A Gravitating Dark-Energy Hypothesis to replace MOND

Remi Cornwall
University of London Alumni Society, Malet Street, London WC1E 7HU
https://cornwall.infinityfreeapp.com/ http://ulondon.academia.edu/RemiCornwall

Abstract
This paper completes a series of earlier papers on the Cosmological Constant as compressible fluid-like zeropoint energy, which acts as a 2nd order perturbation in the stress-energy tensor; this preceded an earlier sketch for a means for Dark-Energy to gravitate. We suggest a replacement to ad-hoc MOND type theories with the theory developed herein, in the light of recent DESI and JWIST discoveries, with a fully covariant version of our earlier gravitating dark-energy model: we show that dark-energy can gravitate if it is considered to be in compression by tidal effects; also it sets a natural cusp-like size limit to galaxies and clusters. A relation is found between the slope of the galactic rotation curve and the slope of the dark-energy/matter zone is found too. An earlier proof by the author that MOND type theories are not Heisenberg Uncertainty Principle compatible is cited here. Finally we ask if an extra-repulsive form of dark-energy may prevent collapse to singularity in Black hole solutions. All-in-all, this paper is a semi-classical treatment of gravity with vacuum corrections that might manifest on the large-scale.

Keywords: Action, Cosmological Constant, Covariance, Dark-Matter, DESI, Gravitating Dark-Energy, Uncertainty Principle, JWIST, ΛCDM, MOND, Renormalisation Group, Rotation Curves, Singularities

1. Introduction

This paper is the result of an initially engineering-science endeavour looking at novel electromagnetic propulsion[1, 2] justification was sought for the claim that the mass-energy of ZPE could be pushed against[3] and then further calculation cleared-up the 120 orders of magnitude problem (really 129 orders with interaction between the modes to make up the factor, see Appendix 1 errata). It then occurred to the author that if a rational means could be found, then Dark-Energy (with modulation by gravitational gradient) might offer an alternative to the Galactic Rotation Curve problem[4].

The consensus is the ΛCDM model, which explains the condition of the Universe in various eras: an Inflation then Radiation Pressure dominated era, Matter with Dark-Matter era giving rise to galactic and intergalactic structure and finally, a dark-energy dominated era, in which we will seem to be doomed to an "island universe" fate of galaxies interspersed with massive voids. This paper won't add anything to Inflation Theory, nor do a more full survey of dark-matter (done somewhat in [4]) with the full gamut of the various fudges expected to explain dark-matter: microscopic primeval black-holes, MACHOs, RAMBOs, WIMPs and MOND/TeVeS type theories, save to say that the latter violate the Uncertainty Principle[5].

Using an "Hypothesis non-fingo" or even an Occam's Razor credo, we shall concentrate on what has been detected, measured, quantified and possibly linked together (Hubble Expansion, Vacuum Energy by our earlier paper [3]), as opposed to what is fanciful (like the arbitrary fix-up à la MOND) and constantly theorised but never measured to explain dark-matter: it shall be demonstrated that dark-energy under the right conditions gravitates and can account for dark-matter. The initial model we developed for Gravitating dark-energy[4] seems to be "giving the right numbers" and here we'll generalise it and compare with other measurements. The recent DESI and JWST results question the constancy of dark-energy and even the need for dark-matter and we shall be saying something about that too (see Appendix 2, which also contains a short sceptical note about its evolution).

2. The Ansatz

Any serious enquiry into the laws of physics must start from the Least Action Principle as it unifies all of fundamental physics with the least amount of "baggage" of concepts[6-10]. The action for General Relativity is written,

$$S = \int \left[\frac{c^3}{16\pi G}(R - 2\Lambda) + L_M \right] \sqrt{-g}\, d^4x \quad \text{eqn. } 1^\dagger$$

The variation with the metric yields the field equations. As usually written, they emphasize the geometric aspects on the LHS and the Physics on the RHS,

$$\mathbf{G}_{\mu\nu} + \Lambda \mathbf{g}_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2} R \mathbf{g}_{\mu\nu} + \Lambda \mathbf{g}_{\mu\nu} = \frac{8\pi G}{c^4} \mathbf{T}_{\mu\nu} \quad \text{eqn. } 2$$

It is an easy matter to jiggle around terms to meet the objective in the earlier papers[3, 4] to the field equations, where it was decided to equate the

† Where c^3 is used in the numerator if the measure $d^4x = d(ct)\,d^3x$ is used, otherwise c^4 for geometrized units.

Cosmological Constant to vacuum energy of Quantum Field Theories to the measured value, by realising that it was a 2nd order fluctuation in $\mathbf{T}_{\mu\nu}$ (but 3rd order in κ) from the electromagnetic field (for U^{-3} term see [3]),

$$\begin{aligned}\mathbf{G}_{\mu\nu} &= \kappa\mathbf{T}_{\mu\nu} - \kappa^3 U^{-3}\mathbf{g}_{\mu\nu}u_{QFT} \\ or \quad &= \kappa\mathbf{T}_{\mu\nu} - \mathbf{g}_{\mu\nu}u_{\Lambda-measured}\end{aligned} \qquad \text{eqn. 3}$$

With, $$u_\Lambda = \frac{\Lambda c^4}{8\pi G} \qquad \text{eqn. 4}$$

The usual constant has been replaced by κ and the RHS once again brings the physics to the fore, imbuing the Cosmological Constant/Dark Energy with a material like properties, i.e. – *Physics*, rather than just Mathematics.

One should be wary of adding terms to the field equations, even if they are tensors and obey the Bianchi identities and so forth, for the trusted method of starting from the Action for the following reasons:-

- Locality
- Invariance/Covariance
- Noether's Theorem (hence conservation laws).
- Well-posedness of the Cauchy problem (solutions exist, are well-behaved and unique, depend on initial conditions, perturbations don't blow-up).
- It will have the correct degrees of freedom and not make-up spurious particles or extra modes (such as longitudinal propagation).
- One can see from the Action what terms matter at low energy, what derivatives are in one's theory (hence new modes of propagation) and so forth…
- Which all leads to a sane theory with which to attempt quantisation, should one so wish. Quantisation proceeds from the Action.

So the following purely algebraic action is proposed,

$$S = \int\left[\frac{c^3}{16\pi G(1-\varepsilon\Lambda)}(R-2\Lambda) + L_M\right]\sqrt{-g}\,d^4x \quad \text{eqn. 5}$$

Which leads to the field equation,

$$\mathbf{G}_{\mu\nu} + \Lambda\left(\mathbf{g}_{\mu\nu} - \varepsilon\,\mathbf{G}_{\mu\nu}\right) = \kappa\mathbf{T}_{\mu\nu} \qquad \text{eqn. 6}$$

Where a constant ε, units m^2, has been introduced. This doesn't appear to do much but a similar route is being taken to the author's "intuitive guess paper"[4], by making the hypothesis that somehow dark-energy is compressed and gravitates; we realise that the Ricci Tensor represents volume compression (Appendix 3). This form is covariant from the get-go and has all the other desirable properties mentioned when starting from the Action.

It must be mentioned that any new hypothesis must not break old theory and results – especially solar system scale results. The scale of our departure is of the order $\varepsilon\Lambda$ and the precision of experimental results in gravity research are of the order of 1 part in 10^{-15} for speed of gravity experiments, such as the GW170817 gravity wave detection in 2017[11], to effects 1 part in 10^{-5} for the Shapiro Delay[12]. So there is considerable leeway in $\varepsilon\Lambda$ with $\Lambda \sim 10^{-52}\,m^{-2}$.

3. Reduction of the Gravitating Dark-Energy Hypothesis to Newtonian Regime

The Einstein Tensor is the trace-reverse of the Ricci Tensor and that this reduces down to Newtonian Gravitation (Appendices 3, 4) in the weak field limit (applicable to solar systems and more so to galaxies and clusters of galaxies): eqn. 6 can be whittled down to a modified Newton's law including the effect of Dark Energy *and* Gravitating Dark Energy[4], thus:

$$\nabla^2\phi - \varepsilon\Lambda\nabla^2\phi = 4\pi G\left(\rho_m - \frac{u_\Lambda}{c^2}\right) \qquad \text{eqn. 7}$$

And the dark energy has been entered in the stress-energy tensor by writing,

$$\nabla^2\phi = \frac{4\pi G}{1-\varepsilon\Lambda}\left(\rho_m - \frac{u_\Lambda}{c^2}\right) \qquad \text{eqn. 8}$$

This thus increases the effect of both gravitating mass-energy and dark-energy but this is not the "tool" to study gravitational collapse, in high field conditions, to a singularity; we will look at this later to see if a compression effect might increase dark-energy and if it could push back. For now, the topic is *gravitating* dark-energy.

And so, illustrating the QFT origin of dark energy[3] too (via eqn. 3 and eqn. 4)[‡],

$$\begin{aligned}&\nabla^2\phi = 4\pi G\left(\rho_m - \frac{u_\Lambda}{c^2}\left(1-\frac{\varepsilon}{c^2}\nabla^2\phi\right)\right) \\ &\Rightarrow \nabla^2\phi = 4\pi G\left(\rho_m - \frac{\kappa^3 U^{-3}}{c^2}u_{QFT}\left(1-\frac{\varepsilon}{c^2}\nabla^2\phi\right)\right)\end{aligned} \qquad \text{eqn. 9}$$

‡ See appendix 4 for the 1/2 factor, which arises when we go from $\mathbf{G}_{\mu\nu} \to \nabla^2$ in the Newtonian limit.

$[m]^{-2}\,\nabla^2\,[m]^2[s]^{-2}\,\phi = [s]^{-2}\,G\rho_{mass}\,[m]^2/([m]^2\,[s]^{-2})\,\frac{\varepsilon}{c^2}\,[s]^{-2}\,\nabla^2\phi$

Overall both these dark energy effects are "materialised" on the RHS as *Physics* (LHS is pure geometry).

