - 2n)k,$$
$$c_1 = 2n - 1,$$
$$c_2 = 2n - 2.$$

and $j, k \geq 0$.

6.1.3 Arithmetic extensions of the first kind with 5-digit repetitions

An arithmetic extension of $1.m + (n+1)k.n$ in base $b = m(n-1) + (n^2 - 1)k$ with repetitions of six digits is of the form

		(c_3)	(c_2)	(c_1)	(1)		
	$A_1.$	$[A_2.$	$A_3.$	$A_4.$	$A_5.$	$A_1].$	1
1.	$[A_1.$	$A_5.$	$A_4.$	$A_3.$	$A_2].$	$A_1.$	n

Base $b = m(n-1) + (n^2 - 1)k$.

where $A_i = a_i + b_i k$ and the carries c_i are integers. Solving the 1-phoenix equations with Maple yields the following. *Warning:* manually edited; hopefully no typos.

$$a_2 = \frac{b_4 n^3 a_1 - a_4 n^4 - a_1 b_4 n^2 - c_1 n^3 + c_1 a_1 n + a_4 n^2 - c_1 a_1 - n b_4 - 1}{(n-1)(n+1)},$$

$$a_3 = -a_1^2 b_4 n^2 + a_4 n^3 a_1 + 2a_1^2 b_4 n - a_1 a_4 n^2 + a_1 c_1 n^2 - a_1^2 b_4 - n a_4 a_1 - c_1 a_1 n + a_4 a_1 + n a_4 + c_1,$$

$$a_5 = \frac{a_1 b_4 n^2 + a_1 c_1 n^2 - a_4 n^3 - n b_4 a_1 - c_1 a_1 n - c_1 n^2 + n a_4 - b_4 - n}{(n-1)(n+1)},$$

$$b_2 = c_1,$$
$$b_3 = -b_4 n^3 a_1 + a_4 n^4 + a_1 b_4 n^2 + c_1 n^3 + n b_4 a_1 - 2a_4 n^2 - a_1 b_4 + n b_4 - c_1 n + a_4,$$
$$b_5 = c_1 n,$$
$$c_2 = n b_4 a_1 - a_4 n^2 - a_1 b_4 - c_1 n + a_4,$$
$$c_3 = -a_1 b_4 n^2 + a_4 n^3 + n b_4 a_1 + c_1 n^2 - n a_4 + b_4,$$

$c_4 = 0,$
$d_1 = 0,$
$d_2 = 0,$
$d_3 = 0,$
$d_4 = 0.$

Table 29 shows all progressions with bases with arithmetic difference $n^2 - 1$ up to $n = 25$.

Table 29: Arithmetic extensions of the first kind with 5-digit repetitions.

$6 + 8k$	$1.[3 + 4k.4 + 6k.0 + 1k.2 + 3k.1 + 2k].3 + 4k.3$	2	0	1
$12 + 48k$	$1.[2 + 8k.5 + 21k.6 + 26k.9 + 38k.0 + 3k].2 + 8k.7$	3	3	5
$18 + 48k$	$1.[3 + 8k.5 + 14k.12 + 33k.14 + 39k.0 + 2k].3 + 8k.7$	2	4	5
$44 + 143k$	$1.[4 + 13k.22 + 72k.21 + 70k.38 + 125k.1 + 6k].4 + 13k.12$	6	5	10
$66 + 143k$	$1.[6 + 13k.22 + 48k.43 + 94k.58 + 127k.1 + 4k].6 + 13k.12$	4	7	10
$42 + 224k$	$1.[3 + 16k.28 + 150k.13 + 73k.37 + 199k.1 + 10k].3 + 16k.15$	10	4	13
$140 + 224k$	$1.[10 + 16k.28 + 45k.111 + 178k.128 + 206k.1 + 3k].10 + 16k.15$	3	11	13
$105 + 255k$	$1.[7 + 17k.46 + 112k.58 + 142k.95 + 232k.2 + 7k].7 + 17k.16$	7	8	14
$85 + 323k$	$1.[5 + 19k.52 + 198k.32 + 124k.77 + 294k.2 + 11k].5 + 19k.18$	11	6	16
$187 + 323k$	$1.[11 + 19k.52 + 90k.134 + 232k.173 + 300k.2 + 5k].11 + 19k.18$	5	12	16
$96 + 624k$	$1.[4 + 26k.73 + 475k.22 + 148k.89 + 580k.2 + 19k].4 + 26k.25$	19	5	23
$144 + 624k$	$1.[6 + 26k.98 + 425k.45 + 198k.134 + 582k.3 + 17k].6 + 26k.25$	17	7	23

No formulas found so far.

6.1.4 Arithmetic extensions of the first kind with 6-digit repetitions

An arithmetic extension of $1.m + (n+1)k.n$ in base $b = m(n-1) + (n^2-1)k$ with repetitions of six digits is of the form

		(c_4)	(c_3)	(c_2)	(c_1)	(1)		
	$A_1.$	$[A_2.$	$A_3.$	$A_4.$	$A_5.$	$A_6.$	$A_1].$	1
1.	$[A_1.$	$A_6.$	$A_5.$	$A_4.$	$A_3.$	$A_2].$	$A_1.$	n

Base $b = m(n-1) + (n^2-1)k.$

where $A_i = a_i + b_i k$ and the carries c_i are integers. The solutions to the 1-phoenix equations are too long to be displayed. Table 30 has to be rotated into landscape in order to display the extensions properly.

Table 30: Arithmetic extensions of the first kind with 6-digit repetitions.

$4 + 8k$	$1.[2 + 4k.1 + 3k.0 + 1k.2 + 4k.1 + 3k.0 + 1k].A_1.3$	1	0	1	1
$4 + 8k$	$1.[2 + 4k.1 + 3k.3 + 7k.1 + 4k.2 + 5k.0 + 1k].A_1.3$	1	2	1	1
$6 + 8k$	$1.[3 + 4k.4 + 6k.0 + 1k.0 + 0k.2 + 3k.1 + 2k].A_1.3$	2	0	0	1
$9 + 15k$	$1.[3 + 5k.2 + 4k.8 + 14k.5 + 10k.6 + 11k.0 + 1k].A_1.4$	1	3	2	2
$12 + 15k$	$1.[4 + 5k.6 + 8k.1 + 2k.4 + 5k.6 + 8k.1 + 2k].A_1.4$	2	0	1	2
$16 + 24k$	$1.[4 + 6k.3 + 5k.15 + 23k.11 + 18k.12 + 19k.0 + 1k].A_1.5$	1	4	3	3
$20 + 24k$	$1.[5 + 6k.8 + 10k.2 + 3k.10 + 12k.12 + 15k.1 + 2k].A_1.5$	2	0	2	3
$15 + 35k$	$1.[3 + 7k.10 + 24k.1 + 4k.3 + 7k.10 + 24k.1 + 4k].A_1.6$	4	0	1	4
$25 + 35k$	$1.[5 + 7k.4 + 6k.24 + 34k.19 + 28k.20 + 29k.0 + 1k].A_1.6$	1	5	4	4
$30 + 35k$	$1.[6 + 7k.10 + 12k.3 + 4k.18 + 21k.20 + 24k.1 + 2k].A_1.6$	2	0	3	4
$36 + 48k$	$1.[6 + 8k.5 + 7k.35 + 47k.29 + 40k.30 + 41k.0 + 1k].A_1.7$	1	6	5	5
$36 + 48k$	$1.[6 + 8k.26 + 35k.3 + 5k.6 + 8k.26 + 35k.3 + 5k].A_1.7$	5	0	1	5
$42 + 48k$	$1.[7 + 8k.12 + 14k.4 + 5k.28 + 32k.30 + 35k.1 + 2k].A_1.7$	2	0	4	5
$28 + 63k$	$1.[4 + 9k.14 + 32k.27 + 62k.11 + 27k.24 + 55k.1 + 4k].A_1.8$	4	7	3	6
$49 + 63k$	$1.[7 + 9k.6 + 8k.48 + 62k.41 + 54k.42 + 55k.0 + 1k].A_1.8$	1	7	6	6
$56 + 63k$	$1.[8 + 9k.14 + 16k.5 + 6k.40 + 45k.42 + 48k.1 + 2k].A_1.8$	2	0	5	6
$24 + 80k$	$1.[3 + 10k.16 + 54k.23 + 79k.5 + 20k.21 + 71k.1 + 6k].A_1.9$	6	8	2	7
$32 + 80k$	$1.[4 + 10k.25 + 63k.2 + 7k.4 + 10k.25 + 63k.2 + 7k].A_1.9$	7	0	1	7
$48 + 80k$	$1.[6 + 10k.16 + 27k.47 + 79k.29 + 50k.42 + 71k.1 + 3k].A_1.9$	3	8	5	7
$64 + 80k$	$1.[8 + 10k.7 + 9k.63 + 79k.55 + 70k.56 + 71k.0 + 1k].A_1.9$	1	8	7	7
$72 + 80k$	$1.[9 + 10k.16 + 18k.6 + 7k.54 + 60k.56 + 63k.1 + 2k].A_1.9$	2	0	6	7
$72 + 99k$	$1.[8 + 11k.58 + 80k.5 + 8k.8 + 11k.58 + 80k.5 + 8k].A_1.10$	8	0	1	8
$81 + 99k$	$1.[9 + 11k.8 + 10k.80 + 98k.71 + 88k.72 + 89k.0 + 1k].A_1.10$	1	9	8	8
$90 + 99k$	$1.[10 + 11k.18 + 20k.7 + 8k.70 + 77k.72 + 80k.1 + 2k].A_1.10$	2	0	7	8
$20 + 120k$	$1.[2 + 12k.9 + 55k.1 + 9k.10 + 60k.16 + 99k.0 + 5k].A_1.11$	5	0	5	9
$100 + 120k$	$1.[10 + 12k.9 + 11k.99 + 119k.89 + 108k.90 + 109k.0 + 1k].A_1.11$	1	10	9	9
$110 + 120k$	$1.[11 + 12k.20 + 22k.8 + 9k.88 + 96k.90 + 99k.1 + 2k].A_1.11$	2	0	8	9
$55 + 143k$	$1.[5 + 13k.46 + 120k.3 + 10k.5 + 13k.46 + 120k.3 + 10k].A_1.12$	10	0	1	10
$121 + 143k$	$1.[11 + 13k.10 + 12k.120 + 142k.109 + 130k.110 + 131k.0 + 1k].A_1.12$	1	11	10	10
$132 + 143k$	$1.[12 + 13k.22 + 24k.9 + 10k.108 + 117k.110 + 120k.1 + 2k].A_1.12$	2	0	9	10

b	N_b	c_1	c_2	c_3	c_4
$36 + 168k$	$1.[3 + 14k.11 + 52k.2 + 11k.24 + 112k.30 + 143k.0 + 4k].A_1.13$	4	0	8	11
$60 + 168k$	$1.[5 + 14k.37 + 104k.59 + 167k.19 + 56k.55 + 155k.2 + 8k].A_1.13$	8	12	4	11
$72 + 168k$	$1.[6 + 14k.50 + 117k.4 + 11k.18 + 42k.61 + 143k.3 + 9k].A_1.13$	9	0	3	11
$96 + 168k$	$1.[8 + 14k.37 + 65k.95 + 167k.55 + 98k.88 + 155k.2 + 5k].A_1.13$	5	12	7	11
$120 + 168k$	$1.[10 + 14k.102 + 143k.7 + 11k.10 + 14k.102 + 143k.7 + 11k].A_1.13$	11	0	1	11
$144 + 168k$	$1.[12 + 14k.11 + 13k.143 + 167k.131 + 154k.132 + 155k.0 + 1k].A_1.13$	1	12	11	11
$156 + 168k$	$1.[13 + 14k.24 + 26k.10 + 11k.130 + 140k.132 + 143k.1 + 2k].A_1.13$	2	0	10	11
$169 + 195k$	$1.[13 + 15k.12 + 14k.168 + 194k.155 + 180k.156 + 181k.0 + 1k].A_1.14$	1	13	12	12
$182 + 195k$	$1.[14 + 15k.26 + 28k.11 + 12k.154 + 165k.156 + 168k.1 + 2k].A_1.14$	2	0	11	12
$84 + 224k$	$1.[6 + 16k.73 + 195k.4 + 13k.6 + 16k.73 + 195k.4 + 13k].A_1.15$	13	0	1	13
$196 + 224k$	$1.[14 + 16k.13 + 15k.195 + 223k.181 + 208k.182 + 209k.0 + 1k].A_1.15$	1	14	13	13
$210 + 224k$	$1.[15 + 16k.28 + 30k.12 + 13k.180 + 192k.182 + 195k.1 + 2k].A_1.15$	2	0	12	13
$225 + 255k$	$1.[15 + 17k.14 + 16k.224 + 254k.209 + 238k.210 + 239k.0 + 1k].A_1.16$	1	15	14	14
$240 + 255k$	$1.[16 + 17k.30 + 32k.13 + 14k.208 + 221k.210 + 224k.1 + 2k].A_1.16$	2	0	13	14
$64 + 288k$	$1.[4 + 18k.49 + 221k.63 + 287k.11 + 54k.60 + 271k.2 + 13k].A_1.17$	13	16	3	15
$112 + 288k$	$1.[7 + 18k.66 + 170k.111 + 287k.41 + 108k.105 + 271k.3 + 10k].A_1.17$	10	16	6	15
$128 + 288k$	$1.[8 + 18k.83 + 187k.6 + 15k.40 + 90k.113 + 255k.4 + 11k].A_1.17$	11	0	5	15
$160 + 288k$	$1.[10 + 18k.66 + 119k.159 + 287k.89 + 162k.150 + 271k.3 + 7k].A_1.17$	7	16	9	15
$208 + 288k$	$1.[13 + 18k.49 + 68k.207 + 287k.155 + 216k.195 + 271k.2 + 4k].A_1.17$	4	16	12	15
$256 + 288k$	$1.[16 + 18k.15 + 17k.255 + 287k.239 + 270k.240 + 271k.0 + 1k].A_1.17$	1	16	15	15
$272 + 288k$	$1.[17 + 18k.32 + 34k.14 + 15k.238 + 252k.240 + 255k.1 + 2k].A_1.17$	2	0	14	15
$119 + 323k$	$1.[7 + 19k.106 + 288k.5 + 16k.7 + 19k.106 + 288k.5 + 16k].A_1.18$	16	0	1	16
$289 + 323k$	$1.[17 + 19k.16 + 18k.288 + 322k.271 + 304k.272 + 305k.0 + 1k].A_1.18$	1	17	16	16
$306 + 323k$	$1.[18 + 19k.34 + 36k.15 + 16k.270 + 285k.272 + 288k.1 + 2k].A_1.18$	2	0	15	16
$108 + 360k$	$1.[6 + 20k.74 + 247k.107 + 359k.29 + 100k.102 + 341k.3 + 13k].A_1.19$	13	18	5	17
$234 + 360k$	$1.[13 + 20k.74 + 114k.233 + 359k.155 + 240k.221 + 341k.3 + 6k].A_1.19$	6	18	12	17
$252 + 360k$	$1.[14 + 20k.226 + 323k.11 + 17k.14 + 20k.226 + 323k.11 + 17k].A_1.19$	17	0	1	17
$324 + 360k$	$1.[18 + 20k.17 + 19k.323 + 359k.305 + 340k.306 + 341k.0 + 1k].A_1.19$	1	18	17	17
$342 + 360k$	$1.[19 + 20k.36 + 38k.16 + 17k.304 + 320k.306 + 323k.1 + 2k].A_1.19$	2	0	16	17
$323 + 399k$	$1.[17 + 21k.178 + 220k.14 + 18k.136 + 168k.291 + 360k.8 + 11k].A_1.20$	11	0	8	18
$361 + 399k$	$1.[19 + 21k.18 + 20k.360 + 398k.341 + 378k.342 + 379k.0 + 1k].A_1.20$	1	19	18	18

b	N_b	c_1	c_2	c_3	c_4
$380 + 399k$	$1.[20 + 21k.38 + 40k.17 + 18k.340 + 357k.342 + 360k.1 + 2k].A_1.20$	2	0	17	18
$160 + 440k$	$1.[8 + 22k.145 + 399k.6 + 19k.8 + 22k].8 + 22k.21$	19	0	1	19
$180 + 440k$	$1.[9 + 22k.103 + 252k.179 + 439k.71 + 176k.171 + 419k.4 + 12k].9 + 22k.21$	12	20	8	19
$200 + 440k$	$1.[10 + 22k.124 + 273k.8 + 19k.70 + 154k.181 + 399k.5 + 13k].10 + 22k.21$	13	0	7	19
$240 + 440k$	$1.[12 + 22k.103 + 189k.239 + 439k.131 + 242k.228 + 419k.4 + 9k].12 + 22k.21$	9	20	11	19
$380 + 440k$	$1.[19 + 22k.145 + 168k.16 + 19k.228 + 264k.344 + 399k.6 + 8k].19 + 22k.21$	8	0	12	19
$400 + 440k$	$1.[20 + 22k.19 + 21k.399 + 439k.379 + 418k.380 + 419k.0 + 1k].20 + 22k.21$	1	20	19	19
$420 + 440k$	$1.[21 + 22k.40 + 42k.18 + 19k.378 + 396k.380 + 399k.1 + 2k].21 + 22k.21$	2	0	18	19

Theorem 62 (Arithmetic extension of Theorem 7)**.** *The number* $1.[A_1.A_6.A_5.A_4.A_3.A_2].A_1.n$
is 1-phoenix in base $b = (n-1)^2 + (n^2-1)k$*, where* $A_i = a_i + b_i k$*, and [CHECK]*

$$a_1 = n - 1, \quad b_1 = n + 1,$$
$$a_2 = 0, \quad b_2 = 1,$$
$$a_3 = n^2 - n, \quad b_3 = n^2 + n - 1,$$
$$a_4 = n^2 - n - 1, \quad b_4 = n^2 + n - 2,$$
$$a_5 = n^2 - 1, \quad b_5 = n^2 - 2,$$
$$a_6 = n - 2, \quad b_6 = n,$$
$$c_1 = 1, \quad c_2 = n, \quad c_3 = n - 1, \quad c_4 = n - 1,$$
$$d_1 = d_2 = d_3 = d_4 = 0.$$

Proof. An exercise for the reader. □

Theorem 63 (Arithmetic extension of Theorem ???)**.** *The number* $1.[A_1.A_6.A_5.A_4.A_3.A_2].A_1.n$
is 1-phoenix in base $b = n(n-1) + (n^2-1)k$*, where* $A_i = a_i + b_i k$*, and*

$$a_1 = n, \quad b_1 = n + 1,$$
$$a_2 = 1, \quad b_2 = 2,$$
$$a_3 = (n-1)(n-2), \quad b_3 = (n-1)^2 - 1,$$
$$a_4 = n^2 - 3n, \quad b_4 = n^2 - 2n - 3,$$
$$a_5 = n - 3, \quad b_5 = n - 2,$$
$$a_6 = 2n - 2, \quad b_6 = 2n,$$
$$c_1 = 2, \quad c_2 = 0, \quad c_3 = n - 3, \quad c_4 = n - 2,$$
$$d_1 = d_2 = d_3 = d_4 = 0.$$

Proof. An exercise for the reader. □

Theorem 64. *If* $n \geq 0$*, then the number* $1.[A_1.A_6.A_5.A_4.A_3.A_2].A_1.5 + 4n$ *is 1-phoenix in
base* $b = a_0 + b_0 k$*, where* $A_i = a_i + b_i k$*, and*

$$
\begin{aligned}
a_0 &= 16 + 24n + 8n^2, & b_0 &= (5 + 4n)^2 - 1 = 24 + 40n + 16n^2 \\
&= 8(2 + n)(1 + n) & &= 8(3 + 2n)(1 + n), \\
a_1 &= 4 + 2n, & b_1 &= 6 + 4n, \\
a_2 &= n, & b_2 &= 1 + 2n, \\
a_3 &= 12 + 22n + 8n^2 & b_3 &= 19 + 36n + 16n^2, \\
&= 2(3 + 4n)(2 + n), & & \\
a_4 &= 11 + 14n + 4n^2, & b_4 &= 18 + 24n + 8n^2 \\
& & &= 2(3 + 2n)^2, \\
a_5 &= 15 + 24n + 8n^2, & b_5 &= 23 + 4n + 16n^2,
\end{aligned}
$$

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$$a_6 = 3 + 9n + 4n^2, \qquad b_6 = 5 + 14n + 8n^2,$$
$$c_1 = 1 + 2n, \qquad c_2 = 4 + 4n,$$
$$c_3 = 3 + 2n, \qquad c_4 = 3 + 4n.$$

6.2   Arithmetic progressions of the second kind

By Theorem 5 the number $1.m.n$ is 1-phoenix in base b if and only if $b = m(n-1)$. The arithmetic extension of the second kind of $1.m.n$ is

$$1 + b_3 k.m + b_2 k.n + b_1 k, \qquad b = m(n-1) + b_0 k.$$

The multiplication is

$$
\begin{array}{|ccc|}
\hline
\multicolumn{3}{|c|}{(c_1 + c_2 k)} \\
 & m + b_2 k. & 1 + b_3 k \\
 & & n + b_1 k \\
\hline
1 + b_3 k. & m + b_2 k. & n + b_1 k \\
\hline
\end{array}
$$
$$\text{Base } b = m(n-1) + b_0 k.$$

and the solutions are

$$b_1 = \frac{n b_0}{m(n-1)}, \tag{9a}$$

$$b_2 = \frac{m^2(n-1)b_0}{m^2(n-1)^2 - n^2}, \tag{9b}$$

$$b_3 = \frac{mn b_0}{m^2(n-1)^2 - n^2}, \tag{9c}$$

$$c_1 = 0, \tag{9d}$$

$$c_2 = \frac{n^2 b_0}{(n-1)(m^2(n-1)^2 - n^2)}. \tag{9e}$$

If b_0 is chosen so that (9) has integer solutions, then an arithmetic extension of the second kind of $1.m.n$ is obtained. A computer search provides us with Table 32. Special cases provide interesting theorems.

