A Geometric Proof of Pi's Irrationality

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Abstract

We give a proof of π's irrationality that references principles of set theory and cardinality within the context of basic geometric properties of a circle.

Introduction

There have been many proofs of the irrationality of π [2, 4, 7]. The first is attributed to Lambert. It's long and complicated. In 1947 Niven gave an entirely different shockingly short half a page proof [9, 10]. Still his proof made various unacknowledged (hence obscure) references to the techniques of Hermite in his transcendence of e proof [8]; difficult. In both proofs the natural connection of π to the circle is quite remote.

The proof here makes this connection. It is geometric in nature. Other geometric proofs of note are Sondow's proof of the irrationality of e [11] and Hardy's of the square root of five [5]. These might be thought of as curiosities, not destined for standard analysis textbooks. But, I suggest, π's origins in geometry might make a geometric proof of its irrationality more natural and attractive (classy) to students and mathematicians.

Of course all these words are premised on the proof being correct. It uses an atypical argument. If lines consist of two types ones with defined slopes and ones with undefined slopes and all defined slopes includes all slopes having rational number values then given all radii specified by arc lengths on a unit circle are lines, then a line with an undefined slope can't have a rational slope associated with it, but it can have a rational arc length unless they've been exhausted by some clever (if I do say so myself) trick.

Background

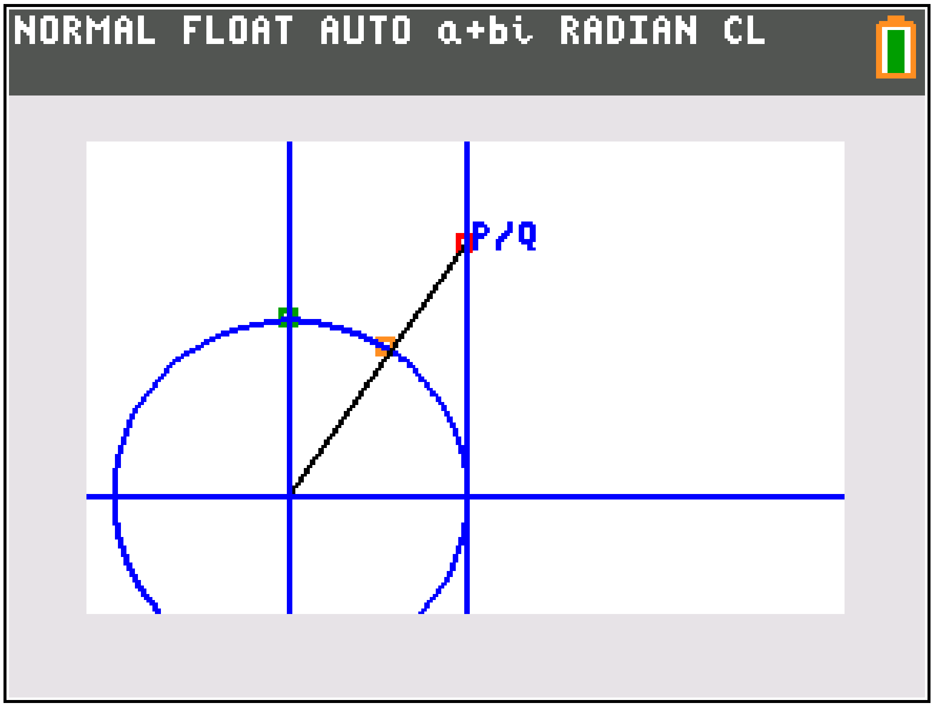

Figure 1: From right to left: Blue box at arc length of $\pi/2$; Orange box at intersection on circle; Red box on line at p/q.

Figure 1: A unit circle has a tangent starting at $(1, 0)$. The tangent line has one representative of all rational numbers; the red box at length p/q from $(1, 0)$. A line is drawn connecting this p/q point with the origin of the circle. This line intersects the circle and generates an arc length, a radian measure of the angle formed. The intersection point is a blue box on the circle. The p/q on the line thus generates a slope and an arc length. The intersection of the positive y-axis and the circle is given by a green box. The radius associated with this intersection has an arc length of $\pi/2$ and doesn't have a defined slope. This contrasts with the other radii of the first quadrant generated by both red and orange boxes.

With this diagram we can prove π is irrational. It's a cardinality (set theory) as well as a geometric proof.

Proof

The positive rational numbers are denumerable. On page 80 of *What is Mathematics?* the authors give a specific mapping from whole numbers to positive rational numbers [3]. Make this mapping the sequence c_n.

We can define a mapping from these points on the tangent line of Figure 1 to associated arc lengths on the circle; the points of intersection (Orange Boxes) on the circle have associated arc lengths; vice versa works too. Let $LtoC(c_n)$ equal such arc lengths. It is clear that

$$Lub(\{LtoC(c_n)\}) = \pi/2,$$

but it is also clear that no c_j is such that $LtoC(c_j) = \pi/2$ as we can always find a $c_j = p/q$ that improves the accuracy of the arc generated, its approximation to $\pi/2$. This statement is reflected in

$$\arctan c_j \neq \pi/2 \tag{1}$$

for all positive integer j. In other words, taking $\tan$ of both sides of (1) gives c_j is infinity, an impossibility, but not quite the needed contradiction.

We need to show that $\tan c_n$ is never infinity, that would imply that c_n, a rational can't equal $\pi/2$. Define the $slope$ function that takes slope intercept forms of lines to their slopes. We note

$$\lim_{n\to\infty} slope(\frac{n}{q}x) = \infty,$$

for all whole number q. Let's also define the function arc that returns the arc length associated with such lines. Then

$$\lim_{n\to\infty} arc(slope(\frac{n}{q}x)) = \pi/2.$$

As the slope of each $\frac{n}{q}x$ is defined (it's n/q), its corresponding arc can't be $\pi/2$. This in turn means that there exists a p for every q such that

$$\frac{p}{q} < \frac{\pi}{2} < \frac{p+1}{q}.$$

Thus $\tan c_n$ is defined for all n and $\pi/2$ and π are proven to be irrational.

Conclusion

We can make Figure 1 a *proof without words*. By geometry (if not geography) the arcs in the first quadrant have stepped through all arcs that generate all possible rational slopes. As the arc $\pi/2$ is different than these (greater) it can't be a rational valued arc.

This seems related to the pigeon hole principle [6]. This principle says that pigeons (dwelling in holes on sides of cliffs) are such that if five pigeons are distributed into four holes at least one hole must have more than one pigeon in it. Given every rational slope from the line can fly into a rational hole on the circle (a tricky flight), every rational valued arc can be occupied [1][1]; that is it has a pigeon from the line occupying it. If we assume $\pi/2$ is rational, it would have to double up in one hole. But each arc length is unique, so that can't happen. It must be $\pi/2$ is irrational.[2]

References

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[8] ______, Discovering and Proving Pi is Irrational, *Monthly*, **53** (2010) 553-557.

[9] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947) 509.

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[1] Benardete develops the paradox of an infinite hotel (holes) without any vacancies (pigeon convention) via a story he attributes to Hilbert.

[2] Note sums of rational numbers are implied by an irrational number. So an infinite sum of rational numbers (slopes from the line) can add up to a vertical, a slope-less line given by arc length $\pi/2$.

[10] ______, *Irrational Numbers*, Carus Mathematical Monographs, no. 11, Mathematical Association of America, Washington, DC, 1985.

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