Let us apply Gauss's Law with spherically symmetric symmetry, which leads to these starting solutions:-

$$\iint_A \nabla\cdot\nabla\phi\, dA = G\iiint_V \rho\, dV$$
$$\Rightarrow -4\pi r^2 g = G\iiint_{\phi=0,2\pi\ \theta=0,\pi\ r=0,R} \rho(\phi,\theta,r)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$g_{\text{point}} = -G\frac{M}{r^2} \text{ where } M = \frac{4}{3}\pi r^3\rho_{const}$$
$$g_{\text{internal}} = -\frac{4}{3}G\pi r\rho_{const} \qquad \text{eqns. 10}$$

(dummy variable called r, but limit, R, replaced with r)

Then with spherical symmetry and no angular variation, with no confusion between the total field potential ϕ_{Total} and the azimuthal angle ϕ, the gravitating dark-energy term is (see eqn. 9 for the "lost" 4π),

$$g_{GDE} = -\frac{1}{r^2}G\frac{u_\Lambda}{c^4}\iiint_{\phi=0,2\pi\ \theta=0,\pi\ r=0,R} \varepsilon\nabla^2\phi_{(Total)}r^2\sin\theta\, dr\, d\theta\, d\phi \qquad \text{eqn. 11}$$

Then with no angular variation, the integral reduces to and writing $\varepsilon_{Newton} = \frac{\varepsilon}{c^2}$ (so units m^2/s^2),

$$g_{GDE} = -\frac{4\pi G}{r^2}\frac{u_\Lambda}{c^4}\varepsilon_{Newton}\int_0^R \nabla^2\phi\, r^2 dr \qquad \text{eqn. 12}$$

Which can be written (reverting to just ε),

$$g_{GDE} = -\frac{4\pi G\varepsilon}{r^2}\frac{u_\Lambda}{c^4}\int_0^R r^2\left(\frac{2}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dr^2}\right)dr \qquad \text{eqn. 13}$$

Performing the first integral $\int\left(2r\frac{d\phi}{dr}\right)dr$ by parts

we obtain: $2\left(r\phi - \int\phi\, dr\right)\Big|_0^R$ then multiply by $\frac{1}{r^2}$

$$\boxed{g_{GDE} \propto -2\left(\phi - \frac{1}{r}\int\phi\, dr\right) + \text{(2nd integral)}}$$

Then the second integral, $\int r^2\frac{d^2\phi}{dr^2}dr$ by parts:-

$$\Rightarrow \int\left(r^2\frac{d\phi}{dr} - \int r\frac{d\phi}{dr}dr\right)dr$$
$$\Rightarrow \int\left(r^2\frac{d\phi}{dr} - \left(r\phi - \int\phi\, dr\right)\right)\text{dr}$$ with inclusion of the previous result

And so,

$$\int\left(r^2\frac{d\phi}{dr}\right)dr - \left(\int(r\phi)\,dr - \iint\phi\, dr\, dr'\right)$$
$$\Rightarrow \int\left(r^2\frac{d\phi}{dr}\right)dr - \left(\left(r\int\phi\, dr - \iint\phi\, dr\, dr'\right) - \iint\phi\, dr\, dr'\right)$$
$$\Rightarrow \int\left(r^2\frac{d\phi}{dr}\right)dr - r\int\phi\, dr + 2\iint\phi\, dr\, dr'$$
$$\Rightarrow \left(r^2\phi - \int r\phi\, dr\right) - r\int\phi\, dr + 2\iint\phi\, dr\, dr'$$
$$\Rightarrow -\left(r\int\phi\, dr - \iint\phi\, dr\, dr'\right) + r^2\phi - r\int\phi\, dr + 2\iint\phi\, dr\, dr'$$
$$\Rightarrow r^2\phi - 2r\int\phi\, dr + 3\iint\phi\, dr\, dr'$$

Upon multiplying by $\frac{1}{r^2}$ the second integral is obtained:

$$\boxed{g_{GDE} \propto -\left(\phi - \frac{2}{r}\int\phi\, dr + \frac{3}{r^2}\iint\phi\, dr\, dr'\right) + \text{(1st integral)}}$$

So overall:

$$g_{GDE} \propto -2\left(\phi - \frac{1}{r}\int\phi\, dr\right) - \left(\phi - \frac{2}{r}\int\phi\, dr + \frac{3}{r^2}\iint\phi\, dr\, dr'\right)$$
$$\Rightarrow g_{GDE} \propto -3\phi + \frac{4}{r}\int\phi\, dr - \frac{3}{r^2}\iint\phi\, dr\, dr'$$

Now, $\nabla\phi = -\mathbf{g} \Rightarrow \phi = -\int g\cdot dr$ thus[§],

$$g_{GDE} = 12\pi G\varepsilon\frac{u_\Lambda}{c^4}\left(\int g(r)\,dr - \frac{4}{3r}\iint g(r)\,dr\,dr' + \frac{1}{r^2}\iiint g(r)\,dr\,dr'\,dr''\right) \qquad \text{eqn. 14}$$

This is interesting because it shows through the moments, by the three integrals, that gravitating dark-energy could be a cumulative effect with distance and as such, an unexpected gravitating term that might explain galactic (and extra-galactic) scale dynamics and structure from an effect wholly attributed to both General Relativity and QFT and absent from Newtonian gravitation.

The double and triple integrals can by eliminated by the Cauchy repeated integral theorem:-

$$f^{(-n)}(x) = \int_a^x\int_a^{\sigma_1}\cdots\int_a^{\sigma_{n-1}} f(\sigma_n)\,d\sigma_n\cdots d\sigma_2\, d\sigma_1 \qquad \text{eqn. 15}$$
$$\Rightarrow f^{(-n)}(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt$$

So we can write,

$$g_{GDE}(R) = 12\pi G\varepsilon\frac{u_\Lambda}{c^4}\left(\int_a^R g(r)\left[1 - \frac{4}{3}\left(\frac{R-r}{R}\right) + \frac{1}{2}\left(\frac{R-r}{R}\right)^2\right]dr\right) \qquad \text{eqn. 16}$$

[§] Potential energy is defined as becoming more negative in a gravity well, compared to infinity, for an attractive potential.

The sign of $g_{GDE}(R)$ must be negative if it gravitates: in general, g(r) is negative for all positive *r* coordinates and the kernel multiplying it is quadratic in $\left(\frac{R-r}{R}\right)$ with discriminant $\Delta=\left(-\frac{4}{3}\right)^2-4\cdot 1\cdot\left(\frac{1}{2}\right)=-\frac{2}{9}$, so the kernel of the integral has no real roots and so the quadratic is positive for all $r \le R$, thus $g_{GDE}(R)$ is indeed negative. The gravitating dark energy hypothesis predicts an increase of the effect at radius, which will be balanced against the repulsive dark energy term

The gravitating dark energy hypothesis predicts an increase of the effect at radius, which will be balanced against the repulsive dark energy term.

4. The Computational Toy Model

A toy spherically symmetric exponentially decaying distribution shall model the density of matter thus,

$$\rho(r)=\rho_0 e^{-r/a} \qquad \text{eqn. 17}$$

Where *a* is a characteristic length scale. The total mass enclosed in this spherically symmetric model is,

$$M = 4\pi\rho_0\int_0^{\infty} r^2 e^{-r/a}dr \qquad \text{eqn. 18}$$

And by change of variables: $x = r/a$, the gamma function $\Gamma(3)=2!$ is recognised[13],

$$M = 4\pi\rho_0 a^3\int_0^{\infty} x^2 e^{-x}dx$$
$$\Rightarrow M = 8\pi\rho_0 a^3 \qquad \text{eqn. 19}$$

Now a toy spherically symmetric galaxy can be set up**,

$$g(r)=\frac{4\pi G\rho_0 a^3}{r^2}\left[2-e^{-r/a}\left[\left(\frac{r}{a}\right)^2+2\left(\frac{r}{a}\right)+2\right]\right]$$

$$g_\Lambda = +\frac{4}{3}G\pi\rho_{\Lambda/c^2}r \qquad \text{Repulsive constant dark energy}$$

eqns. 20

Gravitating dark-energy is a function of sum of eqns. 20,

$$g_{GDE}(R)=12\pi G\varepsilon\frac{u_\Lambda}{c^4}\left(\int_a^R g_{Total}(r)\left[1-\frac{4}{3}\left(\frac{R-r}{R}\right)+\frac{1}{2}\left(\frac{R-r}{R}\right)^2\right]dr\right)$$

(eqn. 16)

** The gamma integral is a standard result and yields the same mass as eqn. 19, when taken to infinity, whereupon we recognise a point source $g(r)=4\pi GM/r^2$.

The shell theorem (effectively eqns. 10) models repulsive dark energy as a constant energy density and we are always inside it. Superposition allows us to sum all the contributions.

The mass equivalence of the gravitating dark-energy is $-g_{GDE}(R)/4\pi G$,

$$\rho_{GDE}(R)=-3\varepsilon\frac{u_\Lambda}{c^4}\left(\int_a^R g_{Total}(r)\left[1-\frac{4}{3}\left(\frac{R-r}{R}\right)+\frac{1}{2}\left(\frac{R-r}{R}\right)^2\right]dr\right)$$

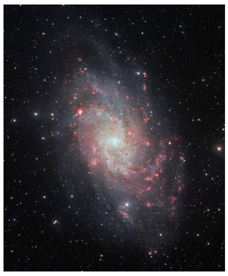
eqn. 21

Once the gravitational field is calculated, it is a simple matter to relate it to the orbital velocity around the galaxy,

$$-\frac{v^2}{r}=g(r)\Rightarrow v=\sqrt{-r\cdot g(r)} \qquad \text{eqn. 22}$$

5. The Toy Model Results

We shall compare the model to actual data of galaxy M33 and then other galaxies. Appendix 5 contains Matlab code, which is very similar to Numpy/Python code (apart from the libraries).