Table 32: Arithmetic extensions of the second kind

$6 + 6k$	$1 + 2k.3 + 4k.3 + 3k$	$1k$
$6 + 15k$	$1 + 6k.2 + 9k.4 + 10k$	$4k$
$12 + 24k$	$1 + 3k.4 + 9k.4 + 8k$	$1k$
$12 + 60k$	$1 + 8k.6 + 32k.3 + 15k$	$2k$
$20 + 60k$	$1 + 4k.5 + 16k.5 + 15k$	$1k$
$10 + 80k$	$1 + 15k.2 + 25k.6 + 48k$	$9k$

b	N_b	$c = c_2k$
$15 + 105k$	$1 + 10k.3 + 25k.6 + 42k$	$4k$
$30 + 120k$	$1 + 5k.6 + 25k.6 + 24k$	$1k$
$18 + 210k$	$1 + 18k.9 + 108k.3 + 35k$	$3k$
$24 + 210k$	$1 + 12k.8 + 72k.4 + 35k$	$2k$
$42 + 210k$	$1 + 6k.7 + 36k.7 + 35k$	$1k$
$28 + 315k$	$1 + 14k.4 + 49k.8 + 90k$	$4k$
$56 + 336k$	$1 + 7k.8 + 49k.8 + 48k$	$1k$
$24 + 440k$	$1 + 24k.3 + 64k.9 + 165k$	$9k$
$24 + 504k$	$1 + 32k.12 + 256k.3 + 63k$	$4k$
$40 + 504k$	$1 + 16k.10 + 128k.5 + 63k$	$2k$
$72 + 504k$	$1 + 8k.9 + 64k.9 + 63k$	$1k$
$18 + 693k$	$1 + 54k.6 + 243k.4 + 154k$	$12k$
$45 + 693k$	$1 + 18k.5 + 81k.10 + 154k$	$4k$
$36 + 720k$	$1 + 27k.12 + 243k.4 + 80k$	$3k$
$90 + 720k$	$1 + 9k.10 + 81k.10 + 80k$	$1k$
$20 + 910k$	$1 + 60k.4 + 200.6 + 273k$	$18k$
$30 + 990k$	$1 + 50k.15 + 500k.3 + 99k$	$5k$
$60 + 990k$	$1 + 20k.12 + 200k.6 + 99k$	$2k$
$110 + 990k$	$1 + 10k.11 + 100k.11 + 99k$	$1k$
$33 + 1155k$	$1 + 44k.3 + 121k.12 + 420k$	$16k$
$44 + 1232k$	$1 + 33k.4 + 121k.12 + 336k$	$9k$
$66 + 1287k$	$1 + 22k.6 + 121k.12 + 234k$	$4k$
$132 + 1320k$	$1 + 11k.12 + 121k.12 + 120k$	$1k$
$36 + 1716k$	$1 + 72k.18 + 864k.3 + 143k$	$6k$
$48 + 1716k$	$1 + 48k.16 + 576k.4 + 143k$	$4k$
$60 + 1716k$	$1 + 36k.15 + 432k.5 + 143k$	$3k$
$84 + 1716k$	$1 + 24k.14 + 288k.7 + 143k$	$2k$
$156 + 1716k$	$1 + 12k.13 + 144k.13 + 143k$	$1k$
$91 + 2145k$	$1 + 26k.7 + 169k.14 + 330k$	$4k$
$182 + 2184k$	$1 + 13k.14 + 169k.14 + 168k$	$1k$
$42 + 2394k$	$1 + 70k.3 + 196k.15 + 855k$	$25k$
$70 + 2618k$	$1 + 42k.5 + 196k.15 + 561k$	$9k$
$42 + 2730k$	$1 + 98k.21 + 1372k.3 + 195k$	$7k$
$112 + 2730k$	$1 + 28k.16 + 392k.8 + 195k$	$2k$
$210 + 2730k$	$1 + 14k.15 + 196k.15 + 195k$	$1k$
$60 + 3135k$	$1 + 60k.4 + 225k.16 + 836k$	$16k$
$30 + 3315k$	$1 + 150k.10 + 1125k.4 + 442k$	$20k$
$45 + 3315k$	$1 + 90k.9 + 675k.6 + 442k$	$12k$
$120 + 3315k$	$1 + 30k.8 + 225k.16 + 442k$	$4k$
$60 + 3360k$	$1 + 75k.20 + 1125k.4 + 224k$	$5k$
$90 + 3360k$	$1 + 45k.18 + 675k.6 + 224k$	$3k$

b	N_b	$c = c_2 k$
$240 + 3360k$	$1 + 15k.16 + 225k.16 + 224k$	$1k$
$48 + 3952k$	$1 + 96k.6 + 512k.9 + 741k$	$18k$
$48 + 4080k$	$1 + 128k.24 + 2048k.3 + 255k$	$8k$
$80 + 4080k$	$1 + 64k.20 + 1024k.5 + 255k$	$4k$
$144 + 4080k$	$1 + 32k.18 + 512k.9 + 255k$	$2k$
$272 + 4080k$	$1 + 16k.17 + 256k.17 + 255k$	$1k$
$51 + 4301k$	$1 + 102k.3 + 289k.18 + 1518k$	$36k$
$102 + 4760k$	$1 + 51k.6 + 289k.18 + 840k$	$9k$
$153 + 4845k$	$1 + 34k.9 + 289k.18 + 570k$	$4k$
$306 + 4896k$	$1 + 17k.18 + 289k.18 + 288k$	$1k$
$36 + 5382k$	$1 + 180k.4 + 648k.10 + 1495k$	$50k$
$54 + 5814k$	$1 + 162k.27 + 2916k.3 + 323k$	$9k$
$72 + 5814k$	$1 + 108k.24 + 1944k.4 + 323k$	$6k$
$126 + 5814k$	$1 + 54k.21 + 972k.7 + 323k$	$3k$
$180 + 5814k$	$1 + 36k.20 + 648k.10 + 323k$	$2k$
$342 + 5814k$	$1 + 18k.19 + 324k.19 + 323k$	$1k$
$76 + 6384k$	$1 + 95k.4 + 361k.20 + 1680k$	$25k$
$95 + 6555k$	$1 + 76k.5 + 361k.20 + 1380k$	$16k$
$190 + 6783k$	$1 + 38k.10 + 361k.20 + 714k$	$4k$
$380 + 6840k$	$1 + 19k.20 + 361k.20 + 360k$	$1k$
$60 + 7020k$	$1 + 140k.3 + 400k.21 + 2457k$	$49k$
$40 + 7820k$	$1 + 240k.8 + 1600k.6 + 1173k$	$36k$
$140 + 7820k$	$1 + 60k.7 + 400k.21 + 1173k$	$9k$
$60 + 7980k$	$1 + 200k.30 + 4000k.3 + 399k$	$10k$
$100 + 7980k$	$1 + 100k.25 + 2000k.5 + 399k$	$5k$
$120 + 7980k$	$1 + 80k.24 + 1600k.6 + 399k$	$4k$
$220 + 7980k$	$1 + 40k.22 + 800k.11 + 399k$	$2k$
$420 + 7980k$	$1 + 20k.21 + 400k.21 + 399k$	$1k$
$84 + 9240k$	$1 + 147k.28 + 3087k.4 + 440k$	$7k$
$168 + 9240k$	$1 + 63k.24 + 1323k.8 + 440k$	$3k$
$462 + 9240k$	$1 + 21k.22 + 441k.22 + 440k$	$1k$
$66 + 10626k$	$1 + 242k.33 + 5324k.3 + 483k$	$11k$
$264 + 10626k$	$1 + 44k.24 + 968k.12 + 483k$	$2k$
$506 + 10626k$	$1 + 22k.23 + 484k.23 + 483k$	$1k$
$69 + 10695k$	$1 + 184k.3 + 529k.24 + 3720k$	$64k$

The progressions of the form $1.n + 1.n + 1$ have an interesting pattern.

b	N_b	c	
2	$6 + 6k$	$1 + 2k.3 + 4k.3 + 3k$	$1k$
3	$12 + 24k$	$1 + 3k.4 + 9k.4 + 8k$	$1k$

4	$12 + 60k$	$1 + 8k.6 + 32k.3 + 15k$	$2k$
4	$20 + 60k$	$1 + 4k.5 + 16k.5 + 15k$	$1k$
5	$30 + 120k$	$1 + 5k.6 + 25k.6 + 24k$	$1k$
6	$18 + 210k$	$1 + 18k.9 + 108k.3 + 35k$	$3k$
6	$24 + 210k$	$1 + 12k.8 + 72k.4 + 35k$	$2k$
6	$42 + 210k$	$1 + 6k.7 + 36k.7 + 35k$	$1k$
7	$56 + 336k$	$1 + 7k.8 + 49k.8 + 48k$	$1k$
8	$24 + 504k$	$1 + 32k.12 + 256k.3 + 63k$	$4k$
8	$40 + 504k$	$1 + 16k.10 + 128k.5 + 63k$	$2k$
8	$72 + 504k$	$1 + 8k.9 + 64k.9 + 63k$	$1k$
9	$36 + 720k$	$1 + 27k.12 + 243k.4 + 80k$	$3k$
9	$90 + 720k$	$1 + 9k.10 + 81k.10 + 80k$	$1k$
10	$30 + 990k$	$1 + 50k.15 + 500k.3 + 99k$	$5k$
10	$60 + 990k$	$1 + 20k.12 + 200k.6 + 99k$	$2k$
10	$110 + 990k$	$1 + 10k.11 + 100k.11 + 99k$	$1k$
11	$132 + 1320k$	$1 + 11k.12 + 121k.12 + 120k$	$1k$
12	$36 + 1716k$	$1 + 72k.18 + 864k.3 + 143k$	$6k$
12	$48 + 1716k$	$1 + 48k.16 + 576k.4 + 143k$	$4k$
12	$60 + 1716k$	$1 + 36k.15 + 432k.5 + 143k$	$3k$
12	$84 + 1716k$	$1 + 24k.14 + 288k.7 + 143k$	$2k$
12	$156 + 1716k$	$1 + 12k.13 + 144k.13 + 143k$	$1k$

The following theorem generalizes Theorem 19. The six-digit repeating pattern of Theorem 21 does not generalize, that is, it holds only for $d = 1$.

Theorem 65. *If d is a proper divisor of n, $n \geq 2$, and $e = n/d$, then*

$$1 + dnk.d(e+1) + dn^2k.(e+1) + (n^2 - 1)k$$

is 1-phoenix in base $b = n(e+1) + n(n^2 - 1)k$.

Proof. The multiplication table is

	(dk)	
$d(e+1) + dn^2k.$		$1 + dnk$
		$(e+1) + (n^2 - 1)k$
$1 + dnk.$	$d(e+1) + dn^2k.$	$(e+1) + (n^2 - 1)k$

Base $b = n(e+1) + n(n^2 - 1)k$.

$$((e+1) + (n^2 - 1)k)(1 + dnk) - ((e+1) + (n^2 - 1)k) =$$
$$= (dnk)((e+1) + (n^2 - 1)k)$$
$$= (dk)(n(e+1) + n(n^2 - 1)k),$$

$$((e + 1) + (n^2 - 1)k)(d(e + 1) + dn^2 k) + (dk) - (d(e + 1) + dn^2 k) =$$
$$= ((e + 1) + (n^2 - 1)k)(d(e + 1) + dn^2 k) - (d(e + 1) + d(n^2 - 1)k)$$
$$= (d(e + 1) + d(n^2 - 1)k)((e + 1) + n^2 k) - (d(e + 1) + d(n^2 - 1)k)$$
$$= (d(e + 1) + d(n^2 - 1)k)(e + n^2 k)$$
$$= (d(e + 1) + d(n^2 - 1)k)(e)(1 + dnk)$$
$$= (1 + dnk)(n(e + 1) + n(n^2 - 1)k). \qquad \square$$

Isolating numbers with carry $4k$ and and remove initial terms of the form $1.n + 1.n + 1$ we obtain the following table.

b	N_b	c
$6 + 15k$	$1 + 6k.2 + 9k.4 + 10k$	$4k$
$15 + 105k$	$1 + 10k.3 + 25k.6 + 42k$	$4k$
$28 + 315k$	$1 + 14k.4 + 49k.8 + 90k$	$4k$
$45 + 693k$	$1 + 18k.5 + 81k.10 + 154k$	$4k$
$66 + 1287k$	$1 + 22k.6 + 121k.12 + 234k$	$4k$
$91 + 2145k$	$1 + 26k.7 + 169k.14 + 330k$	$4k$
$120 + 3315k$	$1 + 30k.8 + 225k.16 + 442k$	$4k$
$153 + 4845k$	$1 + 34k.9 + 289k.18 + 570k$	$4k$

$x \geq 2.$ $b = 2x^2 - x + (2x - 3)(2x - 1)(2x + 1)k.$
$b = x(2x - 1) + (2x - 3)(2x - 1)(2x + 1)k.$
$1 + 2(2x - 1)k.x + (2x - 1)^2 k.2x + 2(2x - 3)(2x + 1)k.$
Isolating those with carry $c = 9k$ and removing bases with b_0 not of the form $n^3 - n$ we obtain.

b	N_b	dk
$10 + 80k$	$1 + 15k.2 + 25k.6 + 48k$	$9k$
$24 + 440k$	$1 + 24k.3 + 64k.9 + 165k$	$9k$
$44 + 1232k$	$1 + 33k.4 + 121k.12 + 336k$	$9k$
$70 + 2618k$	$1 + 42k.5 + 196k.15 + 561k$	$9k$
$102 + 4760k$	$1 + 51k.6 + 289k.18 + 840k$	$9k$

$x \geq 2.$ $b = 3x^2 - x + (3x - 4)(3x - 1)(3x + 2)k.$
$1 + 3(3x - 1)k.x + (3x - 1)^2 k.3x + 3(3x - 4)(3x + 2)k.$ $c = 9k.$
$x \geq 2.$ $b = x(nx - 1) + (nx - (n + 1))(nx - 1)(nx + (n - 1))k.$
$1 + m(mn - 1)k.n + (mn - 1)^2 k.mn + m(mx - (m + 1))(mn + (m - 1))k.$ $c = m^2 k.$

Theorem 66. *Let m and n be positive integers ≥ 2. Then the number*

$$1 + m(mn - 1)k.n + (mn - 1)^2 k.mn + m((mn - 1)^2 - m^2)k$$

is 1-phoenix in base $b = n(mn - 1) + (mn - 1)((mn - 1)^2 - m^2)k$, with carry $c = m^2 k$.

Proof. Rewrite this.

$$(mn + m(mn - (m+1))(mn + (m-1))k)(1 + m(mn - 1)k) -$$
$$- (mn + m(mn - (m+1))(mn + (m-1))k)$$
$$= (m(mn-1)k)(mn + m(mn - (m+1))(mn + (m-1))k)$$
$$= (m^2 k)(mn - 1)(n + (mn - (m+1))(mn + (m-1))k)$$
$$= (m^2 k)(n(mn - 1) + (mn - (m+1))(mn - 1)(mn + (m-1))k)$$
$$(mn + m((mn - 1)^2 - m^2)k)(n + (mn - 1)^2 k) + (m^2 k) - (n + (mn - 1)^2 k)$$
$$= m(n + ((mn - 1)^2 - m^2)k)(n + (mn - 1)^2 k) - (n + ((mn - 1)^2 - m^2)k)$$
$$= (m(n + (mn - 1)^2 k) - 1)(n + ((mn - 1)^2 - m^2)k)$$
$$= ((mn - 1) + m(mn - 1)^2 k)(n + ((mn - 1)^2 - m^2)k)$$
$$= (1 + m(mn - 1)k)(n(mn - 1) + (mn - 1)((mn - 1)^2 - m^2)k). \qquad \square$$

b	N_b	dk
$18 + 693k$	$1 + 54k.6 + 243k.4 + 154k$	$12k$
$45 + 3315k$	$1 + 90k.9 + 675k.6 + 442k$	$12k$
$84 + 9177k$	$1 + 126k.12 + 1323k.8 + 874k$	$12k$

Theorem 67. *The number*

$$1 + 6(6n + 3)k.3(n + 1) + 3(6n + 3)^2 k.2(n + 1) + 2(6n + 1)(6n + 5)k$$

is 1-phoenix in base $b = (n + 1)(6n + 3) + (6n + 1)(6n + 3)(6n + 5)k$, *with carry* $c = 12k$.

b	N_b	dk
$20 + 910k$	$1 + 60k.4 + 200k.6 + 273k$	$18k$
$48 + 3952k$	$1 + 96k.6 + 512k.9 + 741k$	$18k$
$88 + 10450k$	$1 + 132k.8 + 968k.12 + 1425k$	$18k$

Theorem 68. *The number*

$$1 + 6(6n + 4)k.2(n + 1) + 2(6n + 4)^2 k.3(n + 1) + 3(6n + 1)(6n + 7)k$$

is 1-phoenix in base $b = (n + 1)(6n + 4) + (6n + 1)(6n + 4)(6n + 7)$, *with carry* $c = 18k$.

Progressions such that the progression of $1.m.3$ has carry mk and the progression of $1.3.m$ has carry $m^2 k$.

$6+6k$	$1+2k.3+4k.3+3k$	$1k$
$6+6k$	$1+2k.3+4k.3+3k$	$1k$
$12+60k$	$1+8k.6+32k.3+15k$	$2k$
$15+105k$	$1+10k.3+25k.6+42k$	$4k$
$18+210k$	$1+18k.9+108k.3+35k$	$3k$
$24+440k$	$1+24k.3+64k.9+165k$	$9k$
$24+504k$	$1+32k.12+256k.3+63k$	$4k$
$33+1155k$	$1+44k.3+121k.12+420k$	$16k$
$30+990k$	$1+50k.15+500k.3+99k$	$5k$
$42+2394k$	$1+70k.3+196k.15+855k$	$25k$
$36+1716k$	$1+72k.18+864k.3+143k$	$6k$
$51+4301k$	$1+102k.3+289k.18+1518k$	$36k$

Let's break the table into two tables.

$6+6k$	$1+2k.3+4k.3+3k$	$1k$
$12+60k$	$1+8k.6+32k.3+15k$	$2k$
$18+210k$	$1+18k.9+108k.3+35k$	$3k$
$24+504k$	$1+32k.12+256k.3+63k$	$4k$
$30+990k$	$1+50k.15+500k.3+99k$	$5k$
$36+1716k$	$1+72k.18+864k.3+143k$	$6k$

Theorem 69. *The number*

$$1 + 2n^2k.3n + 4n^3k.3 + (4n^2-1)k$$

is 1-phoenix in base $b = 6n + (2n-1)(2n)(2n+1)k$ with carry nk.

$6+6k$	$1+2k.3+4k.3+3k$	$1k$
$15+105k$	$1+10k.3+25k.6+42k$	$4k$
$24+440k$	$1+24k.3+64k.9+165k$	$9k$
$33+1155k$	$1+44k.3+121k.12+420k$	$16k$
$42+2394k$	$1+70k.3+196k.15+855k$	$25k$
$51+4301k$	$1+102k.3+289k.18+1518k$	$36k$

Theorem 70. *The number*

$$1 + n(3n-1)k.3 + (3n-1)^2k.3n + n(2n-1)(4n-1)k$$

is 1-phoenix in base $b = 3(3n-1) + (2n-1)(3n-1)(4n-1)k$ with carry $c = n^2k$.

Let us now consider progressions such that the progression of $1.mn.m$ has carry mk and the progression of $1.n.mn$ has carry m^2k.

$12 + 24k$	$1 + 3k.4 + 9k.4 + 8k$	$1k$
$12 + 24k$	$1 + 3k.4 + 9k.4 + 8k$	$1k$
$24 + 210k$	$1 + 12k.8 + 72k.4 + 35k$	$2k$
$28 + 315k$	$1 + 14k.4 + 49k.8 + 90k$	$4k$
$36 + 720k$	$1 + 27k.12 + 243k.4 + 80k$	$3k$
$44 + 1232k$	$1 + 33k.4 + 121k.12 + 336k$	$9k$
$48 + 1716k$	$1 + 48k.16 + 576k.4 + 143k$	$4k$
$60 + 3135k$	$1 + 60k.4 + 225k.16 + 836k$	$16k$
$60 + 3360k$	$1 + 75k.20 + 1125k.4 + 224k$	$5k$
$76 + 6384k$	$1 + 95k.4 + 361k.20 + 1680k$	$25k$

$12 + 24k$	$1 + 3k.4 + 9k.4 + 8k$	$1k$
$24 + 210k$	$1 + 12k.8 + 72k.4 + 35k$	$2k$
$36 + 720k$	$1 + 27k.12 + 243k.4 + 80k$	$3k$
$48 + 1716k$	$1 + 48k.16 + 576k.4 + 143k$	$4k$
$60 + 3360k$	$1 + 75k.20 + 1125k.4 + 224k$	$5k$

Theorem 71. *The number*

$$1 + 3n^2k.4n + 9n^3k.4 + (9n^2 - 1)k$$

is 1-phoenix in base $b = 12n + (3n - 1)(3n)(3n + 1)k$ with carry $c = nk$.

Now consider $m = 5$.

$20 + 60k$	$1 + 4k.5 + 16k.5 + 15k$	$1k$
$40 + 504k$	$1 + 16k.10 + 128k.5 + 63k$	$2k$
$60 + 1716k$	$1 + 36k.15 + 432k.5 + 143k$	$3k$
$80 + 4080k$	$1 + 64k.20 + 1024k.5 + 255k$	$4k$
$100 + 7980k$	$1 + 100k.25 + 2000k.5 + 399k$	$5k$

Theorem 72. *The number*

$$1 + 4m^2k.5m + 16m^3k.5 + (16m^2 - 1)k$$

is 1-phoenix in base $b = 20m + (4m - 1)(4m)(4m + 1)k$ with carry $c = mk$.

Theorem 73. *Let m and n be positive integers.*

1. Suppose $m \geq 1$ and $n \geq 3$ and let $u = m(n - 1)$. Then the number

$$1 + muk.mn + mu^2k.n + (u^2 - 1)k$$

is 1-phoenix in base $b = nu + u(u^2 - 1)k$ with carry $c = mk$.

2. *Suppose either $m = 1$ and $n \geq 3$ or $m, n \geq 2$. Let $v = mn - 1$. Then the number*

$$1 + mvk.n + v^2k.mn + m(v^2 - m^2)k$$

is 1-phoenix in base $b = nv + v(v^2 - m^2)k$, with carry $c = m^2k$.

The arithmetic extensions of $1.m.n$ and $1.n - 1.m + 1$ in base $b = m(n - 1)$ do not seem to be related.

$6 + 6k$	$1 + 2k.3 + 4k.3 + 3k$	$1k$
$6 + 15k$	$1 + 6k.2 + 9k.4 + 10k$	$4k$
$8 + 312k$	$1 + 80k.2 + 128k.5 + 195k$	$50k$
$8 + 440k$	$1 + 96k.4 + 256k.3 + 165k$	$36k$
$9 + 585k$	$1 + 108k.3 + 243k.4 + 260k$	$48k$
$10 + 80k$	$1 + 15k.2 + 25k.6 + 48k$	$9k$
$10 + 910k$	$1 + 150k.5 + 500k.3 + 273k$	$45k$
$12 + 24k$	$1 + 3k.4 + 9k.4 + 8k$	$1k$
$12 + 1428k$	$1 + 180k.3 + 432k.5 + 595k$	$75k$
$12 + 60k$	$1 + 8k.6 + 32k.3 + 15k$	$2k$
$12 + 1140k$	$1 + 168k.2 + 288k.7 + 665k$	$98k$
$14 + 231k$	$1 + 28k.2 + 49k.8 + 132k$	$16k$
$14 + 2618k$	$1 + 294k.7 + 1372k.3 + 561k$	$63k$
$15 + 105k$	$1 + 10k.3 + 25k.6 + 42k$	$4k$
$15 + 3135k$	$1 + 300k.5 + 1125k.4 + 836k$	$80k$
$16 + 2800k$	$1 + 288k.2 + 512k.9 + 1575k$	$162k$
$16 + 3952k$	$1 + 384k.8 + 2048k.3 + 741k$	$72k$
$16 + 3696k$	$1 + 320k.4 + 1024k.5 + 1155k$	$100k$
$18 + 210k$	$1 + 18k.9 + 108k.3 + 35k$	$3k$
$18 + 504k$	$1 + 45k.2 + 81k.10 + 280k$	$25k$
$18 + 693k$	$1 + 54k.6 + 243k.4 + 154k$	$12k$
$18 + 4950k$	$1 + 378k.3 + 972k.7 + 1925k$	$147k$
$20 + 60k$	$1 + 4k.5 + 16k.5 + 15k$	$1k$
$20 + 910k$	$1 + 60k.4 + 200k.6 + 273k$	$18k$
$20 + 5580k$	$1 + 440k.2 + 800k.11 + 3069k$	$242k$
$20 + 7820k$	$1 + 600k.10 + 4000k.3 + 1173k$	$90k$
$21 + 7917k$	$1 + 504k.3 + 1323k.8 + 3016k$	$192k$
$21 + 8925k$	$1 + 588k.7 + 3087k.4 + 1700k$	$112k$
$22 + 935k$	$1 + 66k.2 + 121k.12 + 510k$	$36k$
$22 + 10450k$	$1 + 726k.11 + 5324k.3 + 1425k$	$99k$
$24 + 210k$	$1 + 12k.8 + 72k.4 + 35k$	$2k$
$24 + 440k$	$1 + 24k.3 + 64k.9 + 165k$	$9k$
$24 + 504k$	$1 + 32k.12 + 256k.3 + 63k$	$4k$

$24 + 9768k$	$1 + 624k.2 + 1152k.13 + 5291k$	$338k$
$24 + 12648k$	$1 + 672k.4 + 2304k.7 + 3689k$	$196k$
$24 + 13224k$	$1 + 720k.6 + 3456k.5 + 2755k$	$150k$

Let's look at $1.2.n$ and $1.n - 1.3$ in base $b = 2n$.

$6 + 15k$	$1 + 6k.2 + 9k.4 + 10k$	$4k$
$6 + 6k$	$1 + 2k.3 + 4k.3 + 3k$	$1k$
$8 + 312k$	$1 + 80k.2 + 128k.5 + 195k$	$50k$
$8 + 440k$	$1 + 96k.4 + 256k.3 + 165k$	$36k$
$10 + 80k$	$1 + 15k.2 + 25k.6 + 48k$	$9k$
$10 + 910k$	$1 + 150k.5 + 500k.3 + 273k$	$45k$
$12 + 1140k$	$1 + 168k.2 + 288k.7 + 665k$	$98k$
$12 + 60k$	$1 + 8k.6 + 32k.3 + 15k$	$2k$
$14 + 231k$	$1 + 28k.2 + 49k.8 + 132k$	$16k$
$14 + 2618k$	$1 + 294k.7 + 1372k.3 + 561k$	$63k$
$16 + 2800k$	$1 + 288k.2 + 512k.9 + 1575k$	$162k$
$16 + 3952k$	$1 + 384k.8 + 2048k.3 + 741k$	$72k$
$18 + 504k$	$1 + 45k.2 + 81k.10 + 280k$	$25k$
$18 + 210k$	$1 + 18k.9 + 108k.3 + 35k$	$3k$
$20 + 5580k$	$1 + 440k.2 + 800k.11 + 3069k$	$242k$
$20 + 7820k$	$1 + 600k.10 + 4000k.3 + 1173k$	$90k$
$22 + 935k$	$1 + 66k.2 + 121k.12 + 510k$	$36k$
$22 + 10450k$	$1 + 726k.11 + 5324k.3 + 1425k$	$99k$
$24 + 9768k$	$1 + 624k.2 + 1152k.13 + 5291k$	$338k$
$24 + 504k$	$1 + 32k.12 + 256k.3 + 63k$	$4k$
$26 + 1560k$	$1 + 91k.2 + 169k.14 + 840k$	$49k$
$26 + 17342k$	$1 + 1014k.13 + 8788k.3 + 2001k$	$117k$
$28 + 15652k$	$1 + 840k.2 + 1568k.15 + 8385k$	$450k$
$28 + 21700k$	$1 + 1176k.14 + 10976k.3 + 2325k$	$126k$
$30 + 2415k$	$1 + 120k.2 + 225k.16 + 1288k$	$64k$
$30 + 990k$	$1 + 50k.15 + 500k.3 + 99k$	$5k$
$32 + 23520k$	$1 + 1088k.2 + 2048k.17 + 12495k$	$578k$
$32 + 32480k$	$1 + 1536k.16 + 16384k.3 + 3045k$	$144k$

7   1-Phoenix with four digits

In this section we examine arithmetic extensions of 1-phoenix numbers of the form $1.x.y.n$ for n a fixed integer. Recall that $1.k.kn.n.$ is 1-phoenix in base $b = k(n^2 - 1)$. See Section 4.

$$
\begin{array}{|c c c c|}
\hline
(c) & & & \\
& y & x & 1 \\
& & & n \\
\hline
1 & x & y & n \\
\hline
\end{array}
$$

If $1.x.y.n$ is to be 1-phoenix in base b, with carry c, then

$$
x = \frac{ncb + b - c}{n^2 - 1}, \tag{10a}
$$

$$
y = \frac{nb + cb - nc}{n^2 - 1}, \tag{10b}
$$

where $n \geq 2$, $3 \leq b \leq n^2 - 1$, $0 \leq c < b$. A computer search yields the following theorem.

