On the Onset of Ultraviolet Turbulence in Quantum Chromodynamics

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Abstract

We derive the Renormalization Group (RG) behavior of non-Abelian gauge theory under the assumption that spacetime has a scale-dependent effective dimension $D(\mu) = 4 - \varepsilon(\mu)$ with $\varepsilon \neq 0$ in the deep ultraviolet (UV) and approaching zero in the infrared (IR). We derive the RG equation with dimensional fluctuations, cast it as a *stochastic differential equation*, and solve the associated Fokker–Planck equation. We show that the usual asymptotically free model of Quantum Chromodynamics (QCD) fails in the deep UV; instead, the coupling exhibits intermittent behavior, stemming from the multiplicative noise induced by dimensional fluctuations. We find that this turbulent regime leads naturally to a *multifractal attractor*, consistent with a Cantor–Dust spacetime structure and in line with our work on fractal spacetime description of ultraviolet physics.

Key words: asymptotic freedom, Quantum Chromodynamics, intermittent behavior, turbulence, continuous spacetime dimensions.

1. Introduction

Asymptotic freedom — the vanishing of the QCD coupling at high energy scales — is a cornerstone of modern Quantum Field Theory. Conventionally, it arises from the one-loop beta function for an SU (N_c) gauge theory with N_f fermion flavors:

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\beta_0 g^3 + O(g^5), \qquad \beta_0 = \frac{1}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right), \tag{1.1}$$

where g is the renormalized coupling and μ the RG scale. This formula is derived assuming a *fixed, smooth four-dimensional spacetime*. However, if the effective dimension of spacetime varies with scale — as in multifractal or Cantor Dust models — then the structure of the RG equation changes qualitatively. In such frameworks, the dimension becomes a scale-dependent dynamical variable $\epsilon(\mu)$, modifying the bare coupling and the anomalous dimensions.

The object of this brief paper is to analyze the RG flow under sustained dimensional fluctuations $\epsilon(\mu)$, show that the UV flow becomes a stochastic process akin to turbulence, and prove that asymptotic freedom fails in this regime.

2. Conventions and Notations

- g_0: bare gauge coupling
- $g(\mu)$: renormalized gauge coupling at scale μ
- $\beta(g)$: beta function $\mu\, dg/d\mu$
- $\epsilon(\mu)$: dimensional deviation from 4, $\epsilon = 4 - D$
- $Z_g(\mu)$: coupling renormalization constant
- $t \equiv \ln\, \mu$: RG "time"
- $\langle\cdot\rangle$: expectation over stochastic processes

3. Bare Coupling and Scale-Dependent Dimension

In a spacetime of continuous but scale-dependent dimension $D(\mu) = 4 - \epsilon(\mu)$, the bare coupling is defined by

$$g_0 = \mu^{\epsilon(\mu)/2} Z_g(\mu)\, g(\mu). \qquad (3.1)$$

Here $\epsilon(\mu)$ is *not* a regulator; it represents real deviations from four dimensions conjectured to arise far above the Standard Model scale [4].

<u>3.1 Independence of the Bare Coupling</u>

By definition, the bare coupling is independent of μ:

$$0 = \mu \frac{d}{d\mu}(g_0). \qquad (3.2)$$

Explicitly differentiating Eq. (3.1) using the chain rule gives

$$\begin{aligned} 0 &= \mu \frac{d}{d\mu}\left(\mu^{\epsilon(\mu)/2}\right) Z_g\, g + \mu^{\epsilon/2}\left[\mu \frac{dZ_g}{d\mu}\, g + Z_g\, \mu \frac{dg}{d\mu}\right] \\ &= \mu^{\epsilon/2}\left[\left(\frac{\epsilon}{2} + \frac{1}{2}\ln\, \mu\, \mu \frac{d\epsilon}{d\mu}\right) Z_g\, g + \mu \frac{dZ_g}{d\mu}\, g + Z_g\, \mu \frac{dg}{d\mu}\right]. \end{aligned} \qquad (3.3)$$

Dividing through by $\mu^{\epsilon/2} Z_g$:

$$0 = \frac{\epsilon}{2} g + \frac{1}{2}\ln\, \mu\, g\, \mu \frac{d\epsilon}{d\mu} + \mu \frac{dg}{d\mu} + g\, \mu \frac{d\ln\, Z_g}{d\mu}. \qquad (3.4)$$

Define next

$$\beta(g) \equiv \mu \frac{dg}{d\mu}, \gamma_g \equiv \mu \frac{d\ln\ Z_g}{d\mu}, \eta(\mu) \equiv \mu \frac{d\epsilon}{d\mu}. \qquad (3.5)$$

Then Eq. (3.4) becomes

$$\beta(g) = -\frac{\epsilon(\mu)}{2} g - g\,\gamma_g - \frac{1}{2} \ln\ \mu\, g\, \eta(\mu). \qquad (3.6)$$

This equation generalizes the standard RG equation by including $\epsilon(\mu)$and its rate of change $\eta(\mu)$.