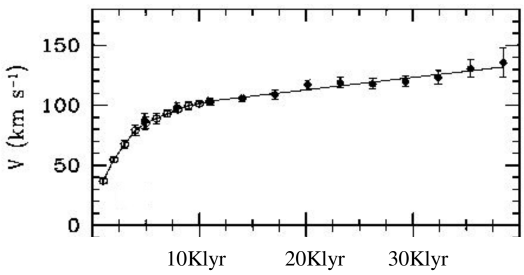


Figure 1 – Image and velocity profile of M33

```
% --- Exponential spherical mass model:
% rho(r) = rho0 * exp(-r/a) ---

% Choose scale length a and total mass
a_ly   = 5e3; % scale length in light-years
a      = a_ly * Lyr;    % meters

M_gal  = 5e10 * Mass_Sun;  % baryonic mass

% 6a of r^2.e^-r/a gets most of mass
Size_gal = 6 * a;

% kg/m^3 central density
rho0   = M_gal / (8*pi*a^3);
```

Figure 2 – Toy model parameters M33

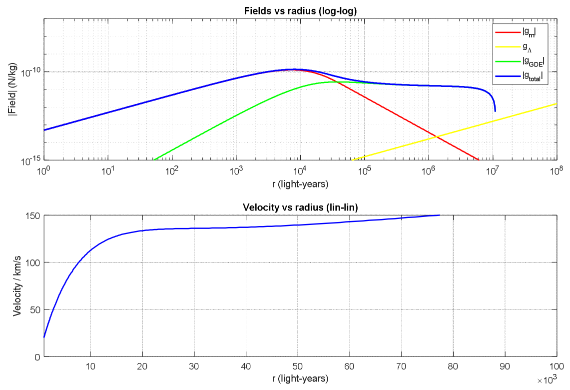


Figure 3 – Toy model results M33

```
>> GravitatingDarkEnergy

Scale length a (kly): = 5.00
Central density rho0 (kg/m^3): = 3.759e-20
Galactic size (kly): = 60.00
Total mass (Kg): = 1.00e+41
Total mass (Msun units): = 5.00e+10
Total mass (MW units): = 0.25
Epsilon:  = 1.5000e+31
Effective GDE mass (Kg) = 1.252e+41
Fraction of baryonic mass = 1.2521 (125.21%)
```

Figure 4 – Numerical output from the code M33

Discussion on first results for M33

Figure 2 shows the parameters entered into the model and figure 3, the graphical output from the code. The first subplot shows the contributions to the gravitational field: the red curve is for our toy baryonic model; yellow is the repulsive dark energy contribution; green is the field from our Gravitating Dark Energy Hypothesis (controlled by the epsilon constant, eqn. 5) and finally, the blue curve is the resultant of the fields.

It is interesting that there is a cusp-like fall in the attraction curve (blue) when the Dark Energy/Hubble Flow takes over. This doesn't necessarily mean that a galaxy could grow to this size (size maybe determined by collision kinetics; galactic density thins exponentially) but it does suggest a limit to local group/cluster sizes, as Table 1/Figure 12 below (sizes in Mlyr and Glyr).

Structure	Gravitationally Bound?	Typical Size
Galaxy	Yes	~0.05–0.2 Mly
Local Group	Yes (gravitationally bound)	~10 Mly
Cluster	Yes (deeply bound)	~10–20 Mly
Supercluster	No (flow bound)	~100–500 Mly
Cosmic Web Filament	No	~100–1000 Mly
Observable Universe	N/A	46.5 Gly radius

Table 1 – Scales of the Cosmos

The value of Epsilon (figure 4) was chosen heuristically (1.5e31 m^2) to get the peak of the GDEH curve (green) near to the peak of the gravitational field of the baryonic curve (red). For sure, the nature of the GDEH function (eqn. 16) integrates (with a kernel) the baryonic curve and should peak soon after, such that the velocity profile doesn't have a dip (there is a very slight dip in the velocity profile graph, figure 3).

Interestingly the effective mass of the gravitating dark energy, at about 125% *and* determined to the size of the galaxy, is right in the ball-park for the believed dark matter.

> We do not posit an additional, unobserved matter component with an arbitrarily assigned halo profile: No new particles, no bespoke density laws, and no coloured-in halos, out to whatever distance in N-body visualisations, are introduced.
>
> Instead, we take only two empirically grounded ingredients—baryonic matter and a uniform dark energy density—and ask how curvature, via the Newtonian limit of GR and a function integrating the field, modifies the proper volume and hence the effective gravitating content.
>
> The additional "dark" mass inferred in this framework is not an invented substance but a re-interpretation of how dark energy contributes to gravity in the presence of baryons.

Further work is required to link eqn. 16 to the peak gravitational field. It is made clear in eqn. 5 that Epsilon should be a universal constant, for the covariant nature of the equation.

We shall now look to see if a similar result is obtained by looking at similar galaxies, near and far (high z/redshift). However for our spherically symmetric model, highly elliptical (more a velocity dispersion relation) and irregular galaxies are out of the picture. Further work needs to be done on a full gravitational model for these cases.

Parameters and results for The Milky Way

```
% --- Exponential spherical mass model:
% rho(r) = rho0 * exp(-r/a) ---

% Choose scale length a and total mass
a_ly = 8.5e3; % scale length in light-years
a    = a_ly * Lyr;   % meters
M_gal  = 2e11 * Mass_Sun;  % baryonic mass

% 6a of r^2.e^-r/a gets most of mass
Size_gal = 6 * a;

% kg/m^3 central density
rho0   = M_gal / (8*pi*a^3);
```

Figure 5 – Toy model parameters MW

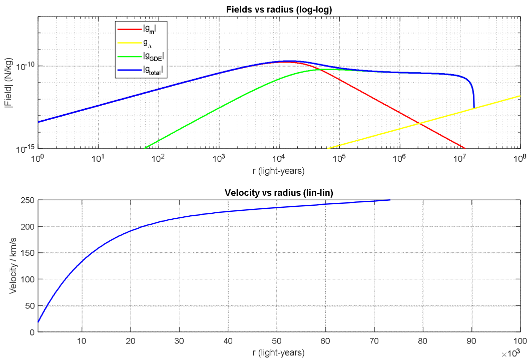


Figure 6 – Toy model results MW

```
>> GravitatingDarkEnergy

Scale length a (kly): = 8.50
Central density rho0 (kg/m^3): = 3.060e-20
Galactic size (kly): = 51.00
Total mass (Kg): = 4.00e+41
Total mass (Msun units): = 2.00e+11
Total mass (MW units): = 1.00
Epsilon:  = 1.5000e+31
Effective GDE mass (Kg) = 2.168e+41
Fraction of baryonic mass = 0.5419 (54.19%)
```

Figure 7 – Numerical output from the code MW

The model for the Milky Way has a slightly longer scale length than M33 and a velocity profile that is a fair version of the real thing.

Parameters and results for Q2343-BX442 (z ≈2.18)

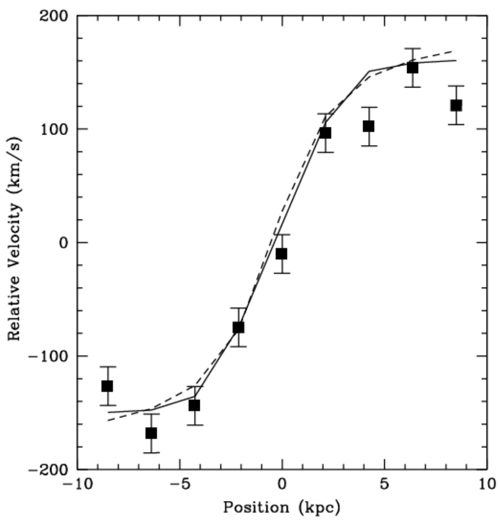


Figure 8 – Rotation curve for Q2343-BX442 (1kpc ≈ 3.26Klyr)

This galaxy[14] is about 10-11 billion light years away and a co-moving distance of 17 billion light years.

```
% --- Exponential spherical mass model:
% rho(r) = rho0 * exp(-r/a) ---

% Choose scale length a and total mass
a_ly = 7.5e3; % scale length in light-years
a = a_ly * Lyr;  % meters
% baryonic mass
M_gal  = 0.8 * Mass_MW*Mass_Sun;

% 6a of r^2.e^-r/a gets most of mass
Size_gal = 6 * a;

% kg/m^3 central density
rho0   = M_gal / (8*pi*a^3);
```

Figure 9 - Toy model parameters Q2343-BX442

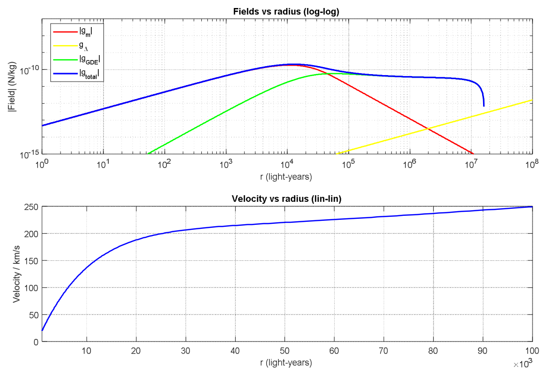


Figure 10 - Toy model results Q2343-BX442

```
>> GravitatingDarkEnergy

Scale length a (kly): = 7.50
Central density rho0 (kg/m^3): = 3.564e-20
Galactic size (kly): = 45.00
Total mass (Kg): = 3.20e+41
Total mass (Msun units): = 1.60e+11
Total mass (MW units): = 0.80
Epsilon:  = 1.5000e+31
Effective GDE mass (Kg) = 1.517e+41
Fraction of baryonic mass = 0.4741 (47.41%)
```

Figure 11 - Numerical output from the code Q2343-BX442

The results for this galaxy's rotation curve are somewhat off from the real data (figure 8) and it is more likely that this young, thick, turbulent and clumpy galaxy hasn't reached mature thin disk equilibrium, rather than Lambda being different in the past! The model is also spherically symmetric.