Theorem 74 (1-Phoenix numbers of the form $1.x.y.n$). *The numbers of the form $1.x.y.n$, with x and y determined by (10), are 1-phoenix in base b with carry c. If $c = 0$, then $[0]$ can be inserted in the middle position and, if $c = n - 1$, then $[B]$ can be inserted in the middle position.*

Table 37: 1-Phoenix numbers of the form $1.x.y.n$, $b \leq 60$.

b	N	N_b	c
3	44	1.1.[0].2.2	0
6	314	1.2.[0].4.2	0
6	412	1.5.[5].2.4	3
6	417	1.5.[5].3.3	2
8	603	1.1.[0].3.3	0
9	1028	1.3.[0].6.2	0
12	2402	1.4.[0].8.2	0
14	5211	1.12.[13].8.3	2
15	3664	1.1.[0].4.4	0
15	4652	1.5.[0].10.2	0
16	4707	1.2.[0].6.3	0
17	6980	1.7.2.10	4
18	7994	1.6.[0].12.2	0
18	9119	1.10.2.11	6
18	11095	1.16.[17].4.7	6
19	9468	1.7.4.6	2
20	12887	1.12.4.7	4
21	12644	1.7.[0].14.2	0
21	17392	1.18.[20].9.4	3
22	15625	1.10.6.5	2

Table 37 continued.

b	N	N_b	c
22	20133	1.19.[21].13.3	2
24	14525	1.1.[0].5.5	0
24	15771	1.3.[0].9.3	0
24	18818	1.8.[0].16.2	0
25	29556	1.22.[24].7.6	5
27	26732	1.9.[0].18.2	0
27	31440	1.16.3.12	7
30	29044	1.2.[0].8.4	0
30	36602	1.10.[0].20.2	0
30	50943	1.26.[29].18.3	2
31	37584	1.8.3.12	3
32	37251	1.4.[0].12.3	0
32	42066	1.9.2.18	5
33	48644	1.11.[0].22.2	0
35	44316	1.1.[0].6.6	0
36	63074	1.12.[0].24.2	0
36	87412	1.31.[35].16.4	3
36	89650	1.33.[35].6.10	9
37	78372	1.20.9.6	3
37	83776	1.24.7.8	5
38	68067	1.9.5.9	2
38	93960	1.27.2.24	17
38	103401	1.33.[37].23.3	2
39	80108	1.13.[0].26.2	0
40	72603	1.5.[0].15.3	0
40	86503	1.14.2.23	8
40	94537	1.19.3.17	8
42	99962	1.14.[0].28.2	0
42	100691	1.15.3.17	6
45	97744	1.3.[0].12.4	0
45	122852	1.15.[0].30.2	0
45	149968	1.29.2.28	18
46	142375	1.21.13.5	2
46	171684	1.35.6.12	9
46	183267	1.40.[45].28.3	2
48	113239	1.1.[0].7.7	0
48	115685	1.2.[0].10.5	0
48	125283	1.6.[0].18.3	0
48	145450	1.15.6.10	3
48	148994	1.16.[0].32.2	0

Table 37 continued.

b	N	N_b	c
48	154760	1.19.8.8	3
51	161392	1.11.2.28	6
51	178604	1.17.[0].34.2	0
51	248272	1.44.[50].23.4	3
52	178888	1.14.8.8	2
52	205929	1.24.8.9	4
54	211898	1.18.[0].36.2	0
54	216438	1.20.12.6	2
54	224868	1.23.6.12	5
54	271375	1.39.3.25	18
54	296301	1.47.[53].33.3	2
55	227164	1.20.5.14	5
56	198747	1.7.[0].21.3	0
56	336177	1.51.[55].11.9	8
57	249092	1.19.[0].38.2	0
60	231364	1.4.[0].16.4	0
60	290402	1.20.[0].40.2	0
60	407886	1.53.[59].18.6	5
60	418093	1.56.[59].8.13	12

End of Table 37.

7.1 1-Phoenix of the form $1.x.n.y$

Note the following examples with $b = x + n$ and $y = 2$.

b	N_b	c
17	1.7.2.10	4
99	1.41.2.58	24
577	1.239.2.338	140
3363	1.1393.2.1970	816

Sequences: A002315, A075870, A001541, A005319.

NOTE: A005319 is $a(n) = 6a(n-1) - a(n-2)$ and also integers n such that $2n^2 - 2$ is a perfect square.

$1.x.2.b - x$ is 1-phoenix in base b if and only if

$$(b - x)(x) - 2 = cb,$$
$$(b - x)(2) + c = b + x.$$

Consequently,

Theorem 75. *The number $1.x.2.b - x$ is 1-phoenix in base b if and only if*

$$b = x + \sqrt{2x^2 - 2}, \quad c = 2x - \sqrt{2x^2 - 2}.$$

Numbers of the form 1.x.3.y have no discernable pattern. $b \leq 200$. The appearance of pairs of bases with the same y is interesting.

b	N	1.x.3.y
6	417	1.5.3.3
8	603	1.1.3.3
27	31440	1.16.3.12
31	37584	1.8.3.12
40	94537	1.19.3.17
42	100691	1.15.3.17
54	271375	1.39.3.25
68	365525	1.11.3.25
89	1259748	1.70.3.42
97	1515184	1.64.3.44
102	1675395	1.59.3.45
116	1978425	1.31.3.45
117	1793652	1.14.3.42
117	1930544	1.24.3.44
121	2855412	1.74.3.54
132	4199643	1.109.3.63
141	3479652	1.34.3.54
167	6721820	1.74.3.70
171	6930700	1.66.3.70
178	6178977	1.17.3.63
178	9854341	1.133.3.83
183	11353408	1.156.3.88

Numbers of the form 1.x.4.y have no discernable pattern. $b \leq 200$.

b	N	1.x.4.y
6	314	1.2.4.2
15	3664	1.1.4.4
18	11095	1.16.4.7
19	9468	1.7.4.6
20	12887	1.12.4.7
62	292410	1.14.4.18
66	374900	1.20.4.20
70	607906	1.54.4.26

Table 40: Fundamental solutions to $r^2 - Ds^2 = 1$.

n	D	r	s
4	13	649	180
5	20	9	2
9	68	33	4
16	229	5848201	386460
25	580	289	12
36	1229	923640201	26346740
49	2308	1153	24
64	3973	31309027849	496718460
81	6404	3201	40
100	9805	471028297609	4756890996

72	430580	1.11.4.20
72	632762	1.50.4.26
84	1094048	1.71.4.32
86	873099	1.32.4.27
91	1225984	1.57.4.32
136	3033927	1.28.4.39
149	3641600	1.15.4.40
150	4478146	1.49.4.46
155	6679628	1.123.4.58
162	7112830	1.109.4.58
168	5363278	1.22.4.46
178	10709973	1.160.4.69
193	10207060	1.81.4.62
198	14506348	1.172.4.76
200	12640869	1.116.4.69

Motivated by 1.2.4.2 in base $b = 6$ and 1.20.4.20 in base $b = 66$, consider finding all numbers of the form $1.x.n.x$ in some base b. If $y^2 - Db^2 = 4n$, $D = (n-1)^2 + 4$, has a solution, then $x = (y - (n-1)b)/2$ and $c = b - (n-1)x$.

If $n = m^2$, then $b = m(m^2 - 1)$ and $y = m(m^4 - 2m^2 + 3)$, then $x = m$, $c = 0$, and $1.m.m^2.m$ is 1-phoenix in base $b = m(m^2 - 1)$. If $n = m^2$, then the fundamental solutions are $(y_0, b_0) = (2m, 0), (m(m^4 - 2m^2 + 3), \pm m(m^2 - 1))$. If $n = 4$, for example, then the fundamental solutions are $(y_0, b_0) = (4, 0), (22, \pm 6)$. If $n = 5$, then $y = 170$ and $b = 38$ but then $x = 9$ and $c = 2$ so 1.9.5.9 in base $b = 38$. There are no other fundamental solutions.

If n is odd square, $n = 2m + 1$, then $r = 2m^2 + 1$ and $s = m$ are fundamental solutions to $r^2 - [(n-1)^2 + 4]s^2 = 1$. However, if n is even square, then, empirically at least, the fundamental solutions get very large, and there appears to be no general formula. Once the fundamental solutions are found, all other solutions are found by Theorem 104.

The sequence of $D = (n-1)^2 + 4$, n a square, is a subsequence of A087475, $a(n) = n^2 + 4$. Note that 13, 229 and 1229 are primes p of the form $4n + 1$, A002144, and solutions to $x^2 - py^2 = 1$ are in A082393 and A081232.

$$b \le 200$$

b	N	$1.x.5.y$
24	14525	1.1.5.5
38	68067	1.9.5.9
55	227164	1.20.5.14
75	613520	1.34.5.20
98	1431513	1.51.5.27
124	2998975	1.71.5.35
129	2380336	1.14.5.28
153	5782832	1.94.5.44
185	10439604	1.120.5.54
199	10020096	1.54.5.48

It turns out that $1.x.5.y$ has a polynomial subsequence with rational coefficients.

Theorem 76 (1-Phoenix numbers of the form $1.x.5.y$)**.** *The number $1.x.5.y$ is 1-phoenix in base $b = \dfrac{1}{2}(n + 2)(3n + 13)$, where*

$$x = (3/2)n^2 + (7/2)n - 4,$$
$$y = (1/2)n^2 + (5/2)n + 2,$$
$$c = (1/2)(n - 1)(n + 2).$$

The exceptions to Theorem 76 for $b \le 500$.

b	N	$1.x.4.y$
129	2380336	1.14.5.28
199	10020096	1.54.5.48
260	18861755	1.19.5.55
284	31699301	1.109.5.73
310	34117119	1.45.5.69
384	83019751	1.179.5.103
408	84732313	1.101.5.97
431	84523500	1.24.5.90
470	134530565	1.139.5.115
499	189990396	1.264.5.138

Theorem 77. *The number*
$$1.9 + 5n.5.9 + 15n + 4n^2$$
is 1-phoenix in base $b = 38 + 71n + 20n^2$ *with carry* $c = 2 + n$, $n \geq 0$.

Proof. Exercise. □

The reader is encouraged to look for more examples!

7.2 1-Phoenix of the form $1.x.y.xy$

It is interesting to observe that $1.9.2.18$ in base $b = 32$ and $1.14.3.42$ in base $b = 117$ have the property that the product of the inner digits is the last digit. We show that these are the first of infinitely many examples resulting from the solution of a generalized Pell equation. The multiplication takes the form

$$
\begin{array}{c|ccc}
(c) & & & \\
 & y & x & 1 \\
 & & & xy \\
\hline
1 & x & y & xy \\
\end{array}
$$

Base b to be determined.

$$
\begin{aligned}
xy \cdot x &= (x^2 y - y) + y \\
&= y(x^2 - 1) + y
\end{aligned}
$$

Let $y(x^2 - 1) = cb$ so that

$$xy \cdot y + c = (xy^2 + c - x) + x$$

Thus, $b = xy^2 + c - x$ for some carry digit c. An examination of the multiplication for our two examples above shows that $c - x = -y^2$ so $c = x - y^2$ and $b = y^2(x - 1)$. Consequently,

$$
\begin{aligned}
y(x^2 - 1) &= (x - y^2) \cdot y^2(x - 1) \\
x + 1 &= (x - y^2)y \\
x &= \frac{y^3 + 1}{y - 1} \\
&= y^2 + y + 1 + \frac{2}{y - 1}.
\end{aligned}
$$

Thus, either $y - 1 = 1$ so $y = 2$, $x = 9$, $b = 2^2(9 - 1) = 32$ or $y - 1 = 2$ so $y = 3$, $x = 14$, $b = 3^2(14 - 1) = 117$.

However, it need not be true that $c = x - y^2$. The equation is

$$y(x^2 - 1) = c(x(y^2 - 1) + c). \tag{11}$$

or equivalently,

$$c^2 + x(y^2 - 1)c - y(x^2 - 1) = 0.$$

Solving for c,

$$\begin{aligned}
c &= \frac{-x(y^2 - 1) \pm \sqrt{x^2(y^2 - 1)^2 + 4y(x^2 - 1)}}{2} \\
&= \frac{-x(y^2 - 1) \pm \sqrt{[(y^2 - 1)^2 + 4y]x^2 - 4y}}{2}
\end{aligned}$$

and so $[(y^2 - 1)^2 + 4y]x^2 - 4y = u^2$, or, equivalently, $[(y^2 - 1)^2 + 4y]x^2 - u^2 = 4y$. If y is even, then

$$c = \frac{-x(y^2 - 1) + u}{2}, \quad [(y^2 - 1)^2 + 4y]x^2 - u^2 = 4y,$$

and if y is odd, then

$$c = -x(y^2 - 1)/2 + u, \quad [((y^2 - 1)/2)^2 + y]x^2 - u^2 = y.$$

The author avoids the task of solving this generally. Fortunately, the excellent web site <http://www.alpertron.com.ar/QUAD.HTM> has a Java applet that will solve your quadratic Diophantine equations for you! In general, the solution is of the form

$$\begin{bmatrix} x_{n+1} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} P & Q \\ R & P \end{bmatrix}^n \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$$

where (x_0, u_0) are fundamental solutions.

Since (11) is true whenever x, c have the same sign, we use the negative x_0 with the understanding that absolute values are taken and that the carry c is the minimum and the base $b = x(y^2 - 1) + c$ is the maximum of the absolute values. Here are some solutions for $y = 2, 3$.

The fundamental solutions for $y = 2$ and $u^2 - 17x^2 = -8$ are $(\pm 3, 1)$.

	$y = 2$ with $(\pm 3, 1)$	
b	c	$1.x.y.xy$
32	5	1.9.2.18
203	32	1.57.2.114
2112	333	1.593.2.1186
13395	2112	1.3761.2.7522
139360	21973	1.39129.2.78258
883867	139360	1.248169.2.496338
9195648	1449885	1.2581921.2.5163842
58321827	9195648	1.16375393.2.32750786

The fundamental solutions for $y = 3$ and $u^2 - 19x^2 = -3$ are $(\pm 4, 1)$.

<div align="center">

$y = 3$ with $(\pm 4, 1)$

b	c	$1.x.y.xy$
117	5	1.14.3.42
2725	117	1.326.3.978
39780	1708	1.4759.3.14277
926492	39780	1.110839.3.332517
13525083	580715	1.1618046.3.4854138
315004555	13525083	1.37684934.3.113054802
4598488440	197441392	1.550130881.3.1650392643
107100622208	4598488440	1.12812766721.3.38438300163

</div>

Just for reference, here is the output from http://www.alpertron.com.ar/QUAD.HTM for $y = 4$, $u^2 - 241x^2 = -16$, with some editing for superscripts and subscripts.

```
x^2 - 241 y^2 + 16 = 0


by Dario Alejandro Alpern


X_0 = -15
Y_0 = 1

and also:
X_0 = 15
Y_0 = 1

and also:
X_0 = 284 044272 (9 digits)
Y_0 = 18 296900 (8 digits)

If (x,y) is a solution, (-x,-y) is also a solution.

X_{n+1} = P X_n + Q Y_n
Y_{n+1} = R X_n + S Y_n

P = 10085 143557 001249 (17 digits)
Q = 156563 530415 748600 (18 digits)
R = 649 641205 044600 (15 digits)
S = 10085 143557 001249 (17 digits)
```

7.3 Arithmetic extension of the first kind $1.x.y.n$

Let us now extend $1.x.y.n$ in base $b = b_0$ to an arithmetic sequence $1.x + x_1k.y + y_1k.n$ of 1-phoenix numbers in base $b = b_0 + b_1k$.

$$
\begin{array}{|ccc|}
\hline
(c + dk) & & \\
y + y_1k. & x + x_1k. & 1 \\
& & n \\
\hline
1. \quad x + x_1k. & y + y_1k. & n \\
\end{array}
$$

Base $b = b_0 + b_1k$.

The equations

$$n(x + x_1k) - (y + y_1k) = (c + dk)(b_0 + b_1k),$$
$$n(y + y_1k) - (x + x_1k) = (1)(b_0 + b_1k).$$

and are true for all k if and only if

$$b_0 = \frac{n^3y_1 + n^2b_1\,y - n^2b_1 - ny_1 - b_1y}{y_1\,(n^2 - 1)} = n + \frac{n^2y - n^2 - y}{y_1}\left(\frac{b_1}{n^2 - 1}\right)$$

$$c = \frac{n^2y_1 - nb_1 - y_1}{b_1} = \left(\frac{n^2 - 1}{b_1}\right)y_1 - n,$$

$$d = 0,$$

$$x = \frac{n^4y_1^2 + n^3b_1yy_1 - 2n^3b_1\,y_1 - n^2b_1^2y + n^2b_1^2 - 2\,n^2y_1^2 - nyy_1b_1 + 2\,nb_1y_1 + b_1^2y + y_1^2}{y_1(n^2 - 1)b_1},$$

$$= \frac{n^4y_1^2 - 2n^2y_1^2 + y_1^2}{y_1(n^2 - 1)b_1} + \frac{n^3b_1yy_1 - 2n^3b_1y_1 - nyy_1b_1 + 2\,nb_1y_1}{y_1(n^2 - 1)b_1} + \frac{-n^2b_1^2y + n^2b_1^2 + b_1^2y}{y_1(n^2 - 1)b_1},$$

$$= \frac{n^4 - 2n^2 + 1}{(n^2 - 1)b_1}y_1 + +\frac{n^3y - 2n^3 - ny + 2\,n}{n^2 - 1} + \frac{-n^2y + n^2 + y}{y_1(n^2 - 1)}b_1,$$

$$= \left(\frac{n^2 - 1}{b_1}\right)y_1 + ny - 2n - \frac{n^2y - n^2 - y}{y_1}\left(\frac{b_1}{n^2 - 1}\right),$$

$$x_1 = ny_1 - b_1.$$

It is clear that if $(n^2 - 1)/b_1$ and $b_1/(n^2 - 1)$ are to be integers, then $b_1 = n^2 - 1$. Thus, once x and y are found from (10) (so that b_0 and c are also known), we have simply

$$b_1 = n^2 - 1, \tag{12a}$$
$$x_1 = cn + 1, \tag{12b}$$
$$y_1 = n + c, \tag{12c}$$
$$d = 0, \tag{12d}$$

where $n \geq 2$, $0 \leq c < b_0$, and $((n^2 - 1)y - n^2) \mod (n + c) = 0$. A computer search yields the following.

Theorem 78 (Arithmetic extension of the first kind of the 1-phoenix numbers $1.x.y.n$). *The following numbers of the form $1.x + x_1k.y + y_1k.n$, $2 \leq n \leq 12$, are 1-phoenix in base $b = b_0 + (n^2 - 1)k$, with carry digit c. If $c = 0$, then [0] can be inserted in the middle position. See Theorem 24. If $c = n - 1$, then [B], where $B = b - 1$ is the maximal digit, can be inserted in the middle position.*

1. $1.1 + 1k.[0].2 + 2k.2$ *in base* $b = 3 + 3k$, *with carry* $c = 0$.

2. $1.5 + 7k.[B].3 + 5k.3$ *in base* $b = 6 + 8k$, *with carry* $c = 2$.

3. $1.1 + 1k.[0].3 + 3k.3$ *in base* $b = 8 + 8k$, *with carry* $c = 0$.

4. $1.5 + 13k.[B].2 + 7k.4$ *in base* $b = 6 + 15k$, *with carry* $c = 3$.

5. $1.1 + 1k.[0].4 + 4k.4$ *in base* $b = 15 + 15k$, *with carry* $c = 0$.

6. $1.10 + 11k.6 + 7k.5$ *in base* $b = 22 + 24k$, *with carry* $c = 2$.

7. $1.16 + 43k.[B].4 + 13k.7$ *in base* $b = 18 + 48k$, *with carry* $c = 6$.

8. $1.1 + 1k.[0].5 + 5k.5$ *in base* $b = 24 + 24k$, *with carry* $c = 0$.

9. $1.7 + 13k.4 + 8k.6$ *in base* $b = 19 + 35k$, *with carry* $c = 2$.

10. $1.22 + 31k.[B].7 + 11k.6$ *in base* $b = 25 + 35k$, *with carry* $c = 5$.

11. $1.1 + 1k.[0].6 + 6k.6$ *in base* $b = 35 + 35k$, *with carry* $c = 0$.

12. $1.12 + 29k.4 + 11k.7$ *in base* $b = 20 + 48k$, *with carry* $c = 4$.

13. $1.1 + 1k.[0].7 + 7k.7$ *in base* $b = 48 + 48k$, *with carry* $c = 0$.

14. $1.24 + 41k.7 + 13k.8$ *in base* $b = 37 + 63k$, *with carry* $c = 5$.

15. $1.19 + 25k.8 + 11k.8$ *in base* $b = 48 + 63k$, *with carry* $c = 3$.

16. $1.14 + 17k.8 + 10k.8$ *in base* $b = 52 + 63k$, *with carry* $c = 2$.

17. $1.1 + 1k.[0].8 + 8k.8$ *in base* $b = 63 + 63k$, *with carry* $c = 0$.

18. $1.9 + 19k.5 + 11k.9$ *in base* $b = 38 + 80k$, *with carry* $c = 2$.

19. $1.24 + 37k.8 + 13k.9$ *in base* $b = 52 + 80k$, *with carry* $c = 4$.

20. $1.51 + 73k.[B].11 + 17k.9$ *in base* $b = 56 + 80k$, *with carry* $c = 8$.

21. $1.1 + 1k.[0].9 + 9k.9$ *in base* $b = 80 + 80k$, *with carry* $c = 0$.

22. $1.7 + 41k.2 + 14k.10$ *in base* $b = 17 + 99k$, *with carry* $c = 4$.

23. $1.33 + 91k.[B].6 + 19k.10$ *in base* $b = 36 + 99k$, *with carry* $c = 9$.

24. $1.15 + 31k.6 + 13k.10$ *in base* $b = 48 + 99k$, *with carry* $c = 3$.

25. $1.53 + 71k.12 + 17k.10$ *in base* $b = 74 + 99k$, *with carry* $c = 7$.

26. $1.48 + 61k.12 + 16k.10$ *in base* $b = 78 + 99k$, *with carry* $c = 6$.

27. $1.1 + 1k.[0].10 + 10k.10$ *in base* $b = 99 + 99k$, *with carry* $c = 0$.

28. $1.10 + 67k.2 + 17k.11$ *in base* $b = 18 + 120k$, *with carry* $c = 6$.

29. $1.18 + 23k.10 + 13k.11$ *in base* $b = 94 + 120k$, *with carry* $c = 2$.

30. $1.83 + 89k.17 + 19k.11$ *in base* $b = 112 + 120k$, *with carry* $c = 8$.

31. $1.1 + 1k.[0].11 + 11k.11$ *in base* $b = 120 + 120k$, *with carry* $c = 0$.

32. $1.16 + 85k.3 + 19k.12$ *in base* $b = 27 + 143k$, *with carry* $c = 7$.

33. $1.8 + 37k.3 + 15k.12$ *in base* $b = 31 + 143k$, *with carry* $c = 3$.

34. $1.35 + 109k.6 + 21k.12$ *in base* $b = 46 + 143k$, *with carry* $c = 9$.

35. $1.23 + 61k.6 + 17k.12$ *in base* $b = 54 + 143k$, *with carry* $c = 5$.

36. $1.42 + 97k.8 + 20k.12$ *in base* $b = 62 + 143k$, *with carry* $c = 8$.

37. $1.11 + 25k.6 + 14k.12$ *in base* $b = 63 + 143k$, *with carry* $c = 2$.

38. $1.92 + 133k.[B].15 + 23k.12$ *in base* $b = 99 + 143k$, *with carry* $c = 11$.

39. $1.1 + 1k.[0].12 + 12k.12$ *in base* $b = 143 + 143k$, *with carry* $c = 0$.

Remark 79. It remains to be seen if patterns similar to those of Section **??** are present whenever $c > 0$.

Proof. By multiplication. Just a few examples are provided.

	(2)	(2)		
	$3 + 5k$.	$[5 + 8k]$.	$5 + 7k$.	1
				3
1.	$5 + 7k$.	$[5 + 8k]$.	$3 + 5k$.	3

Base $b = 6 + 8k$.

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$$3(5 + 7k) - (3 + 5k) = 15 + 21k - 3 - 5k$$
$$= (12 + 16k)$$
$$= (2)(6 + 8k).$$
$$3(5 + 8k) + 2 - (5 + 8k) = 2(5 + 8k) + 2$$
$$= (2)(6 + 8k).$$
$$3(3 + 5k) + 2 - (5 + 7k) = 9 + 15k + 2 - 5 - 7k$$
$$= (1)(6 + 8k).$$

An interesting observation is that while 1.[5].3.3 in base 6, the arithmetic extension is $1.5 + 7k.3 + 5k.3$, and it can be easily checked that $1.5 + 7k.5 + 7k.3.3$ is *not* 1-phoenix. However, $1.5 + 7k.[5 + 8k].3 + 5k.3$ is 1-phoenix in base $b = 6 + 8k$.

