

A Geometric Proof of Pi's Irrationality

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Abstract

We give a proof of π 's irrationality that references the basic geometric properties of a circle and principles of set theory and cardinality: the set of all lines with defined slopes includes all slopes with rational numbers; the radius given by an arc length of $\pi/2$ has an undefined slope; therefore it can't be rational.

Introduction

There have been many proofs of the irrationality of π [1, 3, 6]. The first is attributed to Lambert. It's long and complicated. In 1947 Niven gave an entirely different shockingly short half a page proof [8, 9]. Still his proof made various unacknowledged (hence obscure) references to the techniques of Hermite in his transcendence of e proof [7]; difficult. In both proofs the natural connection of π to the circle is quite remote.

The proof here makes this connection. It is geometric in nature. Other geometric proofs of note are Sondow's proof of the irrationality of e [10] and Hardy's of the square root of five [4]. These might be thought of as curiosities, not destined for standard analysis textbooks. But, I suggest, π 's origins in geometry might make a geometric proof of its irrationality more natural and attractive (classy) to students and mathematicians.

Of course all these words are premised on the proof being correct. It uses an atypical argument. If lines consist of two types ones with defined slopes and ones with undefined slopes and all defined slopes includes all slopes having rational number values then given all radii specified by arc

lengths on a unit circle are lines, then a line with an undefined slope can't have a rational slope associated with it, but it can have a rational arc length unless they've been exhausted by its location. By geometry the arcs in the first quadrant have stepped through all rational arcs that generate all possible rational slopes. As the arc $\pi/2$ is different than these, greater it can't be a rational valued arc.

Background

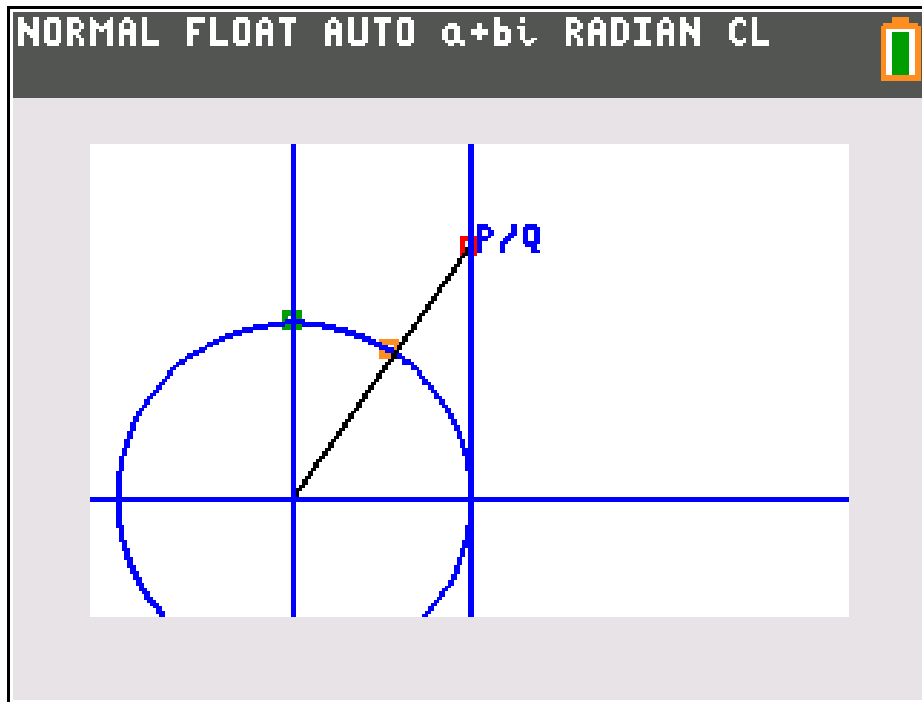


Figure 1: From right to left: BlueBox at $\pi/2$; OrangeBox intersection on circle; RedBox on line at p/q .

All radii, arc lengths, and slopes of radii are in the top semi-circle, the first and second quadrants.