Further work needs to see if the galactic rotation curves have a characteristic slope in the dark-matter zone that is constant or some function of the baryon profile. It is a fair bet that it is, as it is the integral of the baryon gravitational field and this is directly used to calculate the velocity profile.

Epsilon has some physical basis to be a scaling of Lambda (eqn. 6 and $\rho_{\Lambda-measured}$ or $\kappa^3\rho_{QFT}$ if [3] is correct) – and we ask what it might mean? Section 6 will begin to look at this as a "renormalizing group" linked to curvature and vacuum energy.

6. Further work: Extra Repulsive Dark Energy Hypothesis (ERDEH), a possible means to prevent singularities?

Throughout this paper and the proceeding one[4] by the author, puts forth the view of vacuum energy as a compressible fluid. So in this spirit, another algebraic modification is made to the GR-Action thus,

$$S=\int\left[\frac{c^3}{16\pi G\left(1-\varepsilon_+\Lambda+\sum_i {}_i\varepsilon_-\Lambda^i\right)}(R-2\Lambda)+L_M\right]\sqrt{-g}\,d^4x \quad \text{eqn. 23}$$

Where the previous (positive) gravitating dark energy constant is now called ε_+ and a new extra repulsive dark energy terms $+{}_1\varepsilon_-\Lambda+{}_2\varepsilon_-\Lambda^2...$ modify G (with the constants ensuring dimensional and tensorial consistency). Now the field equation is (with dark-energy terms entered onto the RHS eqn. 9),

$$\nabla^2\phi=4\pi G\left(\begin{array}{c}\rho_m-\frac{\kappa^3U^{-3}}{c^2}u_{QFT}\,.\\ \left(1-\varepsilon_+\nabla^2\phi-{}_1\varepsilon_+\nabla^2\phi-{}_2\varepsilon_+\left(\nabla^2\phi\right)^2+...\right)\end{array}\right) \quad \text{eqn. 24}$$

The "Extra Repulsive Dark Energy Hypothesis" (ERDEH) constant terms would be chosen to be negligible at solar, galactic and cosmological levels $(\varepsilon_+\Lambda\ll 1)$ and start to manifest in regions where curvature $\nabla^2\phi$ becomes high $(\varepsilon_-\Lambda\ll 1)$. Thus in a similar manner to eqn. 14 and eqn. 16 we'd have (with the minus sign to make the repulsion explicit, though that would come from $f(r,R)$, the kernel function as well),

$$g_{ERDE}(R)\propto-\left(\int_a^R g(r).f(r,R)\,dr\right) \quad \text{eqn. 25}$$

Then various orders/powers of g_{ERDE} and hence of the field, g, could be introduced. We can't quite proceed along the same tack though as in the high-field regions superposition won't apply and we'd need to use GR from the get-go, even if it were to be approximated or linearised – the very region it is mean to "save" GR occurs at incredibly high curvature near black-holes; at least a flavour of the an "UV limit" (ERDEH) is indicated from the "IR limit" of the Gravitating Dark Energy Hypothesis (GDEH). Any Newtonian-type considerations would begin to fail with this approximation[10, 15],

$$\left|\mathbf{R}_{\mu\nu\rho\sigma}\right|\sim\frac{GM}{c^2r^3} \quad \text{eqn. 26}$$

ERDEH could be introduced in two ways:-

$$S=\int\left[\frac{c^3}{16\pi G(1-\varepsilon_+\Lambda+\varepsilon_-\Lambda K)}(R-2\Lambda)+L_M\right]\sqrt{-g}\,d^4x \quad \text{eqn. 27}$$

With the use of the Kretschmann Invariant (*K*) Scalar[10, 15] in the denominator,

$$K=\mathbf{R}_{\mu\nu\rho\sigma}\mathbf{R}^{\mu\nu\rho\sigma}=\frac{48G^2M^2}{c^4r^6} \quad \text{eqn. 28}$$

And a messy $\mathbf{G}_{\mu\nu}(K)$ in the field equations would result,

$$\mathbf{G}_{\mu\nu}{}^{\alpha\beta}(K)\mathbf{G}_{\alpha\beta}=k\mathbf{T}_{\mu\nu} \quad \text{eqn. 29}$$

Then the other way would be in the numerator,

$$S=\int\left[\frac{c^3}{16\pi G(1-\varepsilon_+\Lambda)}(R-2\Lambda-2\varepsilon_-\Lambda K)+L_M\right]\sqrt{-g}\,d^4x \quad \text{eqn. 30}$$

With the resultant field equations,

$$\mathbf{G}_{\mu\nu}+\Lambda\left(\mathbf{g}_{\mu\nu}-\varepsilon_+\mathbf{G}_{\mu\nu}\right)+\Lambda\varepsilon_-\mathbf{H}_{\mu\nu}=\kappa\mathbf{T}_{\mu\nu} \quad \text{eqn. 31}$$

Where $\mathbf{H}_{\mu\nu}$ would contain derivatives 2nd and higher with respect to the metric of the scalar curvature, *R*; in fact both approaches would, as it is generally acknowledged that a general relativity without singularities must have higher order derivatives[10], anyway.

The other method could be reduced, in field equation form, to eqn. 31 and the point is, a vacuum energy term could be introduced into the stress-energy tensor with this form,

$$\mathbf{T}_{\mu\nu}^{(ERDE)}=-\frac{\Lambda\varepsilon_-}{k}\mathbf{H}_{\mu\nu} \quad \text{eqn. 32}$$

A co-moving observer can observe volume contraction or expansion (hence convergence or divergence of geodesics) dependent on the mass-energy in a region. The Raychaudhuri-Landau equation[16, 17] below describes the rate of change of volume for a co-moving observer with 4-velocity u^μ, with no pressure support (i.e. matter has "failed"),

$$\frac{d\theta}{d\tau}=-\frac{1}{3}\theta^2-\boldsymbol{\sigma}_{\mu\nu}\boldsymbol{\sigma}^{\mu\nu}+\boldsymbol{\omega}_{\mu\nu}\boldsymbol{\omega}^{\mu\nu}-\mathbf{R}_{\mu\nu}u^\mu u^\nu \quad \text{eqn. 33}$$

wrt. affine parameter like wavenumber

$$\frac{d\theta}{dk}=-\frac{1}{2}\hat{\theta}^2-2\hat{\sigma}+2\hat{\omega}-\mathbf{T}_{\mu\nu}u^\mu u^\nu \quad \text{eqn. 34}$$

Null geodesic form with "hats" indicating transverse directions

Where $\sigma_{\mu\nu}$ is the shear tensor and $\omega_{\mu\nu}$ is the vorticity – to the latter one can understand this (and the sign) as the "centrifugal force" of the matter in the potential well. Noting that[††] $u^\mu u^\nu = -1$ it is enough to cast the EFE in this form[15],

$$\mathbf{R}_{\mu\nu} = \kappa\left(\mathbf{T}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}T\right) \qquad \text{eqn. 35}$$

To see that the ERDE term (eqn. 32) could potentially defocus and stop descent to a singularity[‡‡] and more so with "optical"/null-geodesic Raychaudhuri equation (eqn. 34). More work is needed to turn this from a "how long is a piece of string" argument, to something that could, perhaps, fix the scale $\Lambda\varepsilon\_\mathbf{H}_{\mu\nu}$ from work with the Tolman-Oppenheimer-Volkoff[10, 15] equation and matter at exceedingly high density from observational data (beyond neutron stars… strange matter, quark stars before collapse to black-holes?) It is safe to say that a theory with singularities is incomplete. A semi-classical treatment, such as this, might suggest new physics.

It is possible to combine both constants of GDEH and ERDEH,

$$S = \int\left[\frac{c^3}{16\pi G}\left(R - 2\Lambda\left[1 - \varepsilon\tanh\left(\log\frac{K}{K_S}\right)\right]\right) + L_M\right]\sqrt{-g}\,d^4x \qquad \text{eqn. 36}$$

Where over a vast range of curvature and distances, the galactic and cosmological scale would tend to $2\varepsilon\Lambda$ of the gravitating dark-energy hypothesis; then at the solar system scale (K_S) the constant becomes negligible. Systems of extremely large curvature would have a large negative, repulsive constant as a function of curvature preventing singularities. This is a "running constant" or "renormalizing group" linked to curvature and this has been the central tenant of this body of work[3, 4], that the purely geometric considerations of General Relativity can be separated from cosmological constants, which are entered onto the RHS as *physical* stress-energy components, that perhaps make more sense *there*, than as a "bolt-on" to the geometric theory. One sees a semi-classical effective field theory based on vacuum energy emerge by this means.

[††] Using $x^0 = -t$ and geometrized units (-c^2 otherwise) and the signature (-+++).

[‡‡]All vorticity tends to zero with collapse to a point, further simplifying the equations but shear can increase without limit. Our analysis would then start with the shear and vorticity scalars as zero (as well as the pressure term $\nabla_\mu a^\mu - a_\mu a^\mu$, which was left out), just before the inevitable collapse, to see what could arrest it.