	(3)	(3)		
	$2 + 7k$.	$[5 + 15k]$.	$5 + 13k$.	1
				3
1.	$5 + 13k$.	$[5 + 15k]$.	$2 + 7k$.	4

Base $b = 6 + 15k$.

$$4(5 + 13k) - (2 + 7k) = 20 + 52k - 2 - 7k$$
$$= 18 + 45k$$
$$= (3)(6 + 15k).$$
$$4(5 + 15k) + 3 - (5 + 15k) = 3(5 + 15k) + 3$$
$$= (3)(6 + 15k).$$
$$4(2 + 7k) + 3 - (5 + 13k) = 8 + 28k + 3 - 5 - 13k$$
$$= (1)(6 + 15k).$$

	(2)		
	$6 + 7k$.	$10 + 11k$.	1
			5
1.	$10 + 11k$.	$6 + 7k$.	5

Base $b = 22 + 24k$.

$$5(10 + 11k) - (6 + 7k) = 50 + 55k - 6 - 7k$$
$$= 44 + 48k$$
$$= (2)(22 + 24k).$$
$$5(6 + 7k) + 2 = 30 + 35k + 2 - 10 - 11k$$
$$= (1)(22 + 24k).$$

	(5)		(5)		
	$7 + 11k.$	$[24 + 35k].$	$22 + 31k.$	1
				6
1.	$22 + 31k.$	$[24 + 35k].$	$7 + 11k.$	6

Base $b = 25 + 35k.$

$$6(22 + 31k) - (7 + 11k) = 132 + 186k - 7 - 11k$$
$$= 125 + 175k$$
$$= (5)(25 + 35k).$$
$$6(24 + 35k) + 5 - (24 + 35k) = 5(24 + 35k) + 5$$
$$= (5)(25 + 35k).$$
$$6(7 + 11k) + 5 - (22 + 31k) = 42 + 66k + 5 - 22 - 31k$$
$$= 25 + 35k. \qquad \square$$

7.4   Arithmetic extension of the second kind of $1.x.y.n$

Search out to $b_0 \leq 10^5$.

$6 + 12k$	$1 + 3k.2 + 6k.4 + 9k.2 + 4k.$	$1k$	$0 + 2k$
$12 + 210k$	$1 + 24k.4 + 84k.8 + 144k.2 + 35k.$	$4k$	$0 + 14k$
$18 + 360k$	$1 + 27k.6 + 135k.12 + 243k.2 + 40k.$	$3k$	$0 + 15k$
$9 + 693k$	$1 + 108k.3 + 297k.6 + 486k.2 + 154k.$	$24k$	$0 + 66k$
$30 + 1680k$	$1 + 75k.10 + 600k.20 + 1125k.2 + 112k.$	$5k$	$0 + 40k$
$24 + 1716k$	$1 + 96k.8 + 624k.16 + 1152k.2 + 143k.$	$8k$	$0 + 52k$
$15 + 3315k$	$1 + 300k.5 + 1275k.10 + 2250k.2 + 442k.$	$40k$	$0 + 170k$
$42 + 4620k$	$1 + 147k.14 + 1617k.28 + 3087k.2 + 220k.$	$7k$	$0 + 77k$
$36 + 5814k$	$1 + 216k.12 + 2052k.24 + 3888k.2 + 323k.$	$12k$	$0 + 114k$
$21 + 9177k$	$1 + 588k.7 + 3381k.14 + 6174k.2 + 874k.$	$56k$	$0 + 322k$
$54 + 9828k$	$1 + 243k.18 + 3402k.36 + 6561k.2 + 364k.$	$9k$	$0 + 126k$
$48 + 13800k$	$1 + 384k.16 + 4800k.32 + 9216k.2 + 575k.$	$16k$	$0 + 200k$
$66 + 17952k$	$1 + 363k.22 + 6171k.44 + 11979k.2 + 544k.$	$11k$	$0 + 187k$
$27 + 19575k$	$1 + 972k.9 + 7047k.18 + 13122k.2 + 1450k.$	$72k$	$0 + 522k$
$6 + 6k$	$1 + 2k.5 + 6k.3 + 4k.3 + 3k.$	$1k$	$2 + 3k$
$24 + 168k$	$1 + 8k.3 + 24k.9 + 64k.3 + 21k.$	$1k$	$0 + 3k$
$8 + 440k$	$1 + 72k.1 + 88k.3 + 192k.3 + 165k.$	$27k$	$0 + 33k$
$30 + 990k$	$1 + 60k.26 + 880k.18 + 600k.3 + 99k.$	$6k$	$2 + 88k$
$14 + 2618k$	$1 + 336k.12 + 2380k.8 + 1568k.3 + 561k.$	$72k$	$2 + 510k$
$16 + 3952k$	$1 + 288k.2 + 608k.6 + 1536k.3 + 741k.$	$54k$	$0 + 114k$
$48 + 4080k$	$1 + 96k.6 + 544k.18 + 1536k.3 + 255k.$	$6k$	$0 + 34k$
$102 + 5610k$	$1 + 102k.89 + 4930k.63 + 3468k.3 + 165k.$	$3k$	$2 + 145k$

$54 + 5814k$	$1 + 198k.47 + 5130k.33 + 3564k.3 + 323k.$	$11k$	$2 + 285k$
$22 + 10450k$	$1 + 858k.19 + 9350k.13 + 6292k.3 + 1425k.$	$117k$	$2 + 1275k$
$96 + 10912k$	$1 + 128k.12 + 1408k.36 + 4096k.3 + 341k.$	$4k$	$0 + 44k$
$72 + 13800k$	$1 + 216k.9 + 1800k.27 + 5184k.3 + 575k.$	$9k$	$0 + 75k$
$46 + 13846k$	$1 + 552k.40 + 12236k.28 + 8464k.3 + 903k.$	$36k$	$2 + 798k$
$78 + 17550k$	$1 + 416k.68 + 15444k.48 + 10816k.3 + 675k.$	$16k$	$2 + 594k$
$32 + 32480k$	$1 + 1152k.4 + 4480k.12 + 12288k.3 + 3045k.$	$108k$	$0 + 420k$
$38 + 54530k$	$1 + 2622k.33 + 48298k.23 + 33212k.3 + 4305k.$	$207k$	$2 + 3813k$
$168 + 58520k$	$1 + 392k.21 + 7448k.63 + 21952k.3 + 1045k.$	$7k$	$0 + 133k$
$40 + 63640k$	$1 + 1800k.5 + 8600k.15 + 24000k.3 + 4773k.$	$135k$	$0 + 645k$
$120 + 63960k$	$1 + 600k.15 + 8200k.45 + 24000k.3 + 1599k.$	$15k$	$0 + 205k$
$126 + 74046k$	$1 + 1092k.110 + 65016k.78 + 45864k.3 + 1763k.$	$26k$	$2 + 1548k$
$6 + 15k$	$1 + 6k.5 + 15k.2 + 9k.4 + 10k.$	$4k$	$3 + 10k$
$36 + 360k$	$1 + 18k.31 + 315k.16 + 162k.4 + 40k.$	$2k$	$3 + 35k$
$60 + 840k$	$1 + 15k.4 + 60k.16 + 225k.4 + 56k.$	$1k$	$0 + 4k$
$15 + 3135k$	$1 + 240k.1 + 285k.4 + 900k.4 + 836k.$	$64k$	$0 + 76k$
$30 + 3315k$	$1 + 120k.2 + 255k.8 + 900k.4 + 442k.$	$16k$	$0 + 34k$
$66 + 7161k$	$1 + 198k.57 + 6237k.30 + 3267k.4 + 434k.$	$12k$	$3 + 378k$
$21 + 8925k$	$1 + 756k.18 + 7875k.9 + 3969k.4 + 1700k.$	$144k$	$3 + 1500k$
$96 + 13800k$	$1 + 264k.83 + 12000k.44 + 6336k.4 + 575k.$	$11k$	$3 + 500k$
$120 + 26970k$	$1 + 240k.8 + 1860k.32 + 7200k.4 + 899k.$	$8k$	$0 + 62k$
$156 + 29640k$	$1 + 351k.135 + 25740k.72 + 13689k.4 + 760k.$	$9k$	$3 + 660k$
$216 + 31482k$	$1 + 270k.187 + 27324k.100 + 14580k.4 + 583k.$	$5k$	$3 + 506k$
$180 + 45540k$	$1 + 270k.12 + 3105k.48 + 12150k.4 + 1012k.$	$6k$	$0 + 69k$
$70 + 2730k$	$1 + 56k.32 + 1260k.20 + 784k.5 + 195k.$	$4k$	$2 + 90k$
$120 + 2760k$	$1 + 24k.5 + 120k.25 + 576k.5 + 115k.$	$1k$	$0 + 5k$
$22 + 3366k$	$1 + 220k.10 + 1584k.6 + 968k.5 + 765k.$	$50k$	$2 + 360k$
$24 + 13224k$	$1 + 600k.1 + 696k.5 + 2880k.5 + 2755k.$	$125k$	$0 + 145k$
$54 + 360k$	$1 + 9k.20 + 135k.12 + 81k.6 + 40k.$	$1k$	$2 + 15k$
$60 + 990k$	$1 + 30k.53 + 880k.18 + 300k.6 + 99k.$	$3k$	$5 + 88k$
$72 + 1716k$	$1 + 36k.39 + 936k.18 + 432k.6 + 143k.$	$3k$	$3 + 78k$
$19 + 6175k$	$1 + 456k.7 + 2375k.4 + 1444k.6 + 1950k.$	$144k$	$2 + 750k$
$210 + 7140k$	$1 + 35k.6 + 210k.36 + 1225k.6 + 204k.$	$1k$	$0 + 6k$
$282 + 8648k$	$1 + 47k.153 + 4700k.72 + 2209k.6 + 184k.$	$1k$	$3 + 100k$
$25 + 14725k$	$1 + 1050k.22 + 13175k.7 + 4375k.6 + 3534k.$	$252k$	$5 + 3162k$
$264 + 17028k$	$1 + 88k.98 + 6336k.60 + 3872k.6 + 387k.$	$2k$	$2 + 144k$
$70 + 21280k$	$1 + 315k.2 + 665k.12 + 3675k.6 + 1824k.$	$27k$	$0 + 57k$
$35 + 41615k$	$1 + 1260k.1 + 1435k.6 + 7350k.6 + 7134k.$	$216k$	$0 + 246k$
$18 + 4950k$	$1 + 504k.16 + 4500k.4 + 1296k.7 + 1925k.$	$196k$	$6 + 1750k$
$20 + 7020k$	$1 + 560k.12 + 4320k.4 + 1600k.7 + 2457k.$	$196k$	$4 + 1512k$
$336 + 15792k$	$1 + 48k.7 + 336k.49 + 2304k.7 + 329k.$	$1k$	$0 + 7k$
$308 + 17028k$	$1 + 88k.186 + 10296k.70 + 3872k.7 + 387k.$	$2k$	$4 + 234k$

$68 + 20740k$	$1 + 476k.41 + 12580k.15 + 4624k.7 + 2135k.$	$49k$	$4 + 1295k$
$210 + 26970k$	$1 + 240k.188 + 24180k.56 + 7200k.7 + 899k.$	$8k$	$6 + 806k$
$48 + 210k$	$1 + 6k.19 + 84k.8 + 36k.8 + 35k.$	$1k$	$3 + 14k$
$52 + 2145k$	$1 + 52k.14 + 585k.8 + 338k.8 + 330k.$	$8k$	$2 + 90k$
$352 + 9460k$	$1 + 44k.229 + 6160k.72 + 1936k.8 + 215k.$	$1k$	$5 + 140k$
$100 + 15525k$	$1 + 250k.65 + 10125k.20 + 3125k.8 + 1242k.$	$20k$	$5 + 810k$
$304 + 18278k$	$1 + 76k.82 + 4940k.48 + 2888k.8 + 481k.$	$2k$	$2 + 130k$
$504 + 31248k$	$1 + 63k.8 + 504k.64 + 3969k.8 + 496k.$	$1k$	$0 + 8k$
$37 + 48285k$	$1 + 2072k.24 + 31635k.7 + 9583k.8 + 10440k.$	$448k$	$5 + 6840k$

Theorem 80. *Let $n \geq 2$ be a positive integer and let $u = n^2 - 1$. Then the number*

$$1 + uk.n + nuk.n^2 + u^2k.n + n(u - 1)k$$

is 1-phoenix in base $b = nu + nu(u - 1)k$ with carry $c_1 = k$ and $c_2 = nk$. In particular, if $k = 0$ then we have that $1.n.n^2.n$ is 1-phoenix in base $b = n(n^2 - 1)$.

Table 42: tab:1-phx-arith-ext-2nd-kind-base-b=n.n2-1

$24 + 168k$	$1 + 8k.3 + 24k.9 + 64k.3 + 21k$	$1k$	$0 + 3k$
$8 + 440k$	$1 + 72k.1 + 88k.3 + 192k.3 + 165k$	$27k$	$0 + 33k$
$60 + 840k$	$1 + 15k.4 + 60k.16 + 225k.4 + 56k$	$1k$	$0 + 4k$
$15 + 3135k$	$1 + 240k.1 + 285k.4 + 900k.4 + 836k$	$64k$	$0 + 76k$
$120 + 2760k$	$1 + 24k.5 + 120k.25 + 576k.5 + 115k$	$1k$	$0 + 5k$
$24 + 13224k$	$1 + 600k.1 + 696k.5 + 2880k.5 + 2755k$	$125k$	$0 + 145k$

Theorem 81. *Let $n \geq 3$ be a positive integer and let $u = n^2 - 1$. Then the number*

$$1 + u(u + 1)k.1 + u(u + n)k.n + nu^2k.n + n(u^2 - u - 1)k$$

is 1-phoenix in base $b = u + u(u^2 - u - 1)k$ with carries $c_1 = n^3k$ and $c_2 = n(u + n)k$. In particular, if $k = 0$, then we have that $1.1.n.n$ is 1-phoenix in base $b = n^2 - 1$.

Remark 82. There does not appear to be any general pattern for the number $1.m.mn.n$, 1-phoenix in base $b = m(n^2 - 1)$.

7.5 Arithmetic extension of the second kind of $1.x.y.z.n$

8 2-Phoenix numbers

8.1 2-Phoenix with four digits

2-phx, $p.q.x.px$

7	900	2.4.2.4	(4d)	Theorem 88, Theorem 91
19	24592	3.11.2.6	(4d)	Theorem 87
37	271188	5.13.3.15	(4d)	Theorem 88
133	24128368	10.34.4.40	(4d)	Theorem 88
146	19483432	6.38.4.24	(4d)	Theorem 87
248	154070100	10.25.10.100	(4d)	Theorem 91
249	233126400	15.25.10.150	(4d)	Theorem 93
296	371930850	14.101.3.42	(4d)	
361	809295300	17.73.5.85	(4d)	Theorem 88
369	2838957700	56.186.2.112	(4d)	
466	2038446774	20.67.7.140	(4d)	
489	1072221552	9.83.6.54	(4d)	Theorem 87
811	13958160084	26.136.6.156	(4d)	Theorem 88
1156	18732767392	12.146.8.96	(4d)	Theorem 87
1362	129085107327	51.124.11.561	(4d)	
1597	151285172500	37.229.7.259	(4d)	Theorem 88
2857	1168928019648	50.358.8.400	(4d)	Theorem 88
2997				Theorem 91
4753	6991329962628	65.529.9.585	(4d)	Theorem 88
7471	34235762469700	82.748.10.820	(4d)	Theorem 88
10984				Theorem 93

Remark 83. The 2-phoenix numbers 3.11.2.6, $c = 0$, in base $b = 19$ and 56.186.2.112, $c = 16$, in base $b = 369$ seem to be the only examples with their multiplication table. Bases checked out to $b = 10000$.

Theorem 84 (2-Phoenix in base $b = 2n^2 - 1$). *If $n \geq 2$, the number*

$$2(n-1).2n(n-1).\underline{2(n^2 - n - 1).2n}$$

is 2-Phoenix in base $b = 2n^2 - 1$. See Table 43.

Remarks 85.   1. For $n = 2$ we obtain $2.4.2.4 = (4.2)^2$ is 2-Phoenix in base 7. It is easy to show that $n = 2$ is the only case in which $x.y.x.y = (y.x)^2$. Moreover, if $n = 4$, then $6.20.1X.8 = (12.14)^2$ in base $b = 27$.

2. If $n = 2^m$, $m \geq 1$, then $b = 2^{2m+1} - 1$ will be a Mersenne number or Mersenne prime.

Proof. By multiplication.

Table 43: 2-Phoenix numbers for Theorem 84.

n	b	N_b
2	7	2.4.2.4
3	17	4.12.10.6
4	31	6.24.22.8
5	49	8.40.38.10
6	71	10.60.58.12
7	97	12.84.82.14
8	127	14.112.110.16
9	161	16.144.142.18
10	199	18.180.178.20
11	241	20.220.218.22
12	287	22.264.262.24
13	337	24.312.310.26

		(1)	
		$(2n-4)$	
		$2n^2 - 2n.$	$2n - 2.$
		$2n.$	$2n^2 - 2n - 2$
	$2n^2 - 4n + 1.$	$2n - 3.$	$2n$
$2n - 2.$	$2n - 1.$	$2n^2 - 4n + 1.$	
$2n - 2.$	$2n^2 - 2n.$	$2n^2 - 2n - 2.$	$2n$

Theorem 84, base $b = 2n^2 - 1$.

$$
\begin{aligned}
(2n^2 - 2n)(2n - 2) - 2n &= 2\left[2(n-1)(n^2 - n - 1) - n\right] \\
&= 2\left[2n^3 - 4n^2 - n + 2\right] \\
&= 2\left[2n^2(n - 2) - (n - 2)\right] \\
&= (2n - 4)(2n^2 - 1). \\
(2n^2 - 2n - 2)(2n^2 - 2n) + (2n - 4) - (2n - 3) &= 4n(n^2 - n - 1)(n - 1) - 1 \\
&= 4n^4 - 8n^3 + 4n - 1 \\
&= (2n^2 - 4n + 1)(2n^2 - 1). \\
(2n)(2n - 2) - (2n^2 - 4n + 1) &= 4n^2 - 4n - 2n^2 + 4n - 1 \\
&= (1)(2n^2 - 1). \\
(2n)(2n^2 - 2n) + (1) - (2n - 1) &= (2n^2)(2n - 2) - (2n - 2) \\
&= (2n - 2)(2n^2 - 1). \qquad \square
\end{aligned}
$$

Table 44: Multiplication for Theorem 86.

		(n)	
		$8n^2 + 2.$	$2n$
		$4n^2.$	$8n^3 + 3n^2$
	$4n^2 + 2.$	$3n.$	$4n^2$
$2n.$	$4n^2.$	$8n^3$	
$2n.$	$8n^2 + 2.$	$8n^3 + 3n.$	$4n^2$

Base $b = 16n^3 + 2n.$

Table 45: 2-Phoenix for Theorem 86.

b	N	N_b
18	15106	2.10.11.4
132	9801544	4.34.70.16
438	518461074	6.74.225.36

Theorem 86. *The number*

$$2n.8n^2 + 2.\underline{8n^3 + 3n.4n^2}$$

is 2-phoenix in base $b = 16n^3 + 2n$, $n \geq 1$. *See Table 45.*

Proof. By multiplication. See Table 44. $\qquad\square$

Theorem 87. *The number*

$$3n.9n^2 + 2.\underline{2n.6n^2}$$

is 2-phoenix in base $b = 18n^3 + n$, $n \geq 1$. *See Table 47.*

Proof. By multiplication. See Table 46. $\qquad\square$

Theorem 88 (2-Phoenix in base $b = n^4 + n^3 + 2n^2 + 2n + 1$)**.** *The number*

$$n^2 + 1.n^3 + 2n + 1.\underline{n + 1.n^3 + n^2 + n + 1}$$

is 2-Phoenix in base $b = n^4 + n^3 + 2n^2 + 2n + 1$, $n \geq 1$. *See Table 49.*

Proof. By multiplication. See Table 48.

$$(n + 1)(n^2 + 1) = n^3 + n^2 + n + 1,$$
$$(n + 1)(n^3 + 2n + 1) - n = n^4 + 2n^2 + n + n^3 + 2n + 1 - n$$

Table 46: Multiplication for Theorem 87.

		$9n^2 + 2.$	$3n$
		$6n^2.$	$2n$
	(1)		
	1.	$3n.$	$6n^2$
$3n.$	$9n^2.$	$18n^3$	
$3n.$	$9n^2 + 2.$	$2n.$	$6n^2$

Base $b = 18n^3 + n$.

Table 47: 2-Phoenix in base $b = 18n^3 + n$.

b	N	N_b
19	24592	3.11.2.6
146	19483432	6.38.4.24
489	1072221552	9.83.6.54
1156	18732767392	12.146.8.96

Table 48: Multiplication for Theorem 88.

		(n)		
		$n^3 + 2n + 1.$	$n^2 + 1.$	
		$n^3 + n^2 + n + 1.$	$n + 1.$	
	1.	$n.$	$n^3 + n^2 + n + 1$	
$n^2 + 1.$	$n^3 + 2n.$	1.		
$n^2 + 1.$	$n^3 + 2n + 1.$	$n + 1.$	$n^3 + n^2 + n + 1$	

Base $b = n^4 + n^3 + 2n^2 + 2n + 1$, $n \geq 1$.

$$
\begin{aligned}
&= (1)(n^4 + n^3 + 2n^2 + 2n + 1). \\
(n^3 + n^2 + n + 1) \cdot (n^2 + 1) - 1 &= (n+1)(n^2+1)^2 - 1 \\
&= n(n^2+1)^2 + (n^2+1)^2 - 1 \\
&= n^5 + 2n^3 + n + n^4 + 2n^2 \\
&= n^5 + n^4 + 2n^3 + 2n^2 + n \\
&= (n)(n^4 + n^3 + 2n^2 + 2n + 1).
\end{aligned}
$$

$$
\begin{aligned}
(n+1)(n^2+1)\cdot(n^3+2n+1)+n-(n^3+2n) &= (n^2+1)\cdot(n+1)(n^3+2n+1)-n(n^2+1) \\
&= (n^2+1)(n^4+2n^2+n+n^3+2n+1-n) \\
&= (n^2+1)(n^4+n^3+2n^2+2n+1). \qquad \square
\end{aligned}
$$

Sequence 89 (not in OEIS). $a(n) = n^4 + n^3 + 2n^2 + 2n + 1$.

Sequence 90 (not in OEIS). *Primes p of the form $p = n^4 + n^3 + 2n^2 + 2n + 1$:*

$$7, 37, 811, 1597, 2857, 22777, 31123, 70177, 111493, 137941, 204667, 475957$$

Table 49: 2-Phoenix numbers for Theorem 88.

n	b	N_b
1	7	2.4.2.4
2	37	5.13.3.15
3	133	10.34.4.40
4	361	17.73.5.85
5	811	26.136.6.156
6	1597	37.229.7.259
7	2857	50.358.8.400
8	4753	65.529.9.585
9	7471	82.748.10.820
10	11221	101.1021.11.1111
11	16237	122.1354.12.1464
12	22777	145.1753.13.1885

Theorem 91. *The number*

$$n(n^2+1).(n^2+1)^2.\underline{n(n^2+1).n^2(n^2+1)^2}$$

is 2-phoenix in base $b = n(n^2+1)^3 - n$, $n \geq 1$. See Table 51.

Proof. By multiplication. See Table 50. $\qquad\square$

Table 50: Multiplication for Theorem 91.

		(n)	
		$(n^2+1)^2$.	$n(n^2+1)$
		$n^2(n^2+1)^2$.	$n(n^2+1)$
	1.	n.	$n^2(n^2+1)^2$
$n(n^2+1)$.	n^4+2n^2.	n^3	
$n(n^2+1)$.	$(n^2+1)^2$.	$n(n^2+1)$.	$n^2(n^2+1)^2$

Base $b = n(n^2+1)^3 - n$.

Table 51: 2-Phoenix numbers for Theorem 91.

b	N	N_b
7	900	2.4.2.4
248	154070100	10.25.10.100
2997	808470720900	30.100.30.900
19648	515890930582800	68.289.68.4624

8.2 A Fibonacci example

The 2-phoenix number $15.25.\underline{10.150}$ in base $b = 249$ suggests the pattern $x.x+y.\underline{y.xy}$ in base $b = xy + y^2 - 1$. Here is the multiplication table.

	(9)					$(y-1)$		
	(0)					(0)		
	25	15				$x+y$.	x	
	150	10				xy.	y	
1	1	150			1.	1.	xy	
15	24	9		x	$x+y-1$.	$y-1$.		
15	25	10	150	x	$x+y$.	y.	xy	
	Base $b = 249$.				Base $b = xy + y^2 - 1$.			

$$(13)$$

The equations defining the multiplication (13) are

$$x^2 y = (y-1)(xy + y^2 - 1) + (y-1), \tag{14a}$$
$$xy(x+y) + (y-1) = x(xy + y^2 - 1) + (x+y-1). \tag{14b}$$

It is easily verified that (14b) is identically true. However, (14a) is equivalent to the equation

$$y^2 + (x-1)y - (x^2 + x) = 0. \tag{15}$$

Solving for y, we obtain

$$y = \frac{1}{2}\left(\sqrt{5x^2 + 2x + 1} - x + 1\right).\tag{16}$$

Using the computer to generate some examples, we find the 2-Phoenix numbers shown in Table 52.