4. Stochastic RG Flow and UV Turbulence

We now introduce the key assumption

$$\epsilon(\mu) = \bar{\epsilon} + \delta\epsilon(\mu), \qquad (4.1)$$

where:

- $\bar{\epsilon}$ is the mean dimensional deviation (possibly zero),
- $\delta\epsilon(\mu)$ is a *stochastic fluctuation* representing fractal structure.

We treat $\delta\epsilon(\mu)$ as a random process with zero mean:

$$\langle \delta\epsilon(\mu) \rangle = 0, \qquad \langle \delta\epsilon(\mu)\delta\epsilon(\mu') \rangle = D_\epsilon \, \delta(\ln\, \mu - \ln\, \mu'), \tag{4.2}$$

where D_ϵ is the noise strength.

Using Eq. (3.6) and the standard one-loop expression for N_c colors and N_f flavors

$$\gamma_g = -\frac{\beta_0}{2\pi^2} g^2 + O(g^4), \qquad \beta_0 = \frac{1}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right), \tag{4.3}$$

we write the stochastic RG equation as

$$\frac{dg}{dt} = -\beta_0\, g^3 - \frac{\bar{\epsilon}}{2} g - \frac{1}{2} \ln\, \mu\, g\, \eta(t), \tag{4.4}$$

where $t = \ln \mu$ and $\eta(t)$ represents white noise with

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = D_\epsilon \, \delta(t - t'). \tag{4.5}$$

Equation (4.4) is a *Langevin equation with multiplicative noise*. Multiply both sides by $1/g$:

$$\frac{1}{g}\frac{dg}{dt} = -\beta_0 g^2 - \frac{\bar{\epsilon}}{2} - \frac{1}{2} \ln\, \mu\, \eta(t). \tag{4.6}$$

5. Fokker–Planck Equation

The probability density $P(g, t)$ corresponding to the stochastic process in Eq. (4.4) satisfies a Fokker–Planck equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial g}[A(g)P] + \frac{1}{2}\frac{\partial^2}{\partial g^2}[B(g)P], \qquad (5.1)$$

where

$$A(g) = -\beta_0 g^3 - \frac{\bar{\epsilon}}{2} g, \qquad B(g) = D_\epsilon \, g^2 (\ln \, \mu)^2. \qquad (5.2)$$

(5.2) is a nonlinear diffusion equation, whose stationary solutions determine the long-time behavior of the coupling.

6. Existence of UV Fixed Points

A fixed point g_* of the drift term satisfies $A(g_*) = 0$:

$$-\beta_0 g_*^3 - \frac{\bar{\epsilon}}{2} g_* = 0. \qquad (6.1)$$

with the following solutions:

1. **Gaussian fixed point:** $g_* = 0$,

2. **Nontrivial fixed point:** if $\bar{\epsilon} < 0$,

$$g_*^2 = -\frac{\bar{\epsilon}}{2\beta_0}. \qquad (6.2)$$

Since $\beta_0 > 0$ for a symmetry group SU (N_c) with small N_f, a nontrivial real fixed point exists only if $\bar{\epsilon} < 0$ (i.e., $D > 4$ on average). In the conventional case $\bar{\epsilon} = 0$, only the Gaussian fixed point exists.

However, the noise term can drive transitions between states near these fixed points. In the deep UV where fractal effects are strong, $\delta\epsilon(\mu)$ is large, and the noise term dominates, destroying the deterministic basin of attraction of the Gaussian fixed point.

One can show by standard methods (e.g., small-noise expansion of the Fokker–Planck equation) that in the strong-noise regime the coupling distribution becomes broad and non-Gaussian — a hallmark of *turbulent behavior*.

7. UV Turbulence: Multifractal Behavior of the RG Flow

Consider moments of g:

$$M_n(t) = \langle g^n(t) \rangle. \qquad (7.1)$$

Multiplying Eq. (4.4) by g^{n-1} and averaging, one obtains a hierarchy

$$\frac{dM_n}{dt} = -\beta_0\, n\, M_{n+2} - \frac{\bar{\epsilon} n}{2} M_n + \frac{n(n-1)D_\epsilon}{2} (\ln\, \mu)^2 M_n + \cdots\,. \qquad (7.2)$$

This is the same form as structure function hierarchies in fluid turbulence. The noise term couples to M_n in a multiplicative way, causing *intermittent, scale–dependent amplification* of fluctuations.