7. Conclusion and further work

This paper sought to replace MOND, an arbitrary hypothesis which violates the Heisenberg Uncertainty Principle, with a fully covariant theory, worked out from the Action, which is always a good starting point. It follows on from earlier work where the Cosmological Constant was equated to Vacuum Energy of Quantum Field Theories and closed the 120 orders of magnitude problem; it separates the geometric aspect of General Relativity from the physical. What has transpired is a semi-classical theory of General Relativity where vacuum energy is scale dependent and linked to curvature (a renormalizing group theory/running constant theory). This approach also seems to forbid the singularities of bare General Relativity by an Extra Repulsive Dark Energy Hypothesis at Planck Scale.

At the other end of the size scale, with a Gravitating Dark Energy Hypothesis (GDEH), it seems to predict rotation curves of galaxies and give a natural size scale for gravitationally bound clusters against the Hubble Flow. This perhaps confirms JWIST's abeyance of the need for dark-matter. We also think the DESI claim of evolving dark-energy suspect (appendix 2).

Furthermore, hypothesis non-fingo, it doesn't invent an arbitrary fix that can't be measured (dark matter) and no-one has seen, apart from colourising computer simulations and saying "it must be there, around galaxies". The mass equivalent of the GDEH matches the baryonic radial constraints without an appeal to some fix saying, for instance, a dark matter halo must extend out some 10 galactic radii or so. The equivalent mass of the GDEH also comes in at around the mass of galaxies, for the gravitation function is an integral or averaging of the baryon gravitational field: this is a big clue in the author's opinion, as to the nature of this dark component.

We find that the constant of the theory at the galactic and intergalactic scale is about 1.5e31 m^2 and a universal form to the slope of rotation curves is hinted and is further work.

Downsides requiring extra work are: the Epsilon constant size is not fixed by mathematical means at present but heuristically set as a small range; the value of the constant and what it really means is not known (its dimensions of m2 give a clue of a relation to curvature); there is a very slight dip in the generated rotation curves; non-symmetric or even irregular gravitational fields must be accounted for, too. Also the wide discrepancy in real world data as regards to masses, size extents and what is included has begged the question about the data itself.

Re-emphasizing, the Epsilon constants of the theory also need to be either related to other physical constants (much as Lambda was to QFT vacuum energy in an earlier paper by the author[3]) and most of all, to find out what they mean physically at a microscopic level. It hints that General Relativity, even in this semi-classical treatment, is an effective field theory.

The author would also like to return to the ideas of their very early paper[18] where General Relativity is embedded in Euclidean 3+1 space and a tensor field provides the length contraction and time dilation effects of Relativity. This and an Action developed in this paper may provide an alternative means to quantisation.

Dedications

To my mentors, supporters, teachers and funders who gave me much encouragement over the years across all my research areas: Professors Peter Dobson, Daniel Sheehan, Kevin O'Grady, Drs Karla Galdamez, Simon Busbridge and my funder, Mr Richard Balding but especially my first supervisor,

Professor Tim Ellis (dec. 2025)
https://tim-ellis.muchloved.com/

and

Mum (dec. 2014) and Dad.

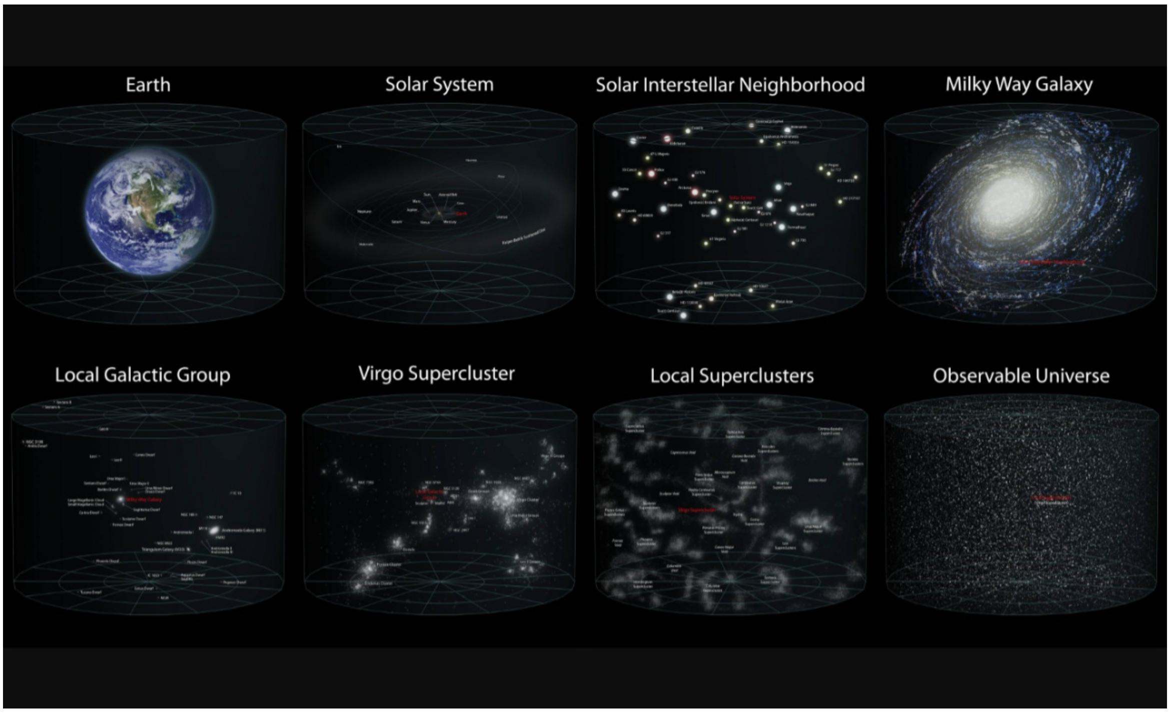


Figure 12 - https://cdn.pmylund.com/blog/content/scale_of_the_universe.jpg

Appendix 1 – Corrections to earlier papers

An earlier paper[3] used in this paper made a mistake on taking the trace of the cosmological constant considered as an entry in the stress energy tensor. The result should be $\Lambda g^{\mu\nu} g_{\mu\nu} = 4\rho_\Lambda$ and this helps the rest of the calculation by a factor of 4.

Next the paper[4] made the quite common mistake of conflating the fluctuations of zeropoint energy with real particle excitations and that it exerts Pauli pressure (electron degeneracy pressure, say). This cannot be: the zeropoint state is the state of zero particles. Only near a horizon acceleration/Rindler and/or black holes does a Bogoliubov transformation of the creation-annihilation operators lead to pair production. However, wrong ideas can lead onto more correct ideas.

The paper[4] is correct in explaining that dark-energy has negative pressure, as it is analogous to surface tension, in that space requires work to expand. Just as a force can be considered the change in energy per unit distance, pressure is the change in energy per unit volume.

Appendix 2 - Cosmology

This section tells how the Universe is really dominated by Dark-Energy – Inflation (some kind of dark-energy field prior to what we call dark-energy now), expansion from that inflation and then, apparently, ceaseless expansion.

What we call the Big Bang is not like a chemical or even nuclear explosion; these wouldn't expand space. Carrying on in the same vein as the previous appendix, the stress energy tensor of any such explosion is positive along the diagonal. The Ricci Scalar (contraction of the Einstein Tensor, eqn. 2) makes it clear that the scalar would be negative and hence space would contract, just as for any mass-energy. Thus the expansion era after the Big Bang is caused by the remnant of the initial "kick" of Inflation which countered the mass-energy of the universe trying to cause an early "Big Crunch"[10, 19].

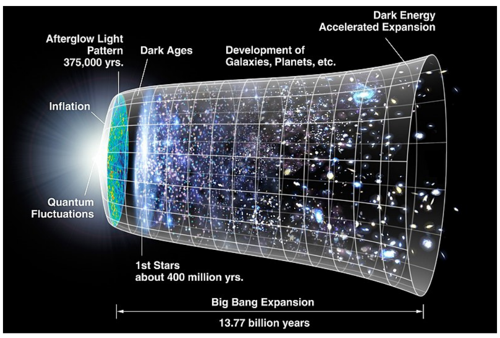


Figure 13 - Courtesy of NASA/WMAP Team

We now live in an era of accelerated dark-energy expansion governed by $\Lambda g_{\mu\nu}$ and $\rho = -p$ which has surmounted the (believed) short time of operation of Inflation and also surmounted the large scale gravitational attraction of mass-energy.

The starting point for understanding Cosmology is the Friedmann–Lemaître–Robertson–Walker (FLRW) metric[10, 19] which describes an homogeneous, isotropic expanding or contracting universe.

$$ds^2 = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) \quad \text{eqn. 37}$$

Where,

a scale factor, a, proportionately governs the separation between galaxy sized objects,

k describes a curvature constant which governs if the universe's overall geometry is: flat = 0 (Euclidean), closed = +1 (like a sphere or a "PacMan" game rolling around at the edges) or open = -1 (hyperbolic)

And $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$

There is no contradiction with the scale factor, a, describing expansion or contraction of space and k describing the geometry of the universe: a sphere is closed and curved, for instance and can expand or contract, as can Euclidean or Hyperbolic spaces.