Table 52: 2-Phoenix Numbers for Theorem 93.

b	N	$N_b = x.x + y.y.xy$
7	900	2.4.2.4
249	233126400	15.25.10.150
10984	137841236442000	104.169.65.6760
510951	95243939022256124100	714.1156.442.315588
23968945	6740614372101838645130240	4895.7921.3026.14812270
1125790992	478731188309509542711486586224800	33552.54289.20737.695767824

Here's where the OEIS proves to be an invaluable resource.

Sequences in Theorem 93 and Table 52.

A081018	$x = F_{2n}F_{2n+1}$	$2, 15, 104, 714, 4895\ldots$
A081068	$x + y = F_{2n+1}^2$	$4, 25, 169, 1156, 7921, \ldots$
A064170	$y = F_{2n-1}F_{2n+1}$	$2, 10, 65, 442, 3026, \ldots$
(not in OEIS)	$xy = F_{2n-1}F_{2n}F_{2n+1}^2$	$4, 150, 6750, 315588, 14812270, \ldots$

Remark 92. The numbers $x = F_{2n}F_{2n+1}$ and $y = F_{2n-1}F_{2n+1}$ are the roots of (15). It is a fun exercise to verify the solution! Consult [14] for appropriate identities.

Theorem 93. *The number*

$$x.x + y.y.xy$$

is 2-Phoenix in base $b = xy + y^2 - 1$, *where* $x = F_{2n}F_{2n+1}$ *and* $y = F_{2n-1}F_{2n+1}$, $n \geq 1$. *Furthermore,* $x + y = F_{2n+1}^2$ *and* $b = F_{2n-1}F_{2n+1}^3 - 1$.

Proof. See multiplication (13) and (14). Apply the identities on page 76 of [14] to simplify $b = xy + y^2 - 1$ with $x = F_{2n}F_{2n+1}$ and $y = F_{2n-1}F_{2n+1}$ to $b = F_{2n-1}F_{2n+1}^3 - 1$. $\square$

8.3 2-Phoenix numbers with five digits

Theorem 94 (2-Phoenix in base $b = n^3 - n^2$)**.** *The number*

$$n.n^2 + 1.n^2 + 1.\underline{n - 1}.n^2 - n$$

is 2-Phoenix in base $b = n^3 - n^2 = n^2(n - 1)$, $n \geq 3$. *See Table 53.*

The sequences that occur are

Sequence 95 ([A045991](#)). $n^3 - n^2$.

Sequence 96 ([A002378](#)). *Oblong (or promic, pronic, or heteromecic) numbers: $n(n+1)$.*

Sequence 97 ([A002522](#)). $n^2 + 1$.

Sequence 98 ([A002061](#)). *Central polygonal numbers: $n^2 - n + 1$.*

Table 53: 2-Phoenix numbers for Theorem 94.

k	b	N_b
3	18	3.10.10.2.6
4	48	4.17.17.3.12
5	100	5.26.26.4.20
6	180	6.37.37.5.30
7	294	7.50.50.6.42
8	448	8.65.65.7.56
9	648	9.82.82.8.72
10	900	10.101.101.9.90
11	1210	11.122.122.10.110
12	1584	12.145.145.11.132

Base $b = n^3 - n^2$.

Proof. By multiplication.

Multiplication for Theorem 94

		(n)	(1)	
		(1)		
		$n^2 + 1.$	$n^2 + 1.$	n
			$n^2 - n.$	$n - 1$
	1.	$n.$	$n - 1.$	$n^2 - n$
$n.$	$n^2.$	$n^2 - n + 1.$	0.	
$n.$	$n^2 + 1.$	$n^2 + 1.$	$n - 1.$	$n^2 - n$

Base $b = n^3 - n^2$.

$$(n-1)(n^2+1) - (n-1) = (n-1)n^2 = (1)(n^3 - n^2),$$
$$(n-1)(n^2+1) + 1 - n = (n-1)n^2 = (1)(n^3 - n^2),$$

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$$(n^2 - n) \cdot n = (1)(n^3 - n^2),$$
$$(n^2 - n) \cdot (n^2 + 1) + 1 - (n^2 - n + 1) = n^4 + n^2 - n^3 - n + 1 - n^2 + n - 1$$
$$= n^4 - n^3 = (n)(n^3 - n^2),$$
$$(n^2 - n) \cdot (n^2 + 1) + n - n^2 = (n^2 - n)(n^2)$$
$$= (n)(n^3 - n^2). \qquad \square$$

Theorem 99 (2-Phoenix numbers in base $b = n^5 + n^3$). *The number*

$$n.n^4.n^5 + n^3 - n.\underline{n^4 + n^2 + 1}.n$$

is 2-Phoenix in base $b = n^5 + n^3$.

Proof. By multiplication.

		(n^3)	(1)	
		$n^5 + n^3 - n.$	$n^4.$	n
			$n.$	$n^4 + n^2 + 1$
(1)				
	$n^4 + n^2.$	$n^3 - n.$	$n^4 + 1.$	n
$n - 1.$	$n^5 + n^3 - n^2.$	$n^5.$	$n^2.$	
$n.$	$n^4.$	$n^5 + n^3 - n.$	$n^4 + n^2 + 1.$	n

Base $b = n^5 + n^3$.

$\square$

Sequence 100 ([A133072](#)). $n^5 + n^3 - n^2$.

$$0, 1, 36, 261, 1072, 3225, 7956, \ldots$$

8.4 2-Phoenix of the form $2.x.6.y.2y$

		(10)		
		(5)		
		6	26	2
			12	6
	(1)	(1)		
	1	12	11	12
2	24	23	24	
2	26	6	6	12

Base $b = 29$.

Observe that $2.26.6.6.12$ in base $b = 29$ and $2.30.6.18.36$ in base $b = 99$. Further observe that both have a multiplication table of the form $2.x.6.y.2y$, then

$$b = 6y - \frac{21}{2} + \frac{63}{2(y+3)}$$

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The only divisors that work in the multiplication are $9, 21, 63$.

Theorem 101. *The number $2.x.6.y.2y$ is 2-Phoenix in base b only if $y + 3$ is one of the following divisors of 63: $9, 21, 63$.*

$$
\begin{array}{ll}
29 & 2.26.6.6.12 \\
99 & 2.30.6.18.36 \\
350 & 2.32.6.60.120
\end{array}
$$

8.5   2-Phoenix of the form $2.b - 3.x.y.2y$

2-Phoenix, five digit. Fourth digit (counting from right) is base minus 3. $2.b - 3.x.y.2y$

29	2053908	2.26.6.6.12
156	1766016112	2.153.28.8.16
196	4407545304	2.193.72.4.8
216	6501143404	2.213.22.14.28
736	879117397812	2.733.18.58.116

From the multiplication table, $xy + (y - 1) = b + p_2$ and $b - 5 = 2p_2 + 1$ so

$$
b = \frac{1}{3}(2xy + 2y + 4) \quad (y \text{ odd})
$$

$$
p_1 = 9y + 38 + \frac{110}{y - 3},
$$
$$
p_2 = 6y + 16 + \frac{55}{y - 3},
$$
$$
q_2 = 6y + 38 + \frac{110}{y - 3},
$$
$$
x = 17 + \frac{55}{y - 3}
$$

and so the only solutions are those corresponding to the four divisors of 55.
   If y is even then

$$
b = \frac{1}{3}(2xy + 2y + 3) \quad (y \text{ even})
$$

and

$$
p_1 = y + \frac{13}{3} + \frac{10}{3(y - 1)},
$$
$$
p_2 = 2y - \frac{1}{3} + \frac{5}{3(y - 1)},
$$

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$$q_2 = 2y + \frac{26}{3} + \frac{20}{3(y-1)},$$
$$x = 5 + \frac{5}{y-1}$$

and only $y - 1 = 5$ gives the solution 2.26.6.6.10 in $b = 29$.

8.6 2-Phoenix numbers of the form $1.x.y.z.n.n$

In this section we consider 2-Phoenix numbers of the form $1.x.y.z.n.n$ in some base b. The first two examples are

		(5)	(4)		
		(5)	(4)		
		4	15	12	1
				6	6
	1	11	4	0	6
1	11	4	0	6	
1	12	15	4	6	6

Base $b = 18$.

		(7)	(6)		
		(7)	(6)		
		3	16	14	1
				9	9
	1	13	3	0	9
1	13	3	0	9	
1	14	16	3	9	9

Base $b = 21$.

The multiplication table is as follows.

		$(c+1)$	(c)		
		$(c+1)$	(c)		
		z	y	x	1
				n	n
	1	p	z	0	n
1	p	z	0	n	
1	x	y	z	n	n

Base b to be determined.

The equations

$$nx = cb$$
$$ny + c = (1 + c)b + z$$
$$nz + (1 + c) = b + p$$
$$p + z = y$$
$$p + 1 = x.$$

have the solution

$$x = \frac{b(n^2 + b + 2n - 2)}{bn - b + n}, \tag{17a}$$

$$y = \frac{2b^2 + bn^2 + 2bn - n^2 - b - 3n}{bn - b + n}, \tag{17b}$$

$$z = \frac{b^2 + bn - n^2 - 2n}{bn - b + n}, \tag{17c}$$

$$p = \frac{b^2 + bn^2 + bn - b - n}{bn - b + n}. \tag{17d}$$

A computer search for bases b, $2 \leq b \leq 144$, and integers n, $1 \leq n < b$, such that (17) has integer solutions yields Table 54. The only relationship between the numbers is that their multiplication tables have the same form.

Table 54: 2-Phoenix numbers of the form $1.x.y.z.n.n$.

b	Base 10	1. x. y. z. n. n
18	3238170	1.12.15. 4. 6. 6
21	6956532	1.14.16. 3. 9. 9
40	155734564	1.20.33.14. 4. 4
52	529825153	1.20.24. 5.13.13
62	1382077242	1.31.33. 3.26.26
66	1679900514	1.22.35.14. 6. 6
78	4837124496	1.52.53. 2.48.48
110	20544060042	1.30.35. 6.22.22
132	58433046595	1.60.62. 3.55.55

8.7 2-Phoenix numbers of the form $1.0.x.y.z.n.n$

REDO THIS SECTION IN BASE 10!!

		(1)				
		(1)				
		2	4	1	0	1
					2	2
(1)	(1)					
		5	2	2	0	2
	5	2	2	0	2	
1	0	1	4	2	2	2

Base $b = 6$.

			(2)	(1)		
			(2)	(1)		
		1	4	2	0	1
					4	4
(1)	(1)					
		6	3	1	0	4
	6	3	1	0	4	
1	0	2	4	1	4	4

Base $b = 7$.

Base 7 has the same carry pattern as $1.x.y.z.n.n$ so let's examine it first.

		(q+1)	(q)			
		(q+1)	(q)			
		z	y	x	0	1
					n	n
(1)	(1)					
		s	r	z	0	n
	s	r	z	0	n	
1	0	x	y	z	n	n

(18)

Base b to be determined.

The equations which define it are

$$nx = qb + z \tag{19a}$$
$$ny + q = (1 + q)b + r \tag{19b}$$
$$nz + (1 + q) = s \tag{19c}$$
$$r + z = y \tag{19d}$$
$$s + r = b + x \tag{19e}$$
$$s + 1 = b \tag{19f}$$

Thus, the last three equations (19d)-(19f) yield $s = b - 1$, $b - 1 + r = b + x$ implies $r = x + 1$ and $y = x + z + 1$. Subtract (19b) from (19c) to obtain

$$n(y - z) - 1 = (1 + q)b + r - s$$
$$n(y - z) - 1 = b + qb + x + 1 - b + 1$$
$$w(y - z) = qb + x + 3$$

and substitute (19a) to obtain

$$n(y - z) = nx - z + x + 3$$

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$$n(y - x - z) = x - z + 3$$
$$n = x - z + 3.$$

Theorem 102. *The number $1.0.x.y.z.n.n$ is 1-phoenix in base b if and only if there exist b and n such that there are integer solutions to the following equations*

$$x = \frac{bn^2 + b^2 - 3bn - 2b - n + 3}{nb + n - 1}, \tag{20}$$

$$y = \frac{bn^2 + 2b^2 - 2bn - n^2 - 4b + 3n + 2}{nb + n - 1}, \tag{21}$$

$$z = \frac{b^2 - n^2 - 2b + 3n}{nb + n - 1}, \tag{22}$$

$$q = \frac{n^3 - 3n^2 + nb - 2n - b + 2}{nb + n - 1}, \tag{23}$$

$$r = \frac{bn^2 + b^2 - 2bn - 2b + 2}{nb + n - 1}, \tag{24}$$

$$s = b - 1, \tag{25}$$

and $0 < x, y, z, q, r, s < b$. See Table 55 for examples.

Note that the two examples $1.0.2.4.1.\underline{4.4}$ in base $b = 7$ and $1.0.23.25.1.\underline{25.25}$ in base $b = 41$ motivate Section 8.8.

By inspection of Table 55 we obtain the following theorem.

Theorem 103 (2-Phoenix numbers in base $b = n^3 - 3n^2 - 3n + 3$)**.** *The number*

$$1.0.n^2 - 2n - 6.2n^2 - 5n - 8.n^2 - 3n - 3.\underline{n.n}$$

is 2-Phoenix in base $b = n^3 - 3n^2 - 3n + 3$. See Table 56.

Proof. By multiplication.

				(2)		(1)		
				(2)		(1)		
			$n^2 - 3n - 3.$	$2n^2 - 5n - 8.$	$n^2 - 2n - 6.$	0.	1
							$n.$	n
(1)	(1)							
			$b - 1.$	$n^2 - 2n - 5.$	$n^2 - 3n - 3.$	0.	n
	$b - 1.$	$n^2 - 2n - 5.$	$n^2 - 3n - 3.$			0.	n
1.	0.	$n^2 - 2n - 6.$	$2n^2 - 5n - 8.$	$n^2 - 3n - 3$	$n.$	n

Base $b = n^3 - 3n^2 - 3n + 3.$

$\square$

Table 55: 2-Phoenix numbers for Theorem 102

b	$1.0.x.y.z.n.n$	q	r	s
7	1.0.2.4.1.4.4	1	3	6
38	1.0.9.17.7.5.5	1	10	37
41	1.0.23.25.1.25.25	14	24	40
93	1.0.18.34.15.6.6	1	19	92
178	1.0.29.55.25.7.7	1	30	177
180	1.0.73.76.2.74.74	30	74	179
274	1.0.167.169.1.169.169	103	168	273
291	1.0.34.46.11.26.26	3	35	290
299	1.0.42.80.37.8.8	1	43	298
346	1.0.41.52.10.34.34	4	42	345
462	1.0.57.109.51.9.9	1	58	461
538	1.0.79.87.7.75.75	11	80	537
657	1.0.67.79.11.59.59	6	68	656
673	1.0.74.142.67.10.10	1	75	672
938	1.0.93.179.85.11.11	1	94	937
1263	1.0.114.220.105.12.12	1	115	1262
1654	1.0.137.265.127.13.13	1	138	1653

Table 56: 2-Phoenix numbers for Theorem 103

b	$1.0.x.y.z.n.n$	q	r	s
7	1.0.2.4.1.4.4	1	3	6
38	1.0.9.17.7.5.5	1	10	37
93	1.0.18.34.15.6.6	1	19	92
178	1.0.29.55.25.7.7	1	30	177
299	1.0.42.80.37.8.8	1	43	298
462	1.0.57.109.51.9.9	1	58	461
673	1.0.74.142.67.10.10	1	75	672
938	1.0.93.179.85.11.11	1	94	937
1263	1.0.114.220.105.12.12	1	115	1262
1654	1.0.137.265.127.13.13	1	138	1653

8.8 An example of 2-phoenix numbers involving a Pell equation

See Section 8.7 for a general discussion of 2-Phoenix numbers of the form $1.0.x.y.z.n.n$ in some base b. Observe that 1.0.2.4.1.4.4 in base 7 and 1.0.23.25.1.25.25 in base 41 are of the form

$$1.0.a^2 - 2.a^2.1.\underline{a^2.a^2} \tag{26}$$

in some base b.

(27)

		(2)	(1)			
		(2)	(1)			
		1	4	2	0	1
					4	4
	(1)					
		6	3	1	0	4
	6	3	1	0	4	
1	0	2	4	1	4	4

Base $b = 7$.

(28)

		(15)	(14)			
		(15)	(14)			
		1	25	23	0	1
					25	25
	(1)					
		40	24	1	0	25
	40	24	1	0	25	
1	0	23	25	1	25	25

Base $b = 41$.

Translating into symbols, we have

		$(c+1)$	(c)			
		$(c+1)$	(c)			
		$1.$	$a^2.$	$a^2-2.$	$0.$	1
					$a^2.$	a^2
	(1)					
		$b-1.$	$a^2-1.$	$1.$	$0.$	a^2
	$b-1.$	$a^2-1.$	$1.$	$0.$	a^2	
$1.$	$0.$	$a^2-2.$	$a^2.$	$1.$	$a^2.$	a^2

Base b to be determined.

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The equations that define the multiplication are

$$a^4 - 2a^2 = cb + 1, \tag{29a}$$
$$a^4 + c = (c+1)b + (a^2 - 1), \tag{29b}$$
$$a^2 + (c+1) = b - 1. \tag{29c}$$

From (29c) we obtain

$$b = c + a^2 + 2 \tag{30}$$

Substitute (30) in (29a), solve for c, and obtain

$$c^2 + (a^2 + 2)c - (a^4 - 2a^2 - 1) = 0. \tag{31}$$

Solve (31) for c, keep only the positive root (32a), and substitute into (30) to obtain

$$c = \frac{\sqrt{a^2(5a^2 - 4)} - (a^2 + 2)}{2}, \tag{32a}$$
$$b = \frac{\sqrt{a^2(5a^2 - 4)} + (a^2 + 2)}{2}. \tag{32b}$$

Since $5a^2 - 4$ must be a square, it follows that we are looking for solutions to the Pell-type equation $x^2 - 5y^2 = -4$.

Theorem 104 ([12]). *Let p and q fundamental solutions to $p^2 - Dq^2 = c$ and let r and s be fundamental solutions to $r^2 - Ds^2 = 1$. The identity*

$$(p^2 - Dq^2)(r^2 - Ds^2 = 1) = (r \cdot p \pm Ds \cdot q)^2 - D(s \cdot p \pm r \cdot q)^2 = c,$$

means that $(x, y) = (r \cdot p \pm Ds \cdot q, s \cdot p \pm r \cdot q)$ are also solutions to $p^2 - Dq^2 = c$. It is easier to use this method if it is written in matrix notation.

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} r & Ds \\ s & r \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}.$$

Now assume $D = 5$ and $c = -4$. Then $(\pm 1, 1)$, $(4, 2)$, are solutions to $p^2 - 5q^2 = -4$ and $(9, 4)$ is a solution to $r^2 - 5s^2 = 1$. Thus,

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}^k \begin{bmatrix} p \\ q \end{bmatrix}$$

Since the growth of solutions is exponential, only the first few examples are provided for each fundamental solution. In Table 57 the fundamental solution $(4, 2)$ starts with the first example (27) with $b = 7$, $a = 2$ and $c = 1$. In Table 58 the fundamental solution $(-1, 1)$ starts with the second example (28) with $b = 41$, $a = 5$ and $c = 14$. In Table 59 the fundamental solution $(1, 1)$ shows new examples outside of my search bound $b \leq 144$.

Table 57: Examples of (26) with fundamental solution $(4, 2)$ to $x^2 - 5y^2 = -4$.

a	b	c	$1.0.a^2 - 2.a^2.1.a^2.a^2$
2	7	1	1.0.2.4.1.4.4
34	1871	713	1.0.1154.1156.1.1156.1156
610	602071	229969	1.0.372098.372100.1.372100.372100
10946	193864607	74049689	1.0.119814914.119814916.1.119814916.119814916

Table 58: Examples of (26) with fundamental solution $(-1, 1)$ to $x^2 - 5y^2 = -4$.

a	b	c	$1.0.a^2 - 2.a^2.1.a^2.a^2$
5	41	14	1.0.23.25.1.25.25
89	12817	4894	1.0.7919.7921.1.7921.7921
1597	4126649	1576238	1.0.2550407.2550409.1.2550409.2550409
28657	1328767777	507544126	1.0.821223647.821223649.1.821223649.821223649

Table 59: Examples of (26) with fundamental solution $(1, 1)$ to $x^2 - 5y^2 = -4$.

a	b	c	$1.0.a^2 - 2.a^2.1.a^2.a^2$
13	274	103	1.0.167.169.1.169.169
233	87842	33551	1.0.54287.54289.1.54289.54289
4181	28284466	10803703	1.0.17480759.17480761.1.17480761.17480761
75025	9107509826	3478759199	1.0.5628750623.5628750625.1.5628750625.5628750625

8.9 Another example of 2-phoenix numbers involving a Pell equation

Let's return to the base 6 example, 1.0.1.4.2.2.2. It turns out that is the first of several examples with third digit equal to 2.

b		c
6	1.0.1.4.2.2.2	0
180	1.0.73.76.2.74.74	30
6090	1.0.2521.2524.2.2522.2522	1044

		$(c+1)$	(c)			
		$(c+1)$	(c)			
		2	$n+2$	$n-1$	0	1
					n	n
(1)	(1)					
		B	n	2	0	n
	B	n	2	0	n	
1	0	$n-1$	$n+2$	2	n	n

Base b to be determined.

The equations

$$n(n-1) = cb + 2$$
$$n(n+2) + c = (c+1)b + n$$
$$2n + (c+1) = b - 1$$

have solution

$$n = \frac{1}{2}\left(\sqrt{8b^2 - 12b + 9} - 2b + 1\right)$$
$$c = 3b - 1 - \sqrt{8b^2 - 12b + 9}$$

Thus, $x = b$ must be a solution to the quadratic Diophantine equation $8x^2 - 12x - y^2 + 9 = 0$. Using the Quadratic Diophantine Equation Solver [1], we obtain

$$x_1 = 6, y_1 = 15,$$
$$x_{n+1} = -3x_n - y_n + 3$$
$$y_{n+1} = -8x_n - 3y_n + 6.$$

The recursion has alternating positive and negative pairs so, keeping only the positive pairs, we obtain

b		c
6	1.0.1.4.2.2.2	0
180	1.0.73.76.2.74.74	30
6090	1.0.2521.2524.2.2522.2522	1044
206856	1.0.85681.85684.2.85682.85682	35490
7026990	1.0.2910673.2910676.2.2910674.2910674	1205640

291	1.0.34.46.11.26.26	3
346	1.0.41.52.10.34.34	4
538	1.0.79.87.7.75.75	11
657	1.0.67.79.11.59.59	6
9359	1.0.633.649.15.621.621	42

The pattern in $m = 10$ and $m = 25$ is clear: n, $x = n + m - 3$, $y = n + 2m - 2$.

CHECK if n, $x = n + m - 3$, $y = n + 2m - 2$ is true in general for $1.0.x.y.m.n.n$.

If y is defined by

$$y^2 = (225k^2 + 300k + 104)c^2 - (450k^2 + 510k + 132) + (225k^2 + 270k + 89)$$

then

$$n = \frac{1}{2}(y + (10 + 15k)c - (7 + 15k))$$
$$b = (10 + 15k)n + c + 2.$$

m	b		1.0.x.y.10.n.n	c
10	346	1716349977139822	1.0.41.52.10.34.34	4
10	605696	M	1.0.357041.357052.10.357034.357034	35354
	$M = 2197529230548236092457070236904059681322$			

m	b		1.0.x.y.25.n.n	c
25	178	31836226001041	1.0.29.55.25.7.7	1
25	60274	M	1.0.2429.2455.25.2407.2407	97
25	70317953	N	1.0.2808254.2808280.25.2808232.2808232	112151
	$M = 4794909040464228538688080625$			
	$N = 120891932009773369821419339320509695147273500416$			

There are numbers with third digit equal to 7, 11, and 15.
Here are the initial terms with $m = 7 + 4k$.

m	b		1.0.x.y.m.n.n	
7	38	3030645735	1.0.9.17.7.5.5	(7d)
11	291	607481535712064	1.0.34.46.11.26.26	(7d)
15	93	648364155504	1.0.18.34.15.6.6	(7d)

Inspection of their multiplication tables reveals the following pattern.

		$(c+1)$	(c)			
		$(c+1)$	(c)			
		$7+4k$.	$n+(12+8k)$.	$n+(4+4k)$.	0.	1
					n.	n
(1)	(1)					
		$b-1$.	$n+(5+4k)$.	$7+4k$.	0.	n
	$b-1$.	$n+(5+4k)$.	$7+4k$.	0.		n
1.	0.	$n+(4+4k)$.	$n+(12+8k)$.	$7+4k$.	n.	n

Base b to be determined.

Since the carry digit c is the smaller than both n and b, let us solve for b and n in terms of c. The digit n must be a positive integer root of

$$n^2 + ((4k+4) - (4k+7)c)n - (c^2 + 2c + (4k+7)) = 0.$$

Thus, if the discriminant y is defined by

$$y^2 = (16k^2 + 56k + 53)c^2 - (32k^2 + 88k + 48)c + (16k^2 + 48k + 44),$$

then we have

$$n = \frac{1}{2}(y + (4k+7)c - (4k+4))$$
$$b = (4k+7)n + c + 2$$

For example, if $k = 0$ and $c = 1$, then $y^2 = 53 - 48 + 44 = 49$ so $y = 7$. Then $n = (1/2)(7 + 7 - 4) = 5$ and $b = 7(5) + 1 + 2 = 38$.

