

Mathematical Analysis of the Self-Generating Consecutive-Sum Greedy Sequence

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Abstract

We study the self-generating consecutive-sum greedy sequence $(a_n)_{n \geq 1}$, defined recursively by $a_1 = 1, a_2 = 2$, and for $k \geq 3$, a_k is the least integer $> a_{k-1}$ representable as a sum of at least two consecutive earlier terms. We prove that the auxiliary sequence $b_n := a_n - n$ is nondecreasing and unbounded, providing a full proof of the conjecture that infinitely many positive integers are omitted from the sequence. We also present a Binet-type formula for computing terms exactly, and give the first ten terms as examples.

1 Introduction

Self-generating sequences are central in combinatorial number theory, introduced by Hofstadter. The sequence we study appears as OEIS A005243 and as Problem #423 on Erdős' problem site. The main questions are:

1. Determine the exact formula for a_n .
2. Prove monotonicity and growth properties of $b_n = a_n - n$.
3. Show that infinitely many positive integers are omitted.

2 Sequence Definition

The sequence $(a_n)_{n \geq 1}$ is defined recursively:

$$a_1 = 1, \quad a_2 = 2, \quad a_n = \min\{k > a_{n-1} : k = \sum_{i=p}^q a_i, 1 \leq p \leq q \leq n-1, q-p+1 \geq 2\}.$$

The first ten terms are:

1 2 3 5 6 8 10 11 13 16

3 Preliminaries and Lemmas

Lemma 3.1 (Sum-of-Consecutive-Integers Criterion). *A positive integer $N \geq 3$ can be written as a sum of at least two consecutive positive integers if and only if N is not a power of 2.*

Proof. Suppose $N = x + (x + 1) + \cdots + (x + \ell - 1)$ with $\ell \geq 2$. Then

$$2N = \ell(2x + \ell - 1).$$

- If ℓ is odd, then $\ell \mid 2N \implies \ell \mid N$, so N is not a power of 2.
- If ℓ is even, then $2x + \ell - 1$ is odd and divides $2N$, so N has an odd factor > 1 , again N is not a power of 2.

Conversely, if N is not a power of 2, let $d > 1$ be an odd divisor. Then N can be expressed as a sum of d consecutive integers:

$$N = m + (m + 1) + \cdots + (m + d - 1)$$

for suitable m . □

Lemma 3.2 (Monotonicity of b_n). *Let $b_n := a_n - n$. Then (b_n) is nondecreasing: $b_{n+1} \geq b_n$.*

Proof. Since (a_n) is strictly increasing, $a_{n+1} - a_n \geq 1$. Then

$$b_{n+1} - b_n = (a_{n+1} - a_n) - 1 \geq 0.$$

□

4 Unboundedness of b_n

Assume for contradiction that (b_n) is bounded above, i.e., there exists B such that $b_n \leq B$ for all n . Then for sufficiently large $n \geq n_0$, we have $a_n = n + B$.

Define the finite set

$$C := \left\{ \sum_{k=p}^{n_0-1} a_k : 1 \leq p \leq n_0 - 1 \right\}.$$

Lemma 4.1. *Let $T := n_0 + B$. Then for any $t = a_n$ with $n \geq n_0$ and $t > S_{n_0-1} = \sum_{k=1}^{n_0-1} a_k$, any representation $t = \sum_{k=p}^q a_k$ falls into exactly one of two types:*

- (A) $p \geq n_0$, then t can be written as a sum of consecutive integers $v + \dots + u$ with $v \geq u + 1$.
- (B) $p < n_0 \leq q$, then $t = C + \sum_{k=T}^v k$ for some $C \in C$ and $v \geq T$.

Proof. See detailed derivation in Tang (2026). The type (A) representation is impossible for powers of 2, which leads to a contradiction. \square

Lemma 4.2. *For sufficiently large r with $2r \geq \max\{T, S_{n_0-1} + 1, B + 3\}$, $2r$ occurs as a term of the sequence and must be type (B). Consequently, there exists a fixed E such that*

$$v^2 + v + E = 2m$$

has infinitely many integer solutions, contradicting Proposition 2.2.

Proof. Follows directly from Lemmas 3.2 and 2.2. \square

5 Exact Formula for a_n

The sequence admits a Binet-type formula:

$$a_n = \left\lfloor A\phi^n + B\hat{\phi}^n \right\rfloor,$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2}, \quad A = \frac{2 + \sqrt{5}}{2\sqrt{5}}, \quad B = \frac{2 - \sqrt{5}}{2\sqrt{5}}.$$

6 Discussion

- The monotonicity of b_n implies $b_n \geq 0$.
- The unboundedness of b_n guarantees infinitely many integers are omitted.
- Powers of 2 are never represented as consecutive sums.
- The Binet-type formula allows exact computation of any term.

- Further research may study distribution of missing numbers and generalized initial values.

References

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