The Hubble parameter H(t) is defined by the scale factor,

$$H(t) = \frac{\dot{a}(t)}{a(t)} \quad \text{eqn. 38}$$

And the Hubble constant H_0 is simply the value evaluated now. Assuming a flat universe and light travel along a null interval,

$$\begin{aligned} ds^2 &= -c^2 dt^2 + a(t)^2 dr^2 = 0 \\ \Rightarrow dr &= \frac{cdt}{a(t)} \end{aligned} \quad \text{eqn. 39}$$

Then the co-moving distance, χ, between objects from a time in the past to now, is given by,

$$\chi = \int_{t_{Emission}}^{t_{Now}} c \frac{dt'}{a(t')} \quad \text{eqn. 40}$$

As the universe expands, the wavelength, λ, of light traveling through space is physically stretched in proportion to the expansion of space. The scale factor describes the relative size of the universe at any given time thus,

$$\frac{\lambda_{observed}}{\lambda_{emitted}} = \frac{a(t_{observed})}{a(t_{emitted})} \qquad \text{eqn. 41}$$

The redshift is the fractional increase in wavelength,

$$z = \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}}$$
$$\Rightarrow z = \frac{a_{observed}}{a_{emitted}} - 1 \qquad \text{eqn. 42}$$

This then is normalised at our current time as $a(t_{observed}) = 1$ thus,

$$a(t) = \frac{1}{1+z} \text{ and } \dot{a}(t) = -\frac{1}{(1+z)^2}\frac{dz}{dt} \qquad \text{eqns. 43}$$

The Hubble parameter (eqn. 38) is then,

$$H(t) = -\frac{1}{1+z}\frac{dz}{dt} \qquad \text{eqn. 44}$$

Substituting into the co-moving distance (eqn. 40),

$$\chi = \int_0^z \frac{c}{(1+z)^{-1}} \frac{dz}{(1+z)H(t)}$$

Notice that the limits have been flipped and time replaced with z, with $t_{Now} \Rightarrow z = 0$ and $t_{Emission} \Rightarrow z$, then *H(t)* evolves with time and the co-moving distance is usually written,

$$\chi = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')} \qquad \text{eqn. 45}$$

Where H_0 is the current Hubble Constant[20][§§] and $E(z')$ is the dimensionless Hubble parameter encapsulating the Universe's expansion history.

The Friedmann equations are derived by applying Einstein's field equations to the FLRW metric, assuming the universe is filled with a perfect fluid (such as dark-energy).

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \qquad \text{eqn. 46}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \qquad \text{eqn. 47}$$

Where ρ is energy density, p is pressure and the other symbols and Λ is dark-energy. The other symbols have their usual meanings or were discussed previously.

The first Friedmann equation is derived from the 00 components (ie. Conservation of Energy) of the EFEs and describes the instantaneous rate ("velocity") of expansion or contraction (the Hubble parameter is squared) with energy density; it "inherits" the initial conditions for $\dot{a}(t)$ which comes from Inflation, which is initially very large. Notice it says that positive energy density or even the negative energy density of dark-energy will cause *change,* much as application of the vectoring of rocket thrust changes the velocity of a craft – the Universe is a dynamical object but rather than its direction, we are talking about its size.

The curvature constant causes a change in inverse proportion to the scale factor squared, if the universe is closing in on itself ($-\frac{kc^2}{a^2} < 1$) or opening up ($-\frac{kc^2}{a^2} > 1$); the closing in on itself is a finite process and the rate would ultimately slow to a stop, whereas becoming more open isn't bounded. The inverse proportion makes sense too, regardless of the sign of *a(t)*: if the universe is big, changes in its scale will be initially slow.

It is interesting to note that the scaling of this obviously is proportional to a^{-2}, whereas the densities of matter, radiation and dark-energy go as: a^{-3}, a^{-4} and constant respectively; radiation being as matter but with an extra inverse scale factor from the redshift, dark-energy is constant.

The first Friedmann equation is scaled and often written with density functions (in similarity to the 2nd term of eqn. 46) as,

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_k a^{-2} + \Omega_m a^{-3} + \Omega_r a^{-4} \qquad \text{eqn. 48}$$

And,

$$\rho_{critical}(t) = \frac{3H^2(t)}{8\pi G} \qquad \text{eqn. 49}$$

Omegas are then defined,

$$\Omega_i = \frac{\rho_i}{\rho_{critical,0}} \qquad \text{eqn. 50}$$

And,

$$\Omega_\Lambda + \Omega_k + \Omega_m + \Omega_r = 1 \qquad \text{eqn. 51}$$

The critical density today (*t = 0*) is related to the Hubble parameter today ($H_0 \approx 70 km/s/Mpc$) and a Critical Density of $\rho_{critical,0} \approx 9.2\times10^{-27} kg/m^3$ and

[§§] In dispute (The "Hubble Tension") at either 73 km/s/Mpc using standard candle Type 1a supernova or 67 km/s/Mpc from Cosmic Microwave Background data and the Planck satellite mission. One parsec is approximately 3.26 light-years.

this corresponds to a few hydrogen atoms per cubic meter. These are measured over a "reasonably small" volume of space-time such that dark-energy and curvature can be neglected. Looking at a wider scale, we can conclude from measurement of the Hubble parameter (from redshift) and density the following,

If $\rho > \rho_{\text{critical}}$ the universe is closed ($k = +1$)

If $\rho < \rho_{\text{critical}}$ the universe is open ($k = -1$)

If $\rho = \rho_{\text{critical}}$ the universe is flat ($k = 0$)

This gives the familiar view of the Universe's composition,

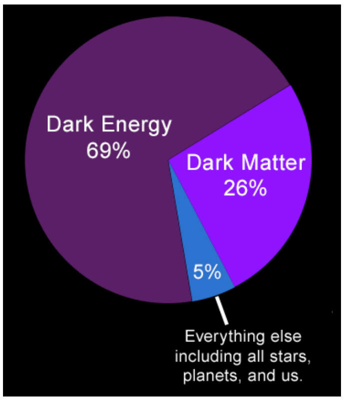


Figure 14 - According to Chandra***

Or,

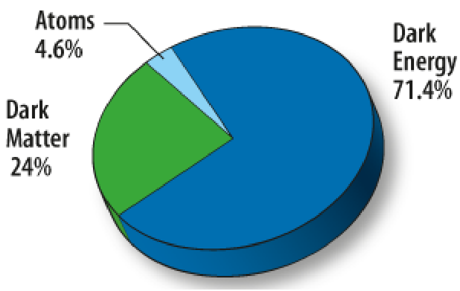


Figure 15 - According to WMAP†††

We are aiming to show that dark-matter isn't needed in this paper and can be explained as being due to Gravitating Dark-Energy.

The second Friedmann equation describes the acceleration of the scale factor. Often an equation of state is written,

$$p = w\rho c^2 \qquad \text{eqn. 52}$$

Such that matter is described as a pressure-less dust with $w = 0$ and radiation as an ultra-relativistic fluid with $w = -1$. Solutions to the Friedmann have the general solution in the spatially flat case are,

$$a(t) \propto t^{\frac{2}{3(w+1)}} \qquad \text{eqn. 53}$$

One can begin to see how the universe has various epochs (figure 13), radiation or matter dominated phases and currently, despite ideas of dark-energy changing through the epochs, such as "Quintessence", w is very close to -1. The equation of state of dark energy (this has been measured by DESI and other surveys) suggests that the Universe is doomed to expand forever and also the curvature constant, k, is exceedingly close to 0, as currently measured. Thus on logarithmic time, we and how we see the Universe and its structure of "fairly close" galaxies, galaxies, star clusters and even solar systems is just a small fraction in time of a rather desolate place, pending new observation and Physics.

Skepticism about the recent DESI claim

A recent press announcement[21] and a more nuanced paper[22] has expressed the view that the equation of state of dark-energy (eqn. 52) might have evolved over a 5 billion year time-scale: it suggests that w has a small correction at a redshift(z) of about 0.5: $w(z) \approx -1 + o_{DE}(z)$ .We find that unlikely and *is* due to seeking agreement between different datasets (BAO, CMB) and is more about politics and statistics than actual discovery, trying to put things diplomatically.

Dark-energy is (primarily[23]) the zero-point field of electromagnetism, whose force carrier has *zero* mass and long range and fixed vacuum energy density. If the photon had mass we'd see: dispersion of light from distant sources, a modified Coulomb Law, a longitudinal photon mode, a violation of gauge invariance and it would not be renormalisable.

Some forms of dark-energy do change from either resonance decay of their field or their vacuum energy level shifts and decays to other fields (The Inflation Field and "reheating"[19]).

*** https://chandra.harvard.edu/darkuniverse/

††† https://map.gsfc.nasa.gov/universe/uni_matter.html

Appendix 3 – Volume element and Ricci Tensor

The essence of dealing with non-Euclidean geometry is the Riemann Tensor[15], which describes the non-commutation of covariant[10, 15] derivatives around an infinitesimal loop (parallel transport of vectors): it relates the starting vector to the change in orientation of the end vector.

The central point of curved space is parallel transport. The figure 16 below shows the surface of a sphere and the movement of a tangent vector always pointing south around it. At the end of the transport, the initial and final vectors (v_i, v_f) do not align; they would on a flat surface.

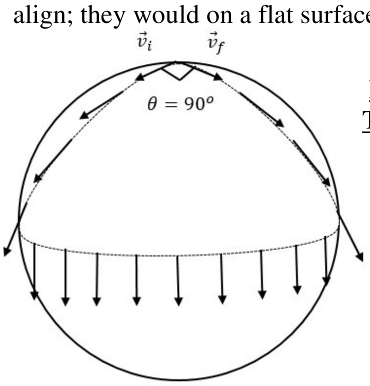


Figure 16 – Parallel Transport of a vector

We can quantify and generalise this notion of parallel transport with the following diagram,

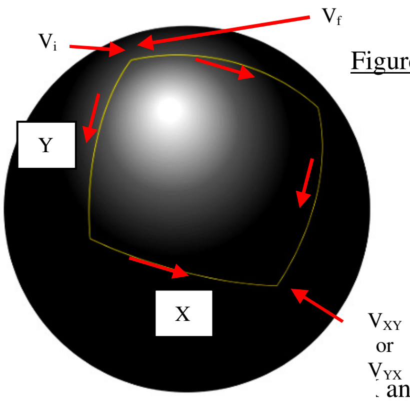


Figure 17 – Parallelogram on a sphere

. and Y which are geodesics (ie. their covariant derivative is zero). Starting from point V_i the far side can be reached by two paths, V_{XY} or V_{YX}.