If $k = 0$, we obtain:

```
                    53 x^2 -  y^2 - 48 x + 44 = 0

X_{n+1} = P X_n + Q Y_n + K
Y_{n+1} = R X_n + S Y_n + L

P = -66249
Q = -9100
K = 30000
R = -482300
S = -66249
L = 218400
```

b		1.0.x.y.7.n.n	c
38	3030645735	1.0.9.17.7.5.5	1
538	24255621974724525	1.0.79.87.7.75.75	11
99949	99694529620775955811692351000	1.0.14002.14010.7.13998.13998	1961

If $k = 1$, we obtain:

```
                 125 x^2 -  y^2 - 168 x + 108 = 0

X_{n+1} = P X_n + Q Y_n + K
Y_{n+1} = R X_n + S Y_n + L

P = -930249
Q = -83204
K = 625128
R = -10 400500 (8 digits)
S = -930249
L = 6 989136 (7 digits)
```

b		1.0.x.y.11.n.n	c
291	607481535712064	1.0.34.46.11.26.26	3
657	80437718459483924	1.0.67.79.11.59.59	6
4412127	M	1.0.397849.397861.11.397841.397841	35874
	$M = 7377140210095892520946774745837121225728$		

If $k = 2$, we obtain:

```
                 229 x^2 -  y^2 - 352 x + 204 = 0

X_{n+1} = P X_n + Q Y_n + K
Y_{n+1} = R X_n + S Y_n + L

P = -5 848201 (7 digits)
Q = -386460
K = 4 494688 (7 digits)
R = -88 499340 (8 digits)
S = -5 848201 (7 digits)
L = 68 016960 (8 digits)
```

b		1.0.x.y.15.n.n	c
93	648364155504	1.0.18.34.15.6.6	1
9359	67201632381120929976960	1.0.633.649.15.621.621	42
4831653	M	1.0.320703.320719.15.320691.320691	21286
	$M = 127225575836188770520590895511690858928884$		

There are no solutions for $k = 3$ ($y^2 = 365c^2 - 600c + 332$) and $k = 4$ ($y^2 = 533c^2 - 912c + 492$), but $k = 5$ ($y^2 = 733c^2 - 1288c + 684$) apparently has a solution, although it breaks the solver:

```
                        733 x^2 -  y^2 - 1288 x + 684 = 0

X_{n+1} = P X_n + Q Y_n + K
Y_{n+1} = R X_n + S Y_n + L


P = -195 307849 (9 digits)
Q = -7 213860 (7 digits)
K = 171 593800 (9 digits)
R = -5287 759380 (10 digits)
S = -195 307849 (9 digits)
L = NaN aN aN 000000 000000 000000 000000 004645 725840 (60 digits)
```

No initial solution has been found.

Just, for fun, if $k = 12$ then

```
                        3029 x2 -  y2 - 5712 x + 2924 = 0

X_{n+1} = P X_n + Q Y_n + K
Y_{n+1} = R X_n + S Y_n + L


P = -13867 785801 (11 digits)
Q = -251 975020 (9 digits)
K = 13075 733328 (11 digits)
R = -763232 335580 (12 digits)
S = -13867 785801 (11 digits)
L = 719640 657120 (12 digits)
```

An initial solution has not been found. It will be left to the reader to find it and generate other solutions using Alpern [1].

8.10 2-Phoenix numbers of the form $1.0.w.x.y.z.n.n$

Consider the following multiplication tables.

		(1)		(1)			
		(1)		(1)			
		2	5	1	4	0	1
						2	2
(1)	(1)	(1)					
		5	4	3	2	0	2
	5	4	3	2	0	2	
1	0	4	1	5	5	2	2

Base $b = 6$. $\qquad(33)$

		(1)		(2)			
		(1)		(2)			
		5	10	1	13	0	1
						3	3
(1)	(1)	(1)					
		16	13	5	5	0	3
	16	13	5	5	0	3	
1	0	13	1	10	5	3	3

Base $b = 17$. $\qquad(34)$

If (33) and (34) are the beginning of a potentially infinite sequence of examples, then the following must be true.

		(1)		(c_1)			
		(1)		(c_1)			
		w	z	y	x	0	1
						n	n
(1)	(1)	(1)					
		B	w	p	z	0	n
	B	w	p	z	0	n	
1	0	w	x	y	z	n	n

Base b to be determined. $\qquad(35)$

It can be checked that (33) and (34) are the only two examples of (35) with $b \leq 2000$. If one assumes $p = 2n - 1$ then it is easy to prove that these are the only two examples.

9 3-Phoenix numbers

Theorem 105 (3-Phoenix in base $b = 14 + 12k$)**.** *The number*

$$3.4.\underline{13 + 11k.1 + 2k.11 + 9k}$$

is 3-Phoenix in base $b = 14 + 12k$, $k \geq 0$.

See Table 61 for the first dozen examples.

Proof. By multiplication, Table 60. □

Table 60: Multiplication for Theorem 105.

			(3)	
			(2)	
		$11 + 9k$.	$1 + 2k$.	$13 + 11k$
			4.	3
	(1)			
	2.	$5 + 3k$.	$5 + 6k$.	$11 + 9k$
3.	2.	$7 + 8k$.	$10 + 8k$.	
3.	4.	$13 + 11k$.	$1 + 2k$.	$11 + 9k$

Base $b = 14 + 12k$.

Theorem 106. *The number*

$$1.0.n + 1.1.n.1.\underline{n^2 - n.0.n^2 - n}$$

is 3-Phoenix in base $b = n^2 - 1$, $n \geq 3$.

Proof. By multiplication.

			$(n-1)$	(1)	(n)			
			1.	n.	1.	$n + 1$.	0.	1
						$n^2 - n$.	0.	$n^2 - n$
		1.	0.	n.	1.	0	0.	$n^2 - n$
1.	0.	n	1.	0	0	$n^2 - n$.		
1.	0.	$n + 1$.	1.	n.	1.	$n^2 - n$.	0.	$n^2 - n$

Base $b = n^2 - 1$, $n \geq 3$.

□

Table 61: 3-Phoenix in base $b = 14 + 12k$

k	b	
0	14	3.4.13.1.11
1	26	3.4.24.3.20
2	38	3.4.35.5.29
3	50	3.4.46.7.38
4	62	3.4.57.9.47
5	74	3.4.68.11.56
6	86	3.4.79.13.65
7	98	3.4.90.15.74
8	110	3.4.101.17.83
9	122	3.4.112.19.92
10	134	3.4.123.21.101
11	146	3.4.134.23.110
12	158	3.4.145.25.119

Table 62: 3-Phoenix numbers for Theorem 106.

b		$\ldots$ $\ldots$ $n.$ $\ldots$ $\ldots$
8	17871750	1.0. 4.1. 3.1. 6.0. 6
15	2620811712	1.0. 5.1. 4.1. 12.0. 12
24	111231578900	1.0. 6.1. 5.1. 20.0. 20
35	2264804855280	1.0. 7.1. 6.1. 30.0. 30
48	28277417323050	1.0. 8.1. 7.1. 42.0. 42
63	248719610720000	1.0. 9.1. 8.1. 56.0. 56
80	1680346686412872	1.0.10.1. 9.1. 72.0. 72
99	9237813698271600	1.0.11.1.10.1. 90.0. 90
120	43034028575472110	1.0.12.1.11.1.110.0.110
143	174970352197080000	1.0.13.1.12.1.132.0.132

9.1 3-Phoenix numbers with five digits

This section is concerned with 3-Phoenix numbers of the form $x.y.q.z.qx$ with base $b = qy$.

9.1.1 $x.y.2.y.2x$ with base $b = 2y$

The multiplication table has the form

		(y/2)	(1)	
		$((x-1)/2)$	(0)	
		$2x$	y	2
			y	x
	$y/2$	1	y	$2x$
x	$y/2$	1	0	
x	y	2	y	$2x$

Base $b = 2y$, y even.

		$((y-1)/2)$	(1)	
		$((x-1)/2)$	(0)	
		$2x$	y	2
			y	x
	(1)			
	$(y-1)/2$	$y+1$	y	$2x$
x	$(y-1)/2$	$y+1$	0	
x	y	2	y	$2x$

Base $b = 2y$, y odd.

The quadratic diophantine equation $4x^2 - 2y^2 + x - 3 = 0$ is obtained in both cases and has fundamental solutions $(1, \pm 1)$. Using the Quadratic Diophantine Solver [1], we obtain

$$x_n = 17x_{n-1} + 12y_{n-1} + 2,$$
$$y_n = 24x_{n-1} + 17y_{n-1} + 3.$$

The fundamental solution $(1, 1)$ gives

88	31.44.2.44.62
2990	1057.1495.2.1495.2114
101572	35911.50786.2.50786.71822
3450458	1219921.1725229.2.1725229.2439842

The fundamental solution $(1, -1)$ gives

20	7.10.2.10.14
682	241.341.2.341.482
23168	8191.11584.2.11584.16382
787030	278257.393515.2.393515.556514

9.1.2 $x.y.4.z.4x$ with base $b = 4y$

Consider a 3-Phoenix number of the form $x.y.4.z.4x$ in base $b = 4y$.

40	5.10.4.30.20
140	11.35.4.126.44
324	19.81.4.306.76
616	19.81.4.306.76
1040	41.260.4.1014.164
1620	55.405.4.1590.220

3.					(c_2)
2.					(c_1)
1.					
				y	x
			$4x$	z	4
		(1)			
1.				1	0	$4x$
2.			v	u	z
3.		x	c_2	w	
	$x.$	$y.$	$4.$	$z.$	$4x$

A quick check on OEIS shows that $5, 11, 19, \ldots$ is A028387, $x = (n+1)^2 + n = n^2 + 3n + 1$, and so there must be something worth investigating. The base and other digit and sequences are not in the OEIS, so we'll have to do it the hard way. The OEIS shows that $3, 9, 17$ is A014209, $c_1 = n^2 + 3n - 1$, and that $c_2 = n + 1$. The sequence of bases $40, 140, 324, 616, 1040, 1620, \ldots$ has third successive differences equal to 24 so we try a cubic fit and find that $b = 4n^3 + 18n^2 + 18n = 2n(n+3)(2n+3)$. Consequently, $y = n(n+3)(2n+3)/2 = n^3 + (9/2)n^2 + (9/2)n$. This is the first example with rational coefficients! The digit z is found by the formula

$$zx = c_1 b + z$$

and so

$$z = 4n^3 + 18n^2 + 14n - 6 = 2(2n+3)(n^2 + 3n - 1).$$

The equation

$$x(4x) = (n+1)b + w$$

gives

$$w = 2n^3 + 8n^2 + 6n + 4.$$

The equation

$$zy + c_1 = vb + u$$

gives

$$v = n^3 + (9/2)n^2 + (7/2)n - 2 \quad \text{and} \quad u = 2n^3 + 10n^2 + 12n - 1.$$

Remark 107. The only 3-Phoenix numbers of the form $x.y.z.1.u.x$ to one found so far are given in the following table.

119	34832758956	14.70.118.1.77.14
129	130149488688	3.82.127.1.88.3
167	3153940849808	24.46.162.1.165.24
1542	8140336889027048007	933.1122.1541.1.1377.933
3611	267082540806026858492	435.67.3606.1.2238.435
3939	27694450357593575936	29.808.3891.1.2222.29

9.2 3-Phoenix with $x.kx.y.z.xy$ with $b = kxy$

Theorem 108. *The number*

$$8n - 3.2(8n - 3).\underline{2(3n - 1).3(3n - 1)(8n - 3).2(3n - 1)(8n - 3)}$$

is 3-Phoenix in base $b = 4(3n - 1)(8n - 3)$.

Theorem 109. *The number*

$$27n - 17.3(27n - 17).\underline{3(8n - 5).8(8n - 5)(27n - 17).3(8n - 5)(27n - 17)}$$

is 3-Phoenix in base $b = 9(8n - 5)(27n - 17)$.

Theorems 108 and 109 are the motivation for the following theorem.

Theorem 110. *The number* $x.kx.\underline{y.z.xy}$ *is 3-Phoenix in base* $b = kxy = k^2ux$ *whenever*

$$x = k^3n - (k^3 - k^2 - 1), \tag{36a}$$
$$u = (k^2 - 1)n - (k^2 - k - 1), \tag{36b}$$
$$y = ku = k(k^2 - 1)n - (k^3 - k^2 - k), \tag{36c}$$
$$z = (k^2 - 1)ux. \tag{36d}$$

where $n \geq 1$ *and* $k \geq 2$. *Furthermore, if* $n = 1$, *then*

$$k^2 + 1.k^3 + k.k^2.\underline{k^5 - k.k^4 + k^2}$$

is 3-Phoenix in base $b = k^5 + k^3$.

Proof. By multiplication.

		$(ky - 1)$	(1)	
		$(y - 1)$	(0)	
		xy	z	y
			kx	x
	(1)			
	$kx - ky$	p_1	z	xy
x	$ky - 1$	q_1	0	
x	kx	y	z	xy

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The intermediates p_1 and q_1 are defined by

$$x(xy) + (y - 1) = (kx - ky)b + p_1,$$
$$kx(z) + 1 = (ky - 1)b + q_1,$$

and so

$$p_1 = k^2 xy^2 - k^2 x^2 y + x^2 y + y - 1,$$
$$q_1 = kxz - kxy^2 + kxy + 1.$$

The identity $p_1 + q_1 = b + y$ is easily verified using your favorite CAS. Furthermore, if $n = 1$, then $p_1 = k^4 + 2k^2 - 1$ and $q_1 = (k^5 + k^3) - (k^4 + k^2 - 1)$. $\qquad\square$

9.3 3-Phoenix with u-property

Definition 111 (u-property). A number base b has the u-property if given u, $1 < u < b$, there exists v such that $uv = rb$, $r = b - u$, or, equivalently, $uv = r.0$ and $u + r = 1.0$ in base b.

The base b has the u-property if and only if $u(b + v) = b^2$ for some v. Neither u nor v are necessarily divisors of b. It is easily verified that neither u nor v can be $b - 1$, and so $2 \leq u, v \leq b - 2$.

Remark 112. The sequence of bases b with the u-property seem to be the same as the sequence A005279, namely, numbers having divisors d, e with $d < e < 2d$. A proof of equality would be nice!

Sequence 113 (A005279). *Numbers having divisors d, e with $d < e < 2d$.*

Comments from A005279: (1) Arithmetic and harmonic means of A046793(n) and $a(n)$ both integers.
(2) n is in this sequence iff n is a multiple of some term in A020886.
(3) $a(n)$ is also a positive integer v for which there exists a smaller positive integer u such that the contraharmonic mean $(uu + vv)/(u + v)$ is an integer c (in fact, there are two distinct values u giving with v the same c). - Pahikkala Jussi, Dec 14 2008
Comments: Note that numbers of the form $n(n+1)$ are in A005279 since one may take $u = n^2$ and $v = (n+1)^2 - n(n+1) = n^2 + 2n + 1 - n^2 - n = n + 1$.

Theorem 114 (3-Phoenix numbers with u-property). *The number $1.0.0.u.\underline{1.v.1}$ is 3-Phoenix in base b if and only if b has the u-property. See Table 64.*

Proof. By multiplication. See Table 63. $\qquad\square$

Theorem 115. *Suppose b is such that there are divisors d, e such that $d < e < 2d$. Then ...*

Table 63: Proof of Theorem 114.

				u	0	0	1
					1	v	1
1.				u	0	0	1
2.		r	0	0	0	v	
3.		u	0	0	1		
	1	0	0	u	1	v	1

Base b has the u-property:
$uv = r.0,\ r = b - u.$

Table 64: 3-Phoenix for Theorem 114

	b	1.0.0.	u.1.	v.1	r
1	6	1.0.0.	4.1.	3.1	2
2	10	1.0.0.	8.1.	6.1	4
3	10	1.0.0.	9.1.	4.1	3
4	13	1.0.0.	9.1.	X.1	6
5	16	1.0.0.	10.1.	9.1	6
6	18	1.0.0.	14.1.	5.1	4
7	20	1.0.0.	14.1.	10.1	8
8	20	1.0.0.	16.1.	8.1	6
9	24	1.0.0.	14.1.	19.1	10
10	26	1.0.0.	16.1.	18.1	10
11	26	1.0.0.	18.1.	13.1	X
12	26	1.0.0.	21.1.	6.1	5
13	2E	1.0.0.	21.1.	12.1	X
14	30	1.0.0.	20.1.	16.1	10
15	30	1.0.0.	23.1.	10.1	9

Kimball: Suppose that b is such that there exists d, $d|b$ and $d^2 < b < 2d^2$. Then $b^2 = u(b + v)$ has a solution where $1 < u, v < b$. *Proof:* Let $u = d^2$. Then $d^2|b^2$ and $b < (b/d)^2 < 2b$. Hence, $0 < b^2/d^2 - b < b$ and so let $v = b^2/d^2 - b$. □

Suppose $d < e < 2d$. Then $d'/2 < e' < d'$ where d' and e' are the conjugate divisors of d, e, respectively. But also $e' \leq \sqrt{n} \leq e$ with equality iff $e = e' = \sqrt{n}$.

9.4   3-Phoenix, 7-digit, $p.q.r.s.x.y.z$

				czr	czq	czp
				cyr	cyq	cyp
				cxr	cxq	cxp
				s	r	q	p
					z	y	x
cq	cr	cs	cx	cy		
		cxs	dxs	dxr	dxq	dxp
	cys	dys	dyr	dyq	dyp	
czs	dzs	dzr	dzq	dzp		
p	q	r	s	x	y	z

Theorem 116.   *1. The number*

$$1.8k.8k - 5.3.\underline{15k.25k} - 1.15k$$

is 3-Phoenix in base $b = 40k$.

2. The number

$$1.55k.55k - 7.4.\underline{28k.49k} - 1.28k$$

is 3-Phoenix in base $b = 77k$.

The following are unaccounted for,

72	191638755392	1.27.3.3.32.36.32
72	209324989376	1.36.13.12.8.63.8
74	246853605304	1.37.18.10.10.63.10
84	614823041184	1.63.1.2.72.62.72
102	1690132993320	1.51.8.22.6.95.6
116	3662393000800	1.58.43.8.20.95.20
119	4883608426900	1.85.77.3.56.2.56
180	61251752403725	1.144.28.61.5.54.5
220	19844621870064	1.165.13.87.4.93.4
234	277872118660227	1.162.15.10.39.80.39

74	246853605304	1.37.18.10.10.63.10
234	277872118660227	1.162.15.10.39.80.39

Note that the fifth decreases from 18 to 15 and so there is no formula.

9.5 3-Phoenix numbers of the form $1.0.0.p.q.x.y.x$

No 3-Phoenix numbers of the form $1.0.p.q.r.\underline{x.y.x}$ have been found.

Here are the first two examples of 3-Phoenix numbers of the form $1.0.0.p.q.\underline{x.y.x}$.

				3	1	0	0	1
						3	2	3
	(1)	(1)	(1)					
1.				9	3	0	0	3
2.			6	2	0	0	2	
3.		9	3	0	0	3		
	1	0	0	1	3	3	2	3

Base $b = 10$.

3.				(1)				
2.				(2)				
1.				(1)				
				3	4	0	0	1
						6	12	6
	(1)	(1)	(1)					
1.				19	3	0	0	6
2.		1	17	6	0	0	12	
3.		19	3	0	0	6		
	1	0	0	4	3	6	12	6

Base $b = 21$.

The general case is as follows.

3.				(c_1)				
2.				(c_2)				
1.				(c_1)				
				q	p	0	0	1
						x	y	x
	(1)	(1)	(1)					
1.				u	q	0	0	x
2.		c_3	w	v	0	0	y	
3.		u	q	0	0	x		
	1	0	0	p	q	x	y	x

Base b to be determined.

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Table 65: 3-Phoenix numbers of the form $1.0.0.p.q.x.y.x$

b		c_1	c_2	c_3	u	v	w
10	1.0.0.1.3.3.2.3	0	0	0	9	2	6
21	1.0.0.4.3.6.12.6	1	2	1	19	6	17
82	1.0.0.8.24.3.33.3	0	3	9	72	18	57
208	1.0.0.174.38.5.74.5	4	61	13	194	188	169
1041	1.0.0.523.50.20.20.20	20	630	10	320	30	1010

The solution is:

$$c_1 = u - qx, \tag{37a}$$

$$c_2 = \frac{b^2qux + b^3u + 2bq^2x + qux - b^2q - b^3qx - b^2u^2 - 3bqu - bu - 2q^2 - q}{bu + q}, \tag{37b}$$

$$c_3 = b - u - 1, \tag{37c}$$

$$p = \frac{bu + q}{x} - bq, \tag{37d}$$

$$v = \frac{bu + q}{x} + b - bq - u, \tag{37e}$$

$$w = b - q - 1, \tag{37f}$$

$$y = \frac{b^3x - b^2ux - 2bqx + bu - ux + q}{bu + q}. \tag{37g}$$

Clearly, we have $u \geq qx$ and $(bu + q) \mod x = 0$. Substituting $q = mx - bu$ we obtain

$$c_1 = bux - mx^2 + u, \tag{38a}$$

$$c_2 = \frac{bu(b^3 + b^2u - u)}{m} + 2bmx^2 - b^3x - 3b^2ux + b^2 + ux + bu - 2mx - 1, \tag{38b}$$

$$c_3 = b - u - 1, \tag{38c}$$

$$p = b^2u - bmx + m, \tag{38d}$$

$$v = b^2u - bmx + b + m - u, \tag{38e}$$

$$w = bu - mx + b - 1, \tag{38f}$$

$$y = \frac{b^3 + b^2u - u}{m} - 2bx + 1. \tag{38g}$$

Observe that we also have $(b^3 + b^2u - u) \mod m = 0$. The results of a computer search up to base $b = 2000$ are shown in Table 65. The author leaves it to the reader to construct more examples. Have fun!

10   4-Phoenix numbers

The only discernable pattern among 4-Phoenix numbers with six digits is $1.p.\underline{x.y.z.x}$.

b	N	N_b
7	29808	1.5.2.6.2.2
112	25496485731	1.50.3.100.26.3

$$
\begin{array}{|ccccc|} \hline
 & (2) & (4) & (1) & \\
 & & & & \\
 & 2. & 2. & 6. & 2 \\
 & & & 5. & 1 \\ \hline
 & & (1) & & \\
 & 2. & 2. & 6. & 2 \\
1. & 5. & 0. & 3. & 3. \\
1. & 5. & 2. & 6. & 2. \quad 2 \\ \hline
\multicolumn{5}{c}{\text{Base } b = 7.}
\end{array}
\qquad
\begin{array}{|ccccc|} \hline
 & (12) & (44) & (1) & \\
 & & & & \\
 & 3. & 26. & 100. & 3 \\
 & & & 50. & 1 \\ \hline
 & & (1) & & \\
 & 3. & 26. & 100. & 3 \\
1. & 50. & 0. & 73. & 3. \\
1. & 50. & 3. & 100. & 26. \quad 3 \\ \hline
\multicolumn{5}{c}{\text{Base } b = 112 = 2^4 \cdot 7.}
\end{array}
\tag{39}
$$

A bit of searching for numbers with multiplication defined by (39) and only one new number was found. See Table 66.

11   Universal 1-phoenix numbers

In Theorem 5 it was observed that for any composite base b the number $1.e.(d+1)$ is 1-phoenix in base b, where d is proper divisor of b and $e = b/d$ is the conjugate divisor of d. But what about prime bases? Are there any 1-phoenix numbers in a prime base? Yes! And they are the key to proving the existence of infinitely many 1-phoenix numbers in any base b, $b \geq 3$.

Here is a table of 1-phoenix numbers in base b for b prime.

3	1.1.0.0.2.2
5	1.3.2.0.2.4
7	1.5.4.0.2.6
11	1.9.8.0.2.10
13	1.11.10.0.2.12

The pattern is clear. If $B = b - 1$, $C = b - 2$, and $D = b - 3$, then a quick calculation shows that $1.C.D.0.2.B$ is 1-phoenix in any base $b \geq 3$. In fact, quite a bit more is true.

Theorem 117 (Universal 1-phoenix numbers). *If $b \geq 3$ is any base, let $B = b - 1$, $C = b - 2$ and $D = b - 3$. Then*

$$1.C.D.0.2.B$$

is 1-phoenix in base b. Furthermore,

1. *$1.C.D.[0.2.1.0.C.D].0.2.B$ is 1-phoenix in base b.*

2. *$1.C.D.[B.1.2.B.D.C].0.2.B$ is 1-phoenix in base b.*

3. *Let $M_0, M_1, \ldots, M_k$ be instances of $[0.2.1.0.C.D]$ and let $P_1, \ldots, P_k$ be instances of $[B.1.2.B.D.C]$. Then*

$$1.C.D. \left[M_k.P_k \ldots M_1.P_1.M_0.P_1.M_1 \ldots P_k.M_k \right].0.2.B$$

   is 1-phoenix in base b.

4. *Let $P_0, P_1, \ldots P_k$ be instances of $[B.1.2.B.D.C]$ and let $M_1, \ldots, M_k$ be instances of $[0.2.1.0.C.D]$. Then*

$$1.C.D. \left[P_k.M_k \ldots P_1.M_1.P_0.M_1.P_1 \ldots M_k.P_k \right].0.2.B$$

   is 1-phoenix in base b.

   Therefore, there are infinitely many 1-phoenix numbers in any base $b \geq 3$.

Remark 118. Note that $1.C.D.0.2.B$ with $b = 3$ is $1.1.0.0.2.2$, an instance of $1.1.[0].n.n$ in base $b = n^2 - 1$ (Theorem 26). Furthermore, note that $1.C.D.0.2.B$ with $b = 4$ is $1.2.1.0.2.3$, an instance of $1.[2.1.0].2.3$ (Theorem 6.2).