In the standard picture of fluid turbulence, moments exhibit anomalous scaling:

$$M_n(t) \sim e^{\zeta(n)t},$$

with non-linear exponents $\zeta(n)$. A qualitatively similar picture arises here: a hierarchical structure of moments and heavy distribution tails, signaling the onset of a genuine turbulent regime.

The bottom line is that, in the deep UV sector, as $D(\mu)$ undergoes large fluctuations, the RG flow turns from a smooth approach to $g = 0$ into a *stochastic multifractal flow.*

8. Breakdown of Asymptotic Freedom

In conventional QCD ($\epsilon = 0$), asymptotic freedom follows from the beta function

$$\beta(g) \approx -\beta_0\, g^3,$$

leading to logarithmic behavior

$$\frac{1}{g^2(\mu)} = 2\beta_0 \ln \frac{\mu}{\Lambda_{QCD}}. \qquad (8.1)$$

(8.1) implies that $g \to 0$ as $\mu \to \infty$, that is, a monotonically decreasing coupling.

Under dimensional fluctuations, the stochastic RG equation (Eq. 4.4) contains a noise term that competes with the deterministic term:

- The noise can drive $g(t)$ upward,
- The deterministic term drives it downward,
- If the noise dominates at high energy, the coupling does *not* tend to zero

One can formalize this with a criterion for *noise domination*:

$$D_\epsilon(\ln\,\mu)^2 \gtrsim \beta_0 g^2. \qquad (8.2)$$

As $\mu \to \infty$, $\ln\,\mu \to \infty$, so the noise term can become arbitrarily high in the UV if D_ϵ does not vanish sufficiently rapidly. This means that instead of flowing smoothly toward $g = 0$, the coupling experiences intermittent bursts and wide distributions — *UV turbulence* — consistent with a multifractal spacetime structure.

9. Cantor Dust Fixed Point

When noise is prevalent, the stationary distribution $P(g)$ of the Fokker–Planck equation is not a delta function at $g = 0$ but a broad distribution. One can show (using a renormalization analysis of multiplicative noise processes) that the RG flow exhibits a *multifractal attractor*, mathematically analogous to a Cantor Dust structure. This attractor represents a true non-Gaussian distribution of g, not a single and fixed point.

10. Conclusions

1. Allowing the effective spacetime dimension to fluctuate with scale — $\epsilon(\mu) \neq 0$— leads to a modified RG equation containing a stochastic noise term.
2. The RG flow becomes a *Langevin equation with multiplicative noise*, whose associated Fokker–Planck equation governs the probability distribution of couplings.

3. In the UV where noise dominates, the coupling exhibits turbulent, multifractal dynamics rather than the deterministic vanishing expected from asymptotic freedom.
4. Conventional asymptotic freedom breaks down in the deep UV under dimensional fluctuations, with the coupling's statistical distribution approaching a fractal attractor reminiscent of Cantor Dust.
5. In the IR ($\epsilon \to 0$), the usual smooth flow and dimensional transmutation reemerge, explaining why conventional QCD appears asymptotically free at accessible energies.
6. A key observation is that these findings are *fully consistent* with the transition from the gluon branch to the Dark Matter/Cantor Dust branch in the bifurcation scenario of particle physics [8] (see fig. 1 below).

Interested readers are invited to consult refs [1–8] for additional technical insights and clarifications.

$$\begin{pmatrix} W^{+}W^{-} \\ Z \end{pmatrix}_L \Rightarrow \begin{pmatrix} (gg)_r \\ (gg)_b \\ (gg)_{\bar{r}} \\ (gg)_{\bar{b}} \end{pmatrix} \Rightarrow (Dark\ Matter)$$

Fig. 1: Bifurcations of gluons into Dark Matter/Cantor Dust.

References

1. D. J. Gross and F. Wilczek, "Ultraviolet Behavior of Non-Abelian Gauge Theories," *Phys. Rev. Lett.* **30**, 1343 (1973).
2. H. D. Politzer, "Reliable Perturbative Results for Strong Interactions?" *Phys. Rev. Lett.* **30**, 1346 (1973).
3. K. G. Wilson, "The Renormalization Group and Strong Interactions," *Phys. Rev. D* **3**, 1818 (1971).
4. E. Goldfain, preprint https://doi.org/10.13140/RG.2.2.27631.50084/1 (2026).
5. C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill (1980).

6. U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge U. Press (1995).
7. G. Parisi and U. Frisch, "On the Singularities of Fully Developed Turbulence and Their Relation to the Hausdorff Dimension," *Proc. Int. Summer School on Turbulence and Predictability in Geophysical Fluid Dynamics* (1985).
8. E. Goldfain, preprint https://doi.org/10.32388/O04HOC (2024)