Mathematically this can be expressed as a Taylor series as a small deviations from the vector fields,

$$
\begin{aligned}
V_{XY} &= V_i + \varepsilon \nabla_X V_i + \varepsilon \nabla_Y V_i \\
&\quad + \varepsilon^2 \nabla_X \nabla_Y V_i + \frac{1}{2}\varepsilon^2 \nabla_X \nabla_X V_i + \frac{1}{2}\varepsilon^2 \nabla_Y \nabla_Y V_i + O(\varepsilon^2) \\
&and \\
V_{YX} &= V_i + \varepsilon \nabla_Y V_i + \varepsilon \nabla_X V_i \\
&\quad + \varepsilon^2 \nabla_Y \nabla_X V_i + \frac{1}{2}\varepsilon^2 \nabla_X \nabla_X V_i + \frac{1}{2}\varepsilon^2 \nabla_Y \nabla_Y V_i + O(\varepsilon^2)
\end{aligned}
$$

eqns. 54

Now this is quite right but doesn't account for the vector fields being different at the far end (indeed, some "flow" in the vector fields) because the Taylor expansion is not at the same point. What we need to add is something called the "Lie Bracket", $\nabla_{[X,Y]}$ which expresses the lack of commutation in doing route XY or route YX first. Now,

$$
\begin{aligned}
&V_f = V_{XY} - V_{YX} \\
&so \\
&\nabla V = V_f - V_i \\
&\quad = \varepsilon^2 \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right) V_i + O(e^3)
\end{aligned}
$$

Dividing by the parallelogram area ε^2 and taking the limit as $\varepsilon \to 0$ the Riemann operator is obtained,

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{\Delta V}{\varepsilon^2} = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right) V \\
&and \qquad \text{eqn. 55} \\
&R(X,Y) = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right)
\end{aligned}
$$

The Riemann Curvature operator describes precisely and infinitesimally the net rotation of a vector when it goes around the loop created by X,Y. If the two vectors are made infinitesimal operators, they will commute and the last term will drop-out just leaving the commutator of covariant derivatives,

$$
\left[\nabla_\mu, \nabla_\nu \right] v^\rho = \nabla_\mu \nabla_\nu v^\rho - \nabla_\nu \nabla_\mu v^\rho \qquad \text{eqn. 56}
$$

So in curved space (not just a surface) is not zero,

$$
\left[\nabla_\mu, \nabla_\nu \right] v^\sigma = R^\sigma_{\rho\mu\nu} v^\rho u^\mu u^\nu
$$

If $\mu = \nabla_\mu$ and $\nu = \nabla_\nu$ then just write, eqn. 57

$$
\left[\nabla_\mu, \nabla_\nu \right] v^\sigma = R^\sigma_{\rho\mu\nu} v^\rho
$$

Which put simply says, the Riemann Tensor is a linear machine that takes an input vector v^ρ in its second slot and outputs the commutator of covariant derivatives of a second vector v^σ for the loop directions μ, ν. Note that it is usually written as outputting a contravariant tensor because it is in a tangent plane. If one wants to use covariant vectors, the indices 1,2 can be swapped but because the Riemann tensor is anti-symmetric, the sign changes,

$$
\left[\nabla_\mu, \nabla_\nu \right] v_\sigma = -R^\rho_{\sigma\mu\nu} v_\sigma \qquad \text{eqn. 58}
$$

Now imagine that the vectors are then basis vectors (unit length vectors) and one can see how basis vectors change orientation in space and time due to curved space time and how the volume element morphs. So taking the trace,

$$
R_{\mu\nu} = \sum_{\sigma=0}^{3} e_\sigma \left(\left[\nabla_\mu, \nabla_\nu \right] e^\sigma \right) = \sum_{\sigma=0}^{3} e_\sigma \left(R^\sigma_{\rho\mu\nu} e^\rho \right) \qquad \text{eqn. 59}
$$

Similarly to how the divergence operator $\nabla\cdot$ considers the infinitesimal volume element and is the dot product (or "trace") of the gradient operator, ∇, on a vector field, which produces a scalar field measuring the infinitesimal rate at which a vector field expands or contracts the volume around a point, the Ricci tensor measures how geodesics emanating from a point tend to converge or diverge, by quantifying the infinitesimal volume distortion produced by curvature: it tells how the curvature of space-time changes the rate at which a small ball of freely falling test particles[10] expands or contracts; a positive value means shrinkage and negative, growth,

$$V(r) = V_{flat}(r)\left[1 - \frac{1}{6}R_{\mu\nu}u^{\mu}u^{\nu}r^{2} + O\left(r^{3}\right)\right] \quad \text{eqn. 60}$$

It was then a step of intuition to relate the 4x4 Ricci tensor (with the Bianchi Identity to ensure local energy conservation, hence $G_{\mu\nu}$ is the trace reverse of the Ricci Scalar) to the 4x4 Stress-energy tensor (energy densities, "momenergy" fluxes and stresses) by Einstein or more systematically and dexterously by Hilbert with the Ricci scalar and the Action.

Appendix 4 – From Ricci Tensor to Newton

The commutator terms in the Ricci tensor (eqn. 59) involving the covariant derivatives expand thus,

$$R_{\mu\nu} = \partial_{\lambda}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda} \quad \text{eqn. 61}$$

The Christoffel symbols (also known as "affine" or "connection" coefficients), are non-tensorial objects (that is, they are not the same under coordinate transformation, unlike say a vector or a tensor) that describe how one coordinate system changes with respect to an old coordinate system, as we move in curved space.

For example, take the change in a vector under parallel displacement[10, 15] in curved space. So for the components of the vector,

$$\delta A^{i} = -\Gamma^{i}_{jk}A^{j}dx^{k} \quad \text{eqn. 62}$$

With,

$$\Gamma^{i}_{jk} = \frac{\partial \mathbf{e}_{j}}{\partial x^{k}}\cdot\mathbf{e}^{i} \quad \text{eqn. 63}$$

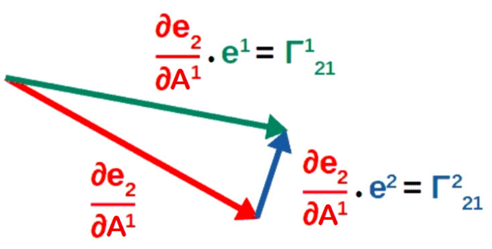


Figure 10 – Parallel displacement of vector x^1

So in figure 10 we parallel displace the component of the vector A in $\mathbf{e}_1$ basis (ie. A^1) in the direction of the $\mathbf{e}_2$ basis vector and that (in this 2D example) causes a *change* in the $\mathbf{e}_2$ component of the parallel displacement that projects onto the old $\mathbf{e}_1$ *and* $\mathbf{e}_2$ basis vectors by the Christoffel symbols given. We could of course look at the *change* in the $\mathbf{e}_1$ component of the parallel displacement too and have Γ^{1}_{12} and Γ^{2}_{12}.

One might think of the Earth at North Pole and setting up 3D basis vectors: z-axis is "up", y-axis is down the Greenwich Meridian towards Europe and the x-axis is negative 90 degrees away from the y-axis. If we rotate or parallel displace relative to the old coordinates, the new x,y,z bases are a linear combination of the old bases.

These kinds of Christoffel symbol with the first index in the upper position are called "the second kind"; these are used to transform the components of vectors and tensors under coordinate transforms.

If we lower the first index with the metric tensor, this leads to "the first kind" and are useful for the covariant derivative. See the references.

The Christoffel symbols are related to the metric tensor,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right) \quad \text{eqn. 64}$$

And one sees that the Einstein Field Equations by eqn. 61 and eqn. 64 are a second-order, non-linear partial differential equation (pde) in the metric tensor g^{uv}. This is indeed similar to other pdes, such as heat diffusion (a scalar), or electromagnetics (a vector potential or even a tensor, if one uses the electromagnetic tensor), so it isn't that daunting; in full the Einstein-Hilbert Field Equations are,

$$\begin{aligned} G_{\mu\nu}\left(g_{\mu\nu}\right) &= \frac{1}{2}g^{\alpha\beta}\left(\partial_{\alpha}\partial_{\beta}g_{\mu\nu} + \partial_{\mu}\partial_{\nu}g_{\alpha\beta} - \partial_{\alpha}\partial_{\mu}g_{\beta\nu} - \partial_{\alpha}\partial_{\nu}g_{\beta\nu}\right) \\ &\quad + Q_{\mu\nu}\left(g_{\mu\nu}, \partial g_{\mu\nu}\right) \\ &= \kappa T_{\mu\nu} + \left(\text{Dark Energy, Putative GDE}\right) \end{aligned} \quad \text{eqn. 65}$$

Where, Q contains quadratic terms in the metric and first order derivatives of the metric.