Proof. By multiplication.

		(D)	(D)			
	2.	0.	D.	C.	1	
					B	
1.	C.	D	0.	2.	B	

Base $b \geq 3$.

$$B \cdot C - 2 = (b-1)(b-2) - 2 = b^2 - 3b = (b-3)b = D.0.$$
$$B \cdot D + D = D \cdot b = D.0.$$
$$B \cdot 2 - C = 2b - 2 - b + 2 = b = 1.0.$$

Verification of 1: $1.C.D.[0.2.1.0.C.D].0.2.B$.

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	⑦	⑥	⑤		④	③			②	①	
	(D)	(D)			(1)	(1)			(D)	(D)	
2.	0.	[D.	C.	0.	1.	2.	0]	D.	C	1	
											B
1.	C.	D.	[0.	2.	1.	0.	C.	D].	0.	2.	B

Base $b \geq 3$.

① $\quad$ B · C − 2 = (b − 1)(b − 2) − 2 = b² − 3b = (b − 3)b = D.0,
② $\quad$ B · D + D = (B + 1)D = D · b = D.0,
③ $\quad$ B · 2 − C = 2b − 2 − b + 2 = b = 1.0,
④ $\quad$ B · 1 + 1 = b = 1.0,
⑤ $\quad\quad$ B · C = ①,
⑥ $\quad$ B · D + D = ②,
⑦ $\quad$ B · 2 − C = ③.

Verification of 2: 1.C.D.[B.1.2.B.D.C].0.2.B.

	⑨	⑧	⑦	⑥	⑤	④	③		②	①	
	(D)	(D)	(C)	(1)	(1)	(C)			(D)	(D)	
2.	0.	[C.	D.	B.	2.	1.	B]	D.	C	1	
											B
1.	C.	D.	[B.	1.	2.	B.	D.	C].	0.	2.	B

Base $b \geq 3$.

① $\quad\quad$ B · C − 2 = (b − 1)(b − 2) − 2 = (b − 3)b = D.0,
② $\quad\quad$ B · D + D = (B + 1)D = D.0,
③ $\quad$ B · B + D − C = b² − 2b + 1 + b − 3 − b + 2 = (b − 2)b = C.0,
④ $\quad$ B · 1 + C − D = b − 1 + b − 2 − b + 3 = b = 1.0,
⑤ $\quad$ B · 2 + 1 − B = b − 1 + 1 = b = 1.0,
⑥ $\quad$ B · B + 1 − 2 = (b − 1)² − 1 = b² − 2b = C.0,
⑦ $\quad$ B · D + C − 1 = (b − 1)(b − 3) + (b − 2) − 1
$\qquad\qquad\qquad = (b − 1)(b − 3) + (b − 3)$
$\qquad\qquad\qquad = (b − 3)b = D.0,$
⑧ $\quad$ B · C + D − B = (b − 1)(b − 2) + (b − 3) − (b − 1)
$\qquad\qquad\qquad = (b − 1)(b − 3) + (b − 3)$
$\qquad\qquad\qquad = (b − 3)b = D.0,$
⑨ $\quad\quad$ B · 2 − C = 2b − 2 − b + 2 = b = 1.0.

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The carry pattern above [0.2.1.0.C.D] is D.[D.0.1.1.0.D] and the carry pattern above [B.1.2.B.D.C] is D.[D.C.1.1.C.D]. Note that the first and last digit in each carry pattern is D so concatenation of carry patterns is well-defined. Let us prove that 1.C.D.[$P.M.P$].0.2.B is 1-phoenix in base b, where M is an instance of [0.2.1.0.C.D] and P is an instance of [B.1.2.B.D.C]. Since all the calculations have already been done, we have

		(D)	([P'	M'	P'])	(D)		1
	2.	0.	[P.	M.	P].	D.	C.	1
								B
1.	C.	D.	[P.	M.	P].	0.	2.	B

Base $b \geq 3$.

where M' and P' are the carry patterns for M and P, respectively. A similar calculation works for MPM and any other pattern with reflection symmetry. □

12 Universal k-phoenix numbers with $k \geq 2$

In this section we demonstrate by construction the existence of infinitely many k-Phoenix numbers in any base b with $k \geq 2$. However, we have the following special case.

Theorem 119. *The number*

$$1.0.B.C.C.B.C.B.B.B.0.1.1.\underline{B.B}$$

is 2-Phoenix in any base $b \geq 2$, where B $= b - 1$ *and* C $= b - 2$.

Proof. By multiplication.

				(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)			
				(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)			
		1.	1.	0.	B.	B.	B.	C.	B.	C.	C.	B.	0.	1
													B.	B
(1)	(1)	(1)	(1)	(1)										
	B.	B.	C.	B.	B	B.	0.	B.	0.	0.	1.	0.	B	
	B.	B.	C.	B.	B.	B	0.	B.	0.	0.	1.	0.	B.	
1.	0.	B.	C.	C.	B.	C	B.	B.	B.	0.	1.	1.	B.	B

Base $b \geq 2$.

□

The author has been unable to find similar k-Phoenix numbers with $k > 2$.

Let us return to the construction of k-Phoenix numbers for any base $b \geq 2$, $k \geq 2$. Scrutiny of many examples reveals the following pattern.

k

2	1.0(1).B(1).0(1).1.0(0).B(2).<u>1.0(0).1</u>
3	1.0(2).B(2).0(3).1.0(1).B(4).<u>1.0(1).1</u>
4	1.0(3).B(3).0(4).1.0(2).B(6).<u>1.0(2).1</u>
5	1.0(4).B(4).0(5).1.0(3).B(8).<u>1.0(3).1</u>

Theorem 120 (Existence of universal k-Phoenix Numbers). *For any base $b \geq 2$ and $k \geq 2$, the $(8k - 7)$-digit number*

$$1.0(k - 1).B(k - 1).0(2k - 3).1.0(k - 2).B(2k - 2).\underline{1.0(k - 2).1} \tag{40}$$

is k-Phoenix in any base b, where $B = b - 1$ and, for example, $B(k - 1)$ denotes the digit B repeated $k - 1$ times.

Proof. By multiplication. To save space, let

$$B' = B(k - 1), \quad B'' = B(k - 2), \quad 0' = 0(k - 1), \quad \text{and} \quad 0'' = 0(k - 2), \quad k \geq 2.$$

			B′	B″	B	0″	1	0″	0′	B′	0	0″	1
											1	0″	1
(1)	(1)	(1)	(1)										
			B′	B″	B	0″	1	0″	0′	B′	0	0″	1
		B′	B′	0″	1	0″	0	0″	B′	0′	1		
	1	0′	B′	0″	0	0″	1	0″	B′	B′	1	0″	1

Base $b \geq 2$.

$\square$

Further scrutiny reveals the following repeating pattern.

Theorem 121 (Infinitely many universal k-phoenix numbers, I). *For any base $b \geq 2$ and $k \geq 2$, let $B = b - 1$, $C = b - 2$, and let*

$$\mathbf{A_0} = B(2k - 3).C.B(k - 2).0(2k - 3).1.0(k - 2). \tag{41}$$

Then the number

$$1.0(k - 1).B(k - 1).0(2k - 3).1.0(k - 2).[\mathbf{A_0}].B(2k - 2).\underline{1.0(k - 2).1} \tag{42}$$

is k-Phoenix in base b. Thus, there are infinitely many k-Phoenix numbers in base b.

Furthermore, if there are m occurrences of $\mathbf{A_0}$, then the number (42) *has $8k - 7 + (6k - 6)m$ digits.*

Due to the width of the multiplication table, it has been moved to its own page in landscape.

Proof. Proof of Theorem 121. To save space, let

$$B' = B(k-1), \quad B'' = B(k-2), \quad 0' = 0(k-1), \quad \text{and} \quad 0'' = 0(k-2), \quad k \geq 2.$$

1.	B′ B″ B ⟨0″ 1 0″ 0″ 0 B″ C B″ B″ B⟩ 0″ 1 0″ 0′ B′ 0 0″ 1
	1 0″ 1
	(1) (1) (1) (1) (1) (1) (1) (1) (1)
2.	B′ B″ B ⟨0″ 1 0″ 0″ 0 B″ C B″ B″ B⟩ 0″ 1 0″ 0′ B′ 0 0″ 1
3.	B′ B′ ⟨0″ 1 0″ 0 0″ B″ C B″ B B″⟩ 0″ 1 0″ 0 0″ B′ 0″ 1
4.	1 0′ B′ 0″ 0 0″ 1 0″ ⟨B″ B B″ C B″ 0″ 0 0″ 1 0″⟩ B′ B′ 1 0″ 1

Base $b \geq 2$.

The original number is in row 4. Since $[PQ]P = P[QP]$, the pattern in the first, second and third rows can be aligned with that in the fourth row.

1.	B′ B″ B 0″ 1 0″ ⟨0″ 0 B″ C B″ B″ B 0″ 1 0″⟩ 0′ B′ 0 0″ 1
	1 0″ 1
	(1) (1) (1) (1) (1) (1) (1) (1) (1)
2.	B′ B″ B 0″ 1 0″ ⟨0″ 0 B″ C B″ B″ B 0″ 1 0″⟩ 0′ B′ 0 0″ 1
3.	B′ B′ 0″ 1 0″ 0 0″ ⟨B″ C B″ B B″ 0″ 1 0″ 0 0″⟩ B′ 0″ 1
4.	1 0′ B′ 0″ 0 0″ 1 0″ ⟨B″ B B″ C B″ 0″ 0 0″ 1 0″⟩ B′ B′ 1 0″ 1

Base $b \geq 2$.

$\square$

Theorem 122 (Infinitely many universal k-phoenix numbers, II). *For any base $b \geq 2$ and $k \geq 2$, let $B = b - 1$, $C = b - 2$, and let*

$$\mathbf{A_1} = B(2k-2).0(k-1), \tag{43}$$
$$\mathbf{A_2} = B(k-1).0(2k-2). \tag{44}$$

Then the number

$$1.0(k-1).[\![\mathbf{A_2}]\!].B(k-1).0(2k-3).1.0(k-2).[\![\mathbf{A_1}]\!].B(2k-2).\underline{1.0(k-2).1} \tag{45}$$

is k-Phoenix in base b. Thus, there are infinitely many k-Phoenix numbers in base b.

    Furthermore, if there are n occurrences each of $\mathbf{A_1}$ and $\mathbf{A_2}$, then the number (45) has $8k - 7 + 6(k-1)n$ digits.

Proof. REWRITE If n is the number of occurrences each of $\mathbf{A_1}$ and $\mathbf{A_2}$, then the proof splits into two cases: $n = 1$ and $n > 1$. The reason is that one cannot shift past the first B in position 1.8 and 2.8. To save space, let

$$B' = B(k-1), \quad B'' = B(k-2), \quad 0' = 0(k-1), \quad \text{and} \quad 0'' = 0(k-2), \quad k \geq 2.$$

    Suppose that

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.
1.		B′	B′	⟪[0′	B′	B″	B]⟫	0″	1	0″	0′	B′	⟪[0′	0′	B′]⟫	0	0″	1
																1	0″	1
	(1)	(1)	(1)	(1)	(1)	(1)	(1)											
2.		B′	B′	⟪[0′	B′	B″	B]⟫	0″	1	0″	0′	B′	⟪[0′	0′	B′]⟫	0	0″	1
3.	B′	B′	⟪[0′	B′	B′]⟫	0″	1	0″	0	0″	B′	⟪[0′	0′	B′]⟫	0′	1			
4.	1	0′	⟨[B′	0′	0′]⟩	B′	0″	0	0″	1	0″	⟨[B′	B′	0′]⟩	B′	B′	1	0″	1
1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.

The original number is reproduced in row 4. Recall that $[PQ]P = P[QP]$. Shift all patterns in rows 1, 2, and 3 to align with their counterparts in row 4, giving the table below.

1.	⟪[B′ B′ 0′]⟫ B′ B″ B 0″ 1 0″ ⟪[0′ B′ 0′]⟫ 0′ B′ 0 0″ 1	
	1 0″ 1	
(1) (1) (1) (1) (1) (1) (1)		
2.	⟪[B′ B′ 0′]⟫ B′ B″ B 0″ 1 0″ ⟪[0′ B′ 0′]⟫ 0′ B′ 0 0″ 1	
3.	B′ ⟪[B′ 0′ B′]⟫ B′ 0″ 1 0″ 0 0″ ⟪[B′ 0′ 0′]⟫ B′ 0′ 1	
4.	1 0′ ⟪[B′ 0′ 0′]⟫ B′ 0″ 0 0″ 1 0″ ⟪[B′ B′ 0′]⟫ B′ B′ 1 0″ 1	

$\square$

Theorem 123 (Infinitely many universal k-phoenix numbers, III). *For any base $b \geq 2$ and $k \geq 2$, let* B $= b - 1$, C $= b - 2$, *and let*

$$\mathbf{A_0} = \text{B}(2k-3).\text{C}.\text{B}(k-2).0(2k-3).1.0(k-2), \tag{46}$$
$$\mathbf{A_1} = \text{B}(2k-2).0(k-1), \tag{47}$$
$$\mathbf{A_2} = \text{B}(k-1).0(2k-2). \tag{48}$$

Then the number

$$1.0(k-1).[\![\mathbf{A_2}]\!].\text{B}(k-1).0(2k-3).1.0(k-2).[\mathbf{A_0}].[\![\mathbf{A_1}]\!].\text{B}(2k-2).\underline{1.0(k-2).1} \tag{49}$$

is k-Phoenix in base b. Thus, there are infinitely many k-Phoenix numbers in base b.

Furthermore, if there are m occurrences of $\mathbf{A_0}$ and n occurrences each of $\mathbf{A_1}$ and $\mathbf{A_2}$, then the number (49) has $8k - 7 + (6k - 6)m + 6(k - 1)n$ digits.

Proof of Theorem 123. Note that the proof of Theorem 122 allows us to shift patterns across the initial B at position 2.8 of the first table. To save space, let

$$B' = B(k-1), \quad B'' = B(k-2), \quad 0' = 0(k-1), \quad \text{and} \quad 0'' = 0(k-2), \quad k \geq 2.$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	
		B′	B′	‹[0′	B′	B″	B]›	‹[0″	1 0″	0″	0″	0	B″	C	B″	B″	B]›	0″	1 0″	0′	B′	‹[0′	0′	B]›	0	0″	1		.1
																									1	0″	1		
(1)	(1)	(1)	(1)	(1)	(1)	(1)						(1)	(1)	(1)	(1)	(1)													
		B′	B′	‹[0′	B′	B″	B]›	‹[0″	1 0″	0″	0″	0	B″	C	B″	B″	B]›	0″	1 0″	0′	B′	‹[0′	0′	B]›	0	0″	1		.2
		B′	B′	‹[0′	B′	B′]›	‹[0″	1	0″	0	0″	B″	C	B″	B	B]›	0″	1	0″	0	0″	B′	‹[0′	0′	B]›	0′	1		.3
1	0′	‹[B′	0′	0′]›	B′	0″	0	0″	1	0″	‹[B″	B	B″	C	B″	0″	0	0″	1	0″]›	‹[B′	B′	0′]›	B′	B′	1	0″	1	.4
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	

The original number is reproduced on row 4. Recall that $P[PQ] = [QP]P$. Observe that all patterns in rows 1, 2, and 3 can be shifted to align with their counterparts in row 4, giving the table below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	
		‹[B′	B′	0′]›	B′	B″	B 0″	1 0″	‹[0″	0	B″	C	B″	B″	B 0″	1 0″]›	‹[0′	B′	0′]›	0′	B′	0	0″	1					.1
																									1	0″	1		
										(1)	(1)	(1)	(1)	(1)															
		‹[B′	B′	0′]›	B′	B″	B 0″	1 0″	‹[0″	0	B″	C	B″	B″	B 0″	1 0″]›	‹[0′	B′	0′]›	0′	B′	0	0″	1					.2
		B′	‹[B′	0′	B′]›	B′	0″	1 0″	0	0″	‹[B″	C	B″	B	B″	0″	1 0″	0	0″]›	‹[B′	0′	0′]›	B′	0′	1				.3
1	0′	‹[B′	0′	0′]›	B′	0″	0	0″	1	0″	‹[B″	B	B″	C	B″	0″	0	0″	1	0″]›	‹[B′	B′	0′]›	B′	B′	1	0″	1	.4
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	

Q.E.D.

119

$$m = [3, 2.0.2]$$
$$n = [3, 1.1.1.0.0.2.0.0.1]$$
$$mn = [3, 1.0.0.2.0.0.1.1.1.2.0.2]$$

3.									(1)			
2.												
1.									(1)			
			1	1	1	0	0	2	0	0	1
									2	0	2
	(1)	(1)	(1)	(1)	(1)						
1.				2	2	2	0	1	1	0	0	2
2.			0	0	0	0	0	0	0	0	0
3.		2	2	2	0	1	1	0	0	2
	1	0	0	2	0	0	1	1	1	2	0	2

			3	0	3	1	1	2	1	3	0	0	1
											1	1	1
	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)				
1.				3	0	3	1	1	2	1	3	0	0	1
2.			3	0	3	1	1	2	1	3	0	0	1
3.		3	0	3	1	1	2	1	3	0	0	1
	1	0	0	3	1	2	1	1	3	0	3	1	1	1

Base $b = 4$.

			B	0	B	1	1	C	1	B	0	0	1
											1	1	1
	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)				
1.				B	0	B	1	1	C	1	B	0	0	1
2.			B	0	B	1	1	C	1	B	0	0	1
3.		B	0	B	1	1	C	1	B	0	0	1
	1	0	0	B	1	C	1	1	B	0	B	1	1	1

Base b.

13 Appendix on the uv-condition and [A005279](#)

See Section 9.3 for 3-Phoenix numbers with the u-property.

Sequence 124 ([A005279](#)). *Integers n with divisors d, e such that $d < e < 2d$.*

Definition 125 (u=property). Integers n such that there exist $1 < u, v < b$ such that $n^2 = u(n + v)$.

Conjecture 126. All integers in [A005279](#) have the u-property and all integers with the u-property are in [A005279](#).

13.1 The v-property

Table ??? shows all numbers n and pairs $[d, E_d]$ where $d|n$ and $E_d = \{e : e|n, d < e < 2d\}$.

n	$E_d, d|n$
6	$[2, \{3\}]$
12	$[2, \{3\}], [3, \{4\}], [4, \{6\}]$
15	$[3, \{5\}]$
18	$[2, \{3\}], [6, \{9\}]$
20	$[4, \{5\}]$
24	$[2, \{3\}], [3, \{4\}], [4, \{6\}], [6, \{8\}], [8, \{12\}]$
28	$[4, \{7\}]$
30	$[2, \{3\}], [3, \{5\}], [5, \{6\}], [6, \{10\}], [10, \{15\}]$
35	$[5, \{7\}]$
36	$[2, \{3\}], [3, \{4\}], [4, \{6\}], [6, \{9\}], [9, \{12\}], [12, \{18\}]$
40	$[4, \{5\}], [5, \{8\}], [8, \{10\}]$
42	$[2, \{3\}], [6, \{7\}], [14, \{21\}]$
45	$[3, \{5\}], [5, \{9\}], [9, \{15\}]$
48	$[2, \{3\}], [3, \{4\}], [4, \{6\}], [6, \{8\}], [8, \{12\}], [12, \{16\}], [16, \{24\}]$
54	$[2, \{3\}], [6, \{9\}], [18, \{27\}]$
56	$[4, \{7\}], [7, \{8\}], [8, \{14\}]$
60	$[2, \{3\}], [3, \{4, 5\}], [4, \{5, 6\}], [5, \{6\}], [6, \{10\}], [10, \{12, 15\}], [12, \{15, 20\}], [15, \{20\}], [20, \{30\}]$
63	$[7, \{9\}]$
66	$[2, \{3\}], [6, \{11\}], [22, \{33\}]$
70	$[5, \{7\}], [7, \{10\}], [10, \{14\}]$
72	$[2, \{3\}], [3, \{4\}], [4, \{6\}], [6, \{8, 9\}], [8, \{9, 12\}], [9, \{12\}], [12, \{18\}], [18, \{24\}], [24, \{36\}]$
75	$[3, \{5\}], [15, \{25\}]$
77	$[7, \{11\}]$
78	$[2, \{3\}], [26, \{39\}]$
80	$[4, \{5\}], [5, \{8\}], [8, \{10\}], [10, \{16\}], [16, \{20\}]$
84	$[2, \{3\}], [3, \{4\}], [4, \{6, 7\}], [6, \{7\}], [7, \{12\}], [12, \{14, 21\}], [14, \{21\}], [21, \{28\}], [28, \{42\}]$
88	$[8, \{11\}]$
90	$[2, \{3\}], [3, \{5\}], [5, \{6, 9\}], [6, \{9, 10\}], [9, \{10, 15\}], [10, \{15, 18\}], [15, \{18\}], [18, \{30\}], [30, \{45\}]$
91	$[7, \{13\}]$
96	$[2, \{3\}], [3, \{4\}], [4, \{6\}], [6, \{8\}], [8, \{12\}], [12, \{16\}], [16, \{24\}], [24, \{32\}], [32, \{48\}]$
99	$[9, \{11\}]$

Integers n not in [A005279](#).
Powerful n not in [A005279](#).

4	2^2
8	2^3
9	3^2

16	2^4
25	5^2
27	3^3
32	2^5
49	7^2
64	2^6
81	3^4
121	11^2
125	5^3
128	2^7
169	13^2
243	3^5
256	2^8
289	17^2
343	7^3
361	19^2
484	$2^2 \cdot 11^2$
512	2^9
529	23^2
625	5^4
676	$2^2 \cdot 13^2$
729	3^6
841	29^2
961	31^2
1024	2^{10}
1156	$2^2 \cdot 17^2$
1331	11^3
1369	37^2
1444	$2^2 \cdot 19^2$
1681	41^2

Powerful n, $n \le 12^4$, not in [A005279](A005279) with two or more prime factors. Observe that they all have only two factors.

484	$2^2 \cdot 11^2$
676	$2^2 \cdot 13^2$
1156	$2^2 \cdot 17^2$
1444	$2^2 \cdot 19^2$
2116	$2^2 \cdot 23^2$
2312	$2^3 \cdot 17^2$
2888	$2^3 \cdot 19^2$
3025	$5^2 \cdot 11^2$

3249	$3^2 \cdot 19^2$
3364	$2^2 \cdot 29^2$
3844	$2^2 \cdot 31^2$
4232	$2^3 \cdot 23^2$
4761	$3^2 \cdot 23^2$
5324	$2^2 \cdot 11^3$
5476	$2^2 \cdot 37^2$
6724	$2^2 \cdot 41^2$
6728	$2^3 \cdot 29^2$
7396	$2^2 \cdot 43^2$
7569	$3^2 \cdot 29^2$
7688	$2^3 \cdot 31^2$
8649	$3^2 \cdot 31^2$
8788	$2^2 \cdot 13^3$
8836	$2^2 \cdot 47^2$
10952	$2^3 \cdot 37^2$
11236	$2^2 \cdot 53^2$
12321	$3^2 \cdot 37^2$
13448	$2^3 \cdot 41^2$
13924	$2^2 \cdot 59^2$
14161	$7^2 \cdot 17^2$
14792	$2^3 \cdot 43^2$
14884	$2^2 \cdot 61^2$
15129	$3^2 \cdot 41^2$
16641	$3^2 \cdot 43^2$
17672	$2^3 \cdot 47^2$
17689	$7^2 \cdot 19^2$
17956	$2^2 \cdot 67^2$
19652	$2^2 \cdot 17^3$
19881	$3^2 \cdot 47^2$
20164	$2^2 \cdot 71^2$

Odd powerful n, $n \le 6 \cdot 12^4$, not in [A005279](#) with two or more prime factors. Observe that they all have only two factors.

3025	$5^2 \cdot 11^2$
3249	$3^2 \cdot 19^2$
4761	$3^2 \cdot 23^2$
7569	$3^2 \cdot 29^2$
8649	$3^2 \cdot 31^2$
12321	$3^2 \cdot 37^2$
14161	$7^2 \cdot 17^2$

15129	$3^2 \cdot 41^2$
16641	$3^2 \cdot 43^2$
17689	$7^2 \cdot 19^2$
19881	$3^2 \cdot 47^2$
25281	$3^2 \cdot 53^2$
25921	$7^2 \cdot 23^2$
31329	$3^2 \cdot 59^2$
33275	$5^2 \cdot 11^3$
33489	$3^2 \cdot 61^2$
40401	$3^2 \cdot 67^2$
45369	$3^2 \cdot 71^2$
47961	$3^2 \cdot 73^2$
56169	$3^2 \cdot 79^2$
61731	$3^2 \cdot 19^3$
62001	$3^2 \cdot 83^2$
64009	$11^2 \cdot 23^2$
70225	$5^2 \cdot 53^2$
71289	$3^2 \cdot 89^2$
84681	$3^2 \cdot 97^2$
87025	$5^2 \cdot 59^2$
91809	$3^2 \cdot 101^2$
93025	$5^2 \cdot 61^2$
93987	$3^3 \cdot 59^2$
95481	$3^2 \cdot 103^2$
100467	$3^3 \cdot 61^2$
101761	$11^2 \cdot 29^2$
103041	$3^2 \cdot 107^2$
106929	$3^2 \cdot 109^2$
109503	$3^2 \cdot 23^3$
112225	$5^2 \cdot 67^2$
114921	$3^2 \cdot 113^2$
116281	$11^2 \cdot 31^2$
121203	$3^3 \cdot 67^2$

Table of powerful n, $n \leq 3 \cdot 12^6$, not in A005279, with powers higher than a square.