In Figure 1 all of the ideas are present. A unit circle has a tangent starting at $(1, 0)$ and one representative of all rational numbers is an RedBox at length p/q from $(1, 0)$. A line is drawn connecting this p/q point with the origin of the circle. This line intersects the circle and generates an arc

length, a radian measure of the angle formed. The intersection point is a BlueBox (a solid disc) on the circle. The unit circle has a circumference of 2π and when this is divided into arc length increments of $\pi/4$, the positive y-axis has an angle measure with the positive x-axis of $\pi/2$. The p/q on the line thus generates a slope and an arc length. The intersection of the positive y-axis and the circle is given by a GreenBox (an open disc). This indicates that the radius at angle $\pi/2$ does not have a defined slope. This contrasts with the solid dots that have defined slopes. The idea for proving $\pi/2$ is irrational consists in noting that ∞ on the line maps to $\pi/2$ on the circle via the solid dot points converging to $\pi/2$.

It's a cardinality as well as a geometric proof. We'll define a few things.

Proof

Definition 1. *The line and radius generated by the asterisk on the line is designated with*

$$\text{LineAsterisk}(p/q)$$

where p/q is a rational length on the line.

Definition 2. *The line and radius generated by the solid dots on the circle is given by*

$$\text{CircleSolidDot}(p/q)$$

Definition 3. *The function $Slope$ and $ArcLength$ return the value of slopes of a line (henceforth radius) and the arc lengths associated with a radius.*

Note that $\tan \pi/4 = 1$ points to $\tan(ArcLength) = Slope$ and that $\arctan(1) = \pi/4$ points to $\arctan(slope) = ArcLength$. Our functions are of the form $Slope(Radius)$ and $ArcLength(Radius)$. The latter function is defined on all radii, but the former is undefined for the radius at $\pi/2$. Thus if $Slope$ is defined for all rational arc lengths, then $\pi/2$ must be irrational.

Note that radii can form a line with a undefined slope but a defined arc length. That is the case of the radius going through $\pi/2$. We'll make some observations and put them in form of lemmas.

Lemma 1. *The arc length of $\text{CircleSolidDot}(p/q)$ is not p/q .*

Proof. Obvious by geometry. □

Lemma 2. *For every positive rational number p/q there exists a radius on the unit circle with a slope of p/q and an associated arc length θ .*

Proof. Create a unit circle and a tangent to it at $(1, 0)$. Mark the length p/q on this tangent and make a line using the origin and this p/q . The resulting line will intersect the circle and form a radius with slope p/q and an associated arc length (angle measurement in radians). \square

Lemma 3. *Given a unit circle has a circumference or total arc length of 2π , a vertical radius occurs at arc length $\pi/2$.*

Proof. When an x and y axis intersect at the origin at right angles, the line connecting $(0, 0)$ and $(0, \pi/2)$ is a vertical radius. \square

Theorem 1. *π is irrational.*

Proof. If π is rational, then $\pi/2$ is also rational, say p/q . Looking at Figure 1, we see that

$$\text{Slope}(\text{LineRedBox}(p/q)) = \text{Slope}(\text{CircleBlueBox}(p/q)). \quad (1)$$

That is the same radius is generated by the redbox and bluebox. If the arc length CircleBlueBox(p/q) is irrational, then it is not equal to $\pi/2$ and is the upper limit for all rational arc lengths in the first quadrant. Thus it is not equal to any rational arc lengths. If the arc length CircleBlueBox(p/q) is rational, then it has a defined slope – can't be radius at $\pi/2$.

The equation (1) implies that all arc lengths that generate defined slopes are exhausted before the arc length $\pi/2$ is reached. Therefore $\pi/2$ can't be rational. There's a flavor of a cardinality proof here: see [2], page 82. \square

Conclusion

The pigeon hole principle might make this proof into a *proof without words*. I'll use some words to explain this. The pigeon hole principle [5] I guess from the idea that if five pigeons exist and desire to go into four holes, two pigeons are going to have share a hole. A version of this might be that if there are five pigeons and four of them are red and one is blue and if all four holes are occupied by a red pigeon, the fifth pigeon is not occupying a hole is not red. Thus in Figure 1, if one could read that all red dots are residing in blue arc length bins (we'll call them) and there is a number not so residing in one of these bin arc length locales, it isn't a red dot – rational.

References

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