The route to Newtonian gravity is first to get rid of the terms that would cause non-linearity – the quadratic terms, leaving,

$$R_{\mu\nu} \approx \partial_{\lambda}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\lambda} \quad \text{eqn. 66}$$

To complete the argument a brief digression on the stress energy tensor of a point particle is needed[10],

$$T_{\mu\nu}(x) = m\int u_\mu(\tau)u_\nu(\tau)\frac{\delta^{(4)}(x - x(\tau))}{\sqrt{-g(x)}}d\tau \quad \text{eqn. 67}$$

Technically an integration over space-time is needed because the Dirac function is a *distribution* and not a density but it is no abuse of notation to project it into a hyperplane of time and 3D space (a so-called "foliation") and arrive at the form most usually used,

$$T_{\mu\nu} = -m\frac{u_\mu(t)u_\nu(t)}{u_0(t)}\delta^{(3)}(x - x(t)) \quad \text{eqn. 68}$$

By definition velocity $u_0 = -\gamma c \approx c$ and $u_\mu = \gamma v_{x,y,z}$

Thus $T_{00} \approx mc^2$, $T_{0i} \sim mcv_\nu$ and $T_{\mu\nu} \sim mv_\mu v_\nu$. So at low speed $\frac{|T_{0i}|}{|T_{00}|} \approx \frac{v}{c}$ and $\frac{|T_{ij}|}{|T_{00}|} \approx \left(\frac{v}{c}\right)^2$ and only the time-time, energy density component is needed in the static, low speed Newtonian regime.

This regime also views the metric tensor as a slight perturbation on the flat-space Minkowski tensor,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu} \ll 1| \quad \text{eqn. 69}$$

The linearised Christoffel symbols are by eqn. 64 then,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\lambda\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) \quad \text{eqn. 70}$$

Returning to eqn. 66 and substituting the connections,

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\nu - \square h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \quad \text{eqn. 71}$$

With $h = \eta^{\mu\nu}h_{\mu\nu}$ and $\square = \eta^{\rho\sigma}\partial_\rho\partial_\sigma$

The scalar curvature is then,

$$R = \eta^{\mu\nu}R_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \square h \quad \text{eqn. 72}$$

If the trace reversed perturbation $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$ is introduced and the harmonic gauge (which rids us of mixed derivatives) $\partial^\mu\bar{h} = 0$ used too, then substitution of the Ricci Tensor and Scalar into the field equations yields, at first order, the wave equation,

$$G_{\mu\nu} = -\frac{1}{2}\square\bar{h}_{\mu\nu} \quad \text{eqn. 73}$$

And with sources,

$$\square\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \quad \text{eqn. 74}$$

Only the static $\bar{h}_{00}$ term needs to be considered in the Newtonian limit and comparing to $\nabla^2\Phi = 4\pi G\rho_m$ (which indeed fixed the constant in the first place in the EFE at their inception) then,

$$\bar{h}_{00} = -\frac{4\Phi}{c^2} \quad \text{eqn. 75}$$

Appendix 5 – Matlab code for the Model

Be careful with line continuation when cutting and pasting. The code has been formatted to fit into the columns of this paper. You made need to edit the lines or use the Matlab "…" continuation formatter.

```
function GravitatingDarkEnergy()

   % --- Constants (floats) ---
   % m^3/(kg*s^2) Gravitational constant
   G = 6.67408e-11;
   % m/s speed of light
   c = 2.99792458e8;
   % J/m^3 Vacuum energy density
   rho_vac = 5.4e-10;
   % m Light years in meters
   Lyr = 9.4607304725808e15;
   Mass_Sun = 2e30; % kg
   % Baryon mass Milky Way in Mass_Sun
   Mass_MW  = 2e11;

   % --- Exponential spherical mass model:
   % rho(r) = rho0 * exp(-r/a) ---
   % Choose scale length a and total mass

   % scale length in light-years
   a_ly = 5e3;
   a = a_ly * Lyr;  % meters
   % target baryonic mass
   M_gal  = 5e10 * Mass_Sun;
   % 6a of r^2.e^-r/a gets most of mass
   % For rho(r) = rho0 e^{-r/a}, M_tot =
   % 8*pi*rho0*a^3
   Size_gal = 6 * a;
   % kg/m^3 central density
   rho0   = M_gal / (8*pi*a^3);

   fprintf('\nScale length a (kly): = %.2f\n',
        a_ly/1e3);
   fprintf('Central density rho0 (kg/m^3): =
        %.3e\n', rho0);
   fprintf('Galactic size (kly): = %.2f\n',
        Size_gal/1e3/Lyr);
   fprintf('Total mass (Kg): = %.2e\n',
        M_gal);
   fprintf('Total mass (Msun units): =
        %.2e\n', M_gal/Mass_Sun);
   fprintf('Total mass (MW units): = %.2f\n',
        M_gal/(Mass_MW*Mass_Sun));

   % --- GDE constant ---
   epsilon  = 1.5e31;
   GDEConst = epsilon .* 12 .* pi .* G .*
        rho_vac ./ (c.^4);
   fprintf('Epsilon:  = %.4e\n',
        epsilon);

   % --- Radius grid: dimensionless (Lyr)
   % on x-axis, meters inside ---

   rmin = 1; rmax = 1e8;
   % in light-years (for x-axis)

   N    = 1000;
   r_dimless = logspace(log10(rmin),
        log10(rmax), N); % light-years

   r_meters  = r_dimless * Lyr; % m

% --- Compute fields ---

% N/kg -ve
gm_vals = g_m_exp(r_meters, G, rho0, a);

% N/kg +ve
gl_vals = g_Lambda(r_meters, G, rho_vac,
     c);

% N/kg -ve
gde_vals  = g_GDE(r_meters, gm_vals
     + gl_vals, GDEConst);

% N/kg
gtot_vals = gm_vals + gde_vals;

% --- Plot: fields (log-log) and velocity
% (lin-lin) ---

create_or_get_figure('Log-Log plots of
     field strengths and lin-lin velocity');

% Top: fields vs radius (log-log)
subplot(2,1,1);
hold on;
box on;
plot(r_dimless, -gm_vals,
     'r', 'LineWidth', 1.5);  % |g_m|
plot(r_dimless,  gl_vals,
     'y', 'LineWidth', 1.5);  % g_Lambda
plot(r_dimless, -gde_vals,
     'g', 'LineWidth', 1.5);  % |g_GDE|
plot(r_dimless, -gtot_vals,
     'b', 'LineWidth', 1.8);  % |g_total|
set(gca, 'XScale', 'log', 'YScale', 'log');
xlim([rmin, rmax]);

% adjust if you want a different window
ylim([1e-15, 1e-7]);
grid on;

legend({'|g_m|','g_\Lambda','|g_{GDE}|',
     '|g_{total}|'}, 'Location', 'best');
xlabel('r (light-years)');
ylabel('|Field| (N/kg)');
title('Fields vs radius (log-log)');

% Bottom: velocity vs radius (lin-lin)
subplot(2,1,2);
v_vals = zeros(size(r_meters));

% Use inward (attractive) part for
% circular velocity
for i = 1:numel(r_meters)

     g_inward = gm_vals(i) + gde_vals(i);
     if g_inward < 0
             v_vals(i) = sqrt(r_meters(i)
             * -g_inward) / 1e3;  % km/s
     else
             v_vals(i) = 0;
     end
end

plot(r_meters./Lyr, v_vals, 'b',
     'LineWidth', 1.5);
axis([1e3 1e5 0 150]);  % tweak as needed
grid on;

ax = gca;
ax.XAxis.Exponent = 3;
xticks(0:10000:100000);
xlabel('r (light-years)');
ylabel('Velocity / km/s');
```

```
    title('Velocity vs radius (lin-lin)');

    % --- Dark-energy mass equivalence
    % calculation

    % Find the index in r_meters
    % closest to Rmax
    [~, idx_R] = min(abs(r_meters - Size_gal));

    % Extract radius and GDE field at that
    % radius
    R_use     = r_meters(idx_R);
    gGDE_R    = gde_vals(idx_R);

    % Effective enclosed GDE mass at Rmax
    M_GDE = - (R_use^2 * gGDE_R) / G;   % kg

    fprintf('Effective GDE mass (Kg) = %.3e
        \n', M_GDE);
    fprintf('Fraction of baryonic mass
        = %.4f (%.2f%%)\n', ...
    M_GDE/M_gal, 100*M_GDE/M_gal);

        disp(' ');
        disp(' ');
end

function g = g_m_exp(r, G, rho0, a)
    % Gravitational field for spherical
    % exponential density:
    % g(r) = - (4*pi*G*rho0*a^3 / r^2)
    % * [1 - e^{-r/a} ...
    % (1 + r/a + r^2/(2a^2))]

    x = r ./ a;
    % Avoid division by zero at r=0
    x = max(x, 1e-12);
    pref = -4*pi*G*rho0*a^3 ./ (r.^2);
    bracket = 1 - exp(-x) .* ...
    (1 + x + 0.5.*x.^2);

    % N/kg, inward negative
    g = pref .* bracket;
end

function g = g_Lambda(r, G, rho_vac, c)
    % Vacuum term g_Lambda(r)
    % using energy density rho_vac (J/m^3)
    % Treat rho_vac/c^2 as mass density
    rho_mass = rho_vac ./ (c.^2);

    % N/kg, outward positive
    g = (4/3).*pi.*G.*rho_mass.*r;
end

function I = g_GDE(r, g, GDEConst)
    % GDE integral: I(R) = GDEConst
    % * ∫_0^R g(r') K(R,r') dr'
    % with K(R,r') = 1 - (4/3)k + 0.5 k^2,
    %k = (R - r')/R

    I = zeros(1, numel(r));
    for i = 2:numel(r)
    R = r(i);
    rr = r(1:i);
    k = (R - rr) ./ R;
    kernel = 1 - (4/3).*k + 0.5.*k.^2;
    integrand = g(1:i) .* kernel;
    I(i) = trapz(rr, integrand);
    end
    I = I .* GDEConst;
end

function handle = create_or_get_figure(name)
    % Reuse same window when run many times

    handle = findobj('Name', name);
        if isempty(handle)
                handle = figure();
                set(handle, 'Name', name);
        else
                figure(handle);
                clf;
        end
end
```

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