39304	$2^3 \cdot 17^3$
54872	$2^3 \cdot 19^3$
97336	$2^3 \cdot 23^3$
195112	$2^3 \cdot 29^3$
238328	$2^3 \cdot 31^3$
405224	$2^3 \cdot 37^3$

551368	$2^3 \cdot 41^3$
636056	$2^3 \cdot 43^3$
668168	$2^3 \cdot 17^4$
810448	$2^4 \cdot 37^3$
830584	$2^3 \cdot 47^3$
1042568	$2^3 \cdot 19^4$
1102736	$2^4 \cdot 41^3$
1191016	$2^3 \cdot 53^3$
1272112	$2^4 \cdot 43^3$
1643032	$2^3 \cdot 59^3$
1661168	$2^4 \cdot 47^3$
1815848	$2^3 \cdot 61^3$
2238728	$2^3 \cdot 23^4$
2382032	$2^4 \cdot 53^3$
2406104	$2^3 \cdot 67^3$
2863288	$2^3 \cdot 71^3$
3112136	$2^3 \cdot 73^3$
3286064	$2^4 \cdot 59^3$
3631696	$2^4 \cdot 61^3$
3944312	$2^3 \cdot 79^3$
4574296	$2^3 \cdot 83^3$
4812208	$2^4 \cdot 67^3$
5545233	$3^3 \cdot 59^3$
5639752	$2^3 \cdot 89^3$
5658248	$2^3 \cdot 29^4$
5726576	$2^4 \cdot 71^3$
6128487	$3^3 \cdot 61^3$
6224272	$2^4 \cdot 73^3$
7301384	$2^3 \cdot 97^3$
7388168	$2^3 \cdot 31^4$
7888624	$2^4 \cdot 79^3$
8120601	$3^3 \cdot 67^3$
8242408	$2^3 \cdot 101^3$
8741816	$2^3 \cdot 103^3$

Table of powerful n, not in A005279, of the form p^3q^3, assuming there are only two prime factors. The primes q depend in general on p^a.

n	$p^3 \cdot q^3$
39304	$2^3 \cdot 17^3$
54872	$2^3 \cdot 19^3$
97336	$2^3 \cdot 23^3$

195112	$2^3 \cdot 29^3$
5545233	$3^3 \cdot 59^3$
6128487	$3^3 \cdot 61^3$
8120601	$3^3 \cdot 67^3$
9663597	$3^3 \cdot 71^3$
18609625	$5^3 \cdot 53^3$
25672375	$5^3 \cdot 59^3$
28372625	$5^3 \cdot 61^3$
1976656375	$5^3 \cdot 251^3$
353393243	$7^3 \cdot 101^3$
374805361	$7^3 \cdot 103^3$
420189749	$7^3 \cdot 107^3$
444194947	$7^3 \cdot 109^3$
16194277	$11^3 \cdot 23^3$
198155287	$11^3 \cdot 53^3$
273359449	$11^3 \cdot 59^3$
21047437081	$11^3 \cdot 251^3$
53582633	$13^3 \cdot 29^3$
65450827	$13^3 \cdot 31^3$
660776311	$13^3 \cdot 67^3$
786330467	$13^3 \cdot 71^3$
248858189	$17^3 \cdot 37^3$
338608873	$17^3 \cdot 41^3$
390617891	$17^3 \cdot 43^3$
510082399	$17^3 \cdot 47^3$
472729139	$19^3 \cdot 41^3$
545338513	$19^3 \cdot 43^3$
712121957	$19^3 \cdot 47^3$
1021147343	$19^3 \cdot 53^3$
1263214441	$23^3 \cdot 47^3$
1811386459	$23^3 \cdot 53^3$
2498846293	$23^3 \cdot 59^3$
2761677827	$23^3 \cdot 61^3$

Table of powerful n, not in A005279, of the form $p^4 q^4$, assuming there are only two prime factors.

n	$p^4 \cdot q^4$
29986576	$2^4 \cdot 37^4$
45212176	$2^4 \cdot 41^4$
54700816	$2^4 \cdot 43^4$

78074896	$2^4 \cdot 47^4$
57178852641	$3^4 \cdot 163^4$
63001502001	$3^4 \cdot 167^4$
72555348321	$3^4 \cdot 173^4$
83156680161	$3^4 \cdot 179^4$
4931550625	$5^4 \cdot 53^4$
7573350625	$5^4 \cdot 59^4$
8653650625	$5^4 \cdot 61^4$
2480703750625	$5^4 \cdot 251^4$
249849022801	$7^4 \cdot 101^4$
270234665281	$7^4 \cdot 103^4$
314722122001	$7^4 \cdot 107^4$
338920744561	$7^4 \cdot 109^4$
115524532321	$11^4 \cdot 53^4$
177410282401	$11^4 \cdot 59^4$
58111973780641	$11^4 \cdot 251^4$
63870929141041	$11^4 \cdot 257^4$
575536166881	$13^4 \cdot 67^4$
725783021041	$13^4 \cdot 71^4$
811082161201	$13^4 \cdot 73^4$
1112453263441	$13^4 \cdot 79^4$
8691228751921	$17^4 \cdot 101^4$
9400362132001	$17^4 \cdot 103^4$
10947899355121	$17^4 \cdot 107^4$
11789670764881	$17^4 \cdot 109^4$
33902309759761	$19^4 \cdot 127^4$
38379524204641	$19^4 \cdot 131^4$
45908877320881	$19^4 \cdot 137^4$
48648964964161	$19^4 \cdot 139^4$
1365534810721	$23^4 \cdot 47^4$
170023892141041	$23^4 \cdot 157^4$
197543053110001	$23^4 \cdot 163^4$
217659300264961	$23^4 \cdot 167^4$

Table of n not in [A005279](A005279) with more than two factors. If squarefree and pqr, then $q > 2p$ and $r > 2pq$.

230	$2 \cdot 5 \cdot 23$
290	$2 \cdot 5 \cdot 29$
310	$2 \cdot 5 \cdot 31$
370	$2 \cdot 5 \cdot 37$

406	$2 \cdot 7 \cdot 29$
410	$2 \cdot 5 \cdot 41$
430	$2 \cdot 5 \cdot 43$
434	$2 \cdot 7 \cdot 31$
470	$2 \cdot 5 \cdot 47$
518	$2 \cdot 7 \cdot 37$
530	$2 \cdot 5 \cdot 53$
574	$2 \cdot 7 \cdot 41$
590	$2 \cdot 5 \cdot 59$
602	$2 \cdot 7 \cdot 43$
610	$2 \cdot 5 \cdot 61$
658	$2 \cdot 7 \cdot 47$
670	$2 \cdot 5 \cdot 67$
710	$2 \cdot 5 \cdot 71$
730	$2 \cdot 5 \cdot 73$
742	$2 \cdot 7 \cdot 53$
790	$2 \cdot 5 \cdot 79$
826	$2 \cdot 7 \cdot 59$
830	$2 \cdot 5 \cdot 83$
854	$2 \cdot 7 \cdot 61$
890	$2 \cdot 5 \cdot 89$
903	$3 \cdot 7 \cdot 43$
938	$2 \cdot 7 \cdot 67$
970	$2 \cdot 5 \cdot 97$
987	$3 \cdot 7 \cdot 47$
994	$2 \cdot 7 \cdot 71$
1010	$2 \cdot 5 \cdot 101$
1022	$2 \cdot 7 \cdot 73$
1030	$2 \cdot 5 \cdot 103$
1034	$2 \cdot 11 \cdot 47$
1070	$2 \cdot 5 \cdot 107$
1090	$2 \cdot 5 \cdot 109$
1106	$2 \cdot 7 \cdot 79$
1113	$3 \cdot 7 \cdot 53$
1130	$2 \cdot 5 \cdot 113$
1162	$2 \cdot 7 \cdot 83$
1166	$2 \cdot 11 \cdot 53$
1239	$3 \cdot 7 \cdot 59$
1246	$2 \cdot 7 \cdot 89$
1265	$5 \cdot 11 \cdot 23$
1270	$2 \cdot 5 \cdot 127$
1281	$3 \cdot 7 \cdot 61$

1298	$2 \cdot 11 \cdot 59$
1310	$2 \cdot 5 \cdot 131$
1342	$2 \cdot 11 \cdot 61$
1358	$2 \cdot 7 \cdot 97$
1370	$2 \cdot 5 \cdot 137$
1378	$2 \cdot 13 \cdot 53$
1390	$2 \cdot 5 \cdot 139$
1407	$3 \cdot 7 \cdot 67$
1414	$2 \cdot 7 \cdot 101$
1442	$2 \cdot 7 \cdot 103$
1474	$2 \cdot 11 \cdot 67$
1490	$2 \cdot 5 \cdot 149$
1491	$3 \cdot 7 \cdot 71$
1498	$2 \cdot 7 \cdot 107$
1510	$2 \cdot 5 \cdot 151$
1526	$2 \cdot 7 \cdot 109$
1533	$3 \cdot 7 \cdot 73$
1534	$2 \cdot 13 \cdot 59$
1562	$2 \cdot 11 \cdot 71$
1570	$2 \cdot 5 \cdot 157$
1582	$2 \cdot 7 \cdot 113$
1586	$2 \cdot 13 \cdot 61$
1606	$2 \cdot 11 \cdot 73$
1630	$2 \cdot 5 \cdot 163$
1659	$3 \cdot 7 \cdot 79$
1670	$2 \cdot 5 \cdot 167$

Table of n, $n < 700000$, not in A005279 with more than two factors, not squarefree.

29095	$5 \cdot 11 \cdot 23^2$
31075	$5^2 \cdot 11 \cdot 113$
34925	$5^2 \cdot 11 \cdot 127$
36025	$5^2 \cdot 11 \cdot 131$
37675	$5^2 \cdot 11 \cdot 137$
38829	$3 \cdot 7 \cdot 43^2$
45129	$3 \cdot 7^2 \cdot 307$
45717	$3 \cdot 7^2 \cdot 311$
46011	$3 \cdot 7^2 \cdot 313$
46389	$3 \cdot 7 \cdot 47^2$
46599	$3 \cdot 7^2 \cdot 317$
48657	$3 \cdot 7^2 \cdot 331$
49539	$3 \cdot 7^2 \cdot 337$

51009	$3 \cdot 7^2 \cdot 347$
51303	$3 \cdot 7^2 \cdot 349$
51891	$3 \cdot 7^2 \cdot 353$
52773	$3 \cdot 7^2 \cdot 359$
53949	$3 \cdot 7^2 \cdot 367$
54665	$5 \cdot 13 \cdot 29^2$
54831	$3 \cdot 7^2 \cdot 373$
55713	$3 \cdot 7^2 \cdot 379$
56301	$3 \cdot 7^2 \cdot 383$
57183	$3 \cdot 7^2 \cdot 389$
58359	$3 \cdot 7^2 \cdot 397$
58947	$3 \cdot 7^2 \cdot 401$
58989	$3 \cdot 7 \cdot 53^2$
59337	$3^2 \cdot 19 \cdot 347$
59679	$3^2 \cdot 19 \cdot 349$
60123	$3 \cdot 7^2 \cdot 409$
60363	$3^2 \cdot 19 \cdot 353$
61389	$3^2 \cdot 19 \cdot 359$
61593	$3 \cdot 7^2 \cdot 419$
61887	$3 \cdot 7^2 \cdot 421$
62465	$5 \cdot 13 \cdot 31^2$
62757	$3^2 \cdot 19 \cdot 367$
63357	$3 \cdot 7^2 \cdot 431$
63651	$3 \cdot 7^2 \cdot 433$
63783	$3^2 \cdot 19 \cdot 373$
64533	$3 \cdot 7^2 \cdot 439$
64809	$3^2 \cdot 19 \cdot 379$
65121	$3 \cdot 7^2 \cdot 443$
65493	$3^2 \cdot 19 \cdot 383$
66003	$3 \cdot 7^2 \cdot 449$
66519	$3^2 \cdot 19 \cdot 389$
67179	$3 \cdot 7^2 \cdot 457$
67767	$3 \cdot 7^2 \cdot 461$
67887	$3^2 \cdot 19 \cdot 397$
68061	$3 \cdot 7^2 \cdot 463$
68571	$3^2 \cdot 19 \cdot 401$
68649	$3 \cdot 7^2 \cdot 467$
69939	$3^2 \cdot 19 \cdot 409$

13.2   The uv-condition

That's about enough on A005279. Let's look at the condition $n^2 = u(n + v)$ for some u and v, $1 < u, v < n$.

Table of square solutions (some u is a square).

n	$[u, v]$ with some u a square
6	$[4, 3]$
12	$[8, 6], [9, 4]$
15	$[9, 10]$
20	$[16, 5]$
24	$[16, 12], [18, 8]$
28	$[16, 21]$
30	$[18, 20], [20, 15], [25, 6]$
35	$[25, 14]$
40	$[25, 24], [32, 10]$
42	$[28, 21], [36, 7]$
45	$[25, 36], [27, 30]$
48	$[32, 24], [36, 16]$
54	$[36, 27]$
56	$[32, 42], [49, 8]$
60	$[36, 40], [40, 30], [45, 20], [48, 15], [50, 12]$
63	$[49, 18]$
66	$[36, 55], [44, 33]$
70	$[49, 30], [50, 28]$
72	$[48, 36], [54, 24], [64, 9]$
77	$[49, 44]$
80	$[50, 48], [64, 20]$
84	$[48, 63], [49, 60], [56, 42], [63, 28], [72, 14]$
88	$[64, 33]$
90	$[50, 72], [54, 60], [60, 45], [75, 18], [81, 10]$
91	$[49, 78]$
96	$[64, 48], [72, 32]$
99	$[81, 22]$
104	$[64, 65]$
108	$[72, 54], [81, 36]$
110	$[100, 11]$
112	$[64, 84], [98, 16]$
117	$[81, 52]$
120	$[64, 105], [72, 80], [75, 72], [80, 60], [90, 40], [96, 30], [100, 24]$
126	$[81, 70], [84, 63], [98, 36], [108, 21]$
130	$[100, 39]$

132 $[72, 110], [88, 66], [99, 44], [121, 12]$
135 $[75, 108], [81, 90]$
140 $[80, 105], [98, 60], [100, 56], [112, 35]$
143 $[121, 26]$
144 $[81, 112], [96, 72], [108, 48], [128, 18]$

Table of n, $n \leq 432$, with no square solutions (no u is a square).

n	$[u, v]$ with no u a square	
18	$2 \cdot 3^2$	$[12, 9]$
36	$2^2 \cdot 3^2$	$[24, 18], [27, 12]$
75	$3 \cdot 5^2$	$[45, 50]$
78	$2 \cdot 3 \cdot 13$	$[52, 39]$
100	$2^2 \cdot 5^2$	$[80, 25]$
102	$2 \cdot 3 \cdot 17$	$[68, 51]$
105	$3 \cdot 5 \cdot 7$	$[63, 70], [75, 42]$
114	$2 \cdot 3 \cdot 19$	$[76, 57]$
138	$2 \cdot 3 \cdot 23$	$[92, 69]$
162	$2 \cdot 3^4$	$[108, 81]$
174	$2 \cdot 3 \cdot 29$	$[116, 87]$
175	$5^2 \cdot 7$	$[125, 70]$
186	$2 \cdot 3 \cdot 31$	$[124, 93]$
189	$3^3 \cdot 7$	$[147, 54]$
196	$2^2 \cdot 7^2$	$[112, 147]$
200	$2^3 \cdot 5^2$	$[125, 120], [160, 50]$
222	$2 \cdot 3 \cdot 37$	$[148, 111]$
225	$3^2 \cdot 5^2$	$[125, 180], [135, 150]$
245	$5 \cdot 7^2$	$[175, 98]$
246	$2 \cdot 3 \cdot 41$	$[164, 123]$
258	$2 \cdot 3 \cdot 43$	$[172, 129]$
282	$2 \cdot 3 \cdot 47$	$[188, 141]$
297	$3^3 \cdot 11$	$[243, 66]$
318	$2 \cdot 3 \cdot 53$	$[212, 159]$
324	$2^2 \cdot 3^4$	$[216, 162], [243, 108]$
348	$2^2 \cdot 3 \cdot 29$	$[232, 174], [261, 116]$
351	$3^3 \cdot 13$	$[243, 156]$
354	$2 \cdot 3 \cdot 59$	$[236, 177]$
366	$2 \cdot 3 \cdot 61$	$[244, 183]$
372	$2^2 \cdot 3 \cdot 31$	$[248, 186], [279, 124]$
385	$5 \cdot 7 \cdot 11$	$[245, 220], [275, 154]$
392	$2^3 \cdot 7^2$	$[224, 294], [343, 56]$
402	$2 \cdot 3 \cdot 67$	$[268, 201]$

426   $2 \cdot 3 \cdot 71$ $[284, 213]$
429   $3 \cdot 11 \cdot 13$ $[363, 78]$

Table of n such that some uv pair has u and v relatively prime. There are no v relatively prime to n.

6      $[4, 3]$
12    $[8, 6], [9, 4]$
15    $[9, 10]$
20    $[16, 5]$
28    $[16, 21]$
30    $[18, 20], [20, 15], [25, 6]$
35    $[25, 14]$
40    $[25, 24], [32, 10]$
42    $[28, 21], [36, 7]$
45    $[25, 36], [27, 30]$
56    $[32, 42], [49, 8]$
63    $[49, 18]$
66    $[36, 55], [44, 33]$
70    $[49, 30], [50, 28]$
72    $[48, 36], [54, 24], [64, 9]$
77    $[49, 44]$
84    $[48, 63], [49, 60], [56, 42], [63, 28], [72, 14]$
88    $[64, 33]$
90    $[50, 72], [54, 60], [60, 45], [75, 18], [81, 10]$
91    $[49, 78]$
99    $[81, 22]$
104   $[64, 65]$
110   $[100, 11]$
117   $[81, 52]$
120   $[64, 105], [72, 80], [75, 72], [80, 60], [90, 40], [96, 30], [100, 24]$
126   $[81, 70], [84, 63], [98, 36], [108, 21]$
130   $[100, 39]$
132   $[72, 110], [88, 66], [99, 44], [121, 12]$
143   $[121, 26]$
144   $[81, 112], [96, 72], [108, 48], [128, 18]$

Table of n such that all uv pairs have a common factor.

18    $2 \cdot 3^2$ $[2^2 \cdot 3, 3^2]$
24    $2^3 \cdot 3$ $[2^4, 2^2 \cdot 3], [2 \cdot 3^2, 2^3]$
36    $2^2 \cdot 3^2$ $[2^3 \cdot 3, 2 \cdot 3^2], [3^3, 2^2 \cdot 3]$

48	$2^4 \cdot 3$	$[2^5, 2^3 \cdot 3], [2^2 \cdot 3^2, 2^4]$
54	$2 \cdot 3^3$	$[2^2 \cdot 3^2, 3^3]$
60	$2^2 \cdot 3 \cdot 5$	$[2^2 \cdot 3^2, 2^3 \cdot 5], [2^3 \cdot 5, 2 \cdot 3 \cdot 5], [3^2 \cdot 5, 2^2 \cdot 5], [2^4 \cdot 3, 3 \cdot 5], [2 \cdot 5^2, 2^2 \cdot 3]$
75	$3 \cdot 5^2$	$[3^2 \cdot 5, 2 \cdot 5^2]$
78	$2 \cdot 3 \cdot 13$	$[2^2 \cdot 13, 3 \cdot 13]$
80	$2^4 \cdot 5$	$[2 \cdot 5^2, 2^4 \cdot 3], [2^6, 2^2 \cdot 5]$
96	$2^5 \cdot 3$	$[2^6, 2^4 \cdot 3], [2^3 \cdot 3^2, 2^5]$
100	$2^2 \cdot 5^2$	$[2^4 \cdot 5, 5^2]$
102	$2 \cdot 3 \cdot 17$	$[2^2 \cdot 17, 3 \cdot 17]$
105	$3 \cdot 5 \cdot 7$	$[3^2 \cdot 7, 2 \cdot 5 \cdot 7], [3 \cdot 5^2, 2 \cdot 3 \cdot 7]$
108	$2^2 \cdot 3^3$	$[2^3 \cdot 3^2, 2 \cdot 3^3], [3^4, 2^2 \cdot 3^2]$
112	$2^4 \cdot 7$	$[2^6, 2^2 \cdot 3 \cdot 7], [2 \cdot 7^2, 2^4]$
114	$2 \cdot 3 \cdot 19$	$[2^2 \cdot 19, 3 \cdot 19]$
135	$3^3 \cdot 5$	$[3 \cdot 5^2, 2^2 \cdot 3^3], [3^4, 2 \cdot 3^2 \cdot 5]$
138	$2 \cdot 3 \cdot 23$	$[2^2 \cdot 23, 3 \cdot 23]$
140	$2^2 \cdot 5 \cdot 7$	$[2^4 \cdot 5, 3 \cdot 5 \cdot 7], [2 \cdot 7^2, 2^2 \cdot 3 \cdot 5], [2^2 \cdot 5^2, 2^3 \cdot 7], [2^4 \cdot 7, 5 \cdot 7]$
150	$2 \cdot 3 \cdot 5^2$	$[2 \cdot 3^2 \cdot 5, 2^2 \cdot 5^2], [2^2 \cdot 5^2, 3 \cdot 5^2], [5^3, 2 \cdot 3 \cdot 5]$
160	$2^5 \cdot 5$	$[2^2 \cdot 5^2, 2^5 \cdot 3], [2^7, 2^3 \cdot 5]$
162	$2 \cdot 3^4$	$[2^2 \cdot 3^3, 3^4]$
168	$2^3 \cdot 3 \cdot 7$	$[2^5 \cdot 3, 2 \cdot 3^2 \cdot 7], [2 \cdot 7^2, 2^3 \cdot 3 \cdot 5], [2^4 \cdot 7, 2^2 \cdot 3 \cdot 7], [2 \cdot 3^2 \cdot 7, 2^3 \cdot 7], [2^4 \cdot 3^2, 2^2 \cdot 7], [3 \cdot 7^2, 2^3$
174	$2 \cdot 3 \cdot 29$	$[2^2 \cdot 29, 3 \cdot 29]$
175	$5^2 \cdot 7$	$[5^3, 2 \cdot 5 \cdot 7]$
180	$2^2 \cdot 3^2 \cdot 5$	$[2^2 \cdot 5^2, 2^4 \cdot 3^2], [2^2 \cdot 3^3, 2^3 \cdot 3 \cdot 5], [2^3 \cdot 3 \cdot 5, 2 \cdot 3^2 \cdot 5], [3^3 \cdot 5, 2^2 \cdot 3 \cdot 5], [2^4 \cdot 3^2, 3^2 \cdot 5], [2 \cdot 3 \cdot$
186	$2 \cdot 3 \cdot 31$	$[2^2 \cdot 31, 3 \cdot 31]$
189	$3^3 \cdot 7$	$[3 \cdot 7^2, 2 \cdot 3^3]$
192	$2^6 \cdot 3$	$[2^7, 2^5 \cdot 3], [2^4 \cdot 3^2, 2^6]$
196	$2^2 \cdot 7^2$	$[2^4 \cdot 7, 3 \cdot 7^2]$
200	$2^3 \cdot 5^2$	$[5^3, 2^3 \cdot 3 \cdot 5], [2^5 \cdot 5, 2 \cdot 5^2]$

References

[1] Dario Alpern's Quadratic Diophantine Solver, https://www.alpertron.com.ar/QUAD.HTM.

[2] Chebolu, Sunil K., *What's so special about the divisors of 24?*, http://arxiv.org/abs/1104.5052

[3] Weisstein, Eric W., "Pillai's Conjecture." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/PillaisConjecture.html

[4] *Binary Quadratic Forms and OEIS*, https://oeis.org/wiki/Binary_Quadratic_Forms_and_OEIS.

[5] P. Erdos and J. L. Selfridge, *The Product Of Consecutive Integers Is Never A Power*, http://projecteuclid.org/euclid.ijm/1256050816.

[6] Guy, Richard K., Unsolved Problems in Number Theory, Third Edition, Problem Books in Mathematics, Springer-Verlag, 2004.

[7] *On Pillai's Diophantine Equation*, Illinois J. Math., Volume 19, Issue 2 (1975), 292-301.

[8] *Differences Between Powers*, http://www.mathpages.com/home/kmath104/kmath104.htm.

[9] Crescenzo, Peter, *A Diophantine Equation Which Arises in the Theory of Finite Groups*, Advances in Mathematics 17, 25-29 (1975).

[10] W. Ljunggren, Noen setninger om ubestemte likninger av formen $(x^n - 1)/(x - 1) = y^q$ [New propositions about the indeterminate equation $(x^n - 1)/(x - 1) = y^q$], Norsk Mat. Tidsskr. 25 (1943), 17-20.

[11] Le Mao-Hua, A note on perfect powers of the form $x^{m-1} + \cdots + x + 1$, Acta Arithmetica, 69, 91-98 (1995).

[12] Pell's Equation, https://en.wikipedia.org/wiki/Pell%27s_equation.

[13] Cassini and Catalan identities, https://en.wikipedia.org/wiki/Cassini_and_Catalan_identities

[14] Koshy, Thomas, *Fibonacci and Lucas Numbers with Applications*, John-Wiley & Sons, 2001.