A Derivation of the Alternating Power Sum Formula

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March 2026

Abstract

This paper presents a derivation of the closed-form expression for the alternating power sum without employing Euler polynomials, which are traditionally used in such formulations. Instead, the derivation utilizes the shifted series method together with differentiation techniques to construct a polynomial representation for the sum. The resulting expression is then connected to classical number-theoretic constants through the Riemann zeta function and Bernoulli numbers. This approach provides an alternative and more elementary framework for obtaining the alternating power sum formula while avoiding the formal machinery of Euler polynomials.

1 Introduction

Power sums of the form $\sum_{n=1}^{x} n^k$ have long been studied in number theory and are known to admit polynomial representations in x with coefficients related to Bernoulli numbers. When an alternating factor is introduced, the sums take the form $\sum_{n=1}^{x}(-1)^n n^k$, which arise in the study of alternating series and analytic number theory. Closed-form expressions for such sums are commonly derived using Euler polynomials; however, this approach introduces additional theoretical machinery. In this paper, we present a derivation of the formula for the alternating power sum without employing Euler polynomials. Instead, the method relies on shifted series techniques and differentiation to construct a polynomial representation of the sum and determine its coefficients, ultimately expressing the result in terms of classical number-theoretic quantities such as Bernoulli numbers.

2 Derivation

$$\sum_{n=1}^{x}(-1)^n n^k \tag{1}$$

We use the shifted series method to express (1) into a different form.

$$\sum_{n=1}^{x}(-1)^n n^k = \sum_{n=1}^{\infty}(-1)^n n^k - \sum_{n=1}^{\infty}(-1)^{n+x}(n+x)^k$$

$$\sum_{n=1}^{x}(-1)^n n^k = (-1)^x \sum_{n=1}^{\infty}(-1)^{n+1}(n+x)^k + \sum_{n=1}^{\infty}(-1)^n n^k \tag{2}$$

$$\sum_{n=1}^{x}(-1)^n n^k = (-1)^x P_k(x) + C_k \tag{3}$$

Set $x = 0$ on equation (3) to determine C_k.

$$0 = (-1)^0 P_k(0) + C_k$$

$$C_k = -P_k(0)$$

$$\sum_{n=1}^{x}(-1)^n n^k = (-1)^x P_k(x) - P_k(0) \tag{4}$$

Apply the shifted series method to the left-hand side of equation (4) and differentiate both sides with respect to x. Let $w = ln(-1)$.

$$w\left(-\sum_{n=1}^{\infty}(-1)^{n+x}(n+x)^k\right)+k\left(-\sum_{n=1}^{\infty}(-1)^{n+x}(n+x)^{k-1}\right)$$
$$=w(-1)^xP_k(x)+(-1)^xP'_k(x) \tag{5}$$

Divide both sides of equation (5) by $(-1)^x$.

$$w\left(-\sum_{n=1}^{\infty}(-1)^{n}(n+x)^k\right)+k\left(-\sum_{n=1}^{\infty}(-1)^{n}(n+x)^{k-1}\right)$$
$$=wP_k(x)+P'_k(x) \tag{6}$$

From equation (6), it is easy to see that:

$$P'_k(x)=kP_{k-1}(x) \tag{7}$$

From equation (7), the most suitable expression for $P_k(x)$ is:

$$P_k(x)=\sum_{j=1}^{k}\binom{k}{j}a_jx^{k-j} \tag{8}$$

Substitute (8) into equation (4).

$$\sum_{n=1}^{x}(-1)^n n^k=(-1)^x\sum_{j=1}^{k}\binom{k}{j}a_jx^{k-j}-a_k \tag{9}$$

Set $x=1$ on equation (9).

$$-1=-\sum_{j=1}^{k}\binom{k}{j}a_j-a_k$$
$$-1=-\sum_{j=1}^{k-1}\binom{k}{j}a_j-2a_k$$
$$a_k=\frac{1}{2}-\frac{1}{2}\sum_{j=1}^{k-1}\binom{k}{j}a_j \tag{10}$$

By looking back at equations (2) and (3), we see that:

$$a_k=-\sum_{n=1}^{\infty}(-1)^n n^k$$
$$a_{-k}=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^k}$$
$$a_{-k}=(1-\frac{1}{2^{k-1}})\zeta(k) \tag{11}$$

Where $\zeta(k)$ is the zeta function evaluated at k. We can use the zeta function representation of Bernoulli numbers ($B_k=-k\zeta(1-k)$) to define a_k in terms of the Bernoulli numbers.

$$B_k=-k\zeta(1-k)$$
$$\zeta(k)=\frac{B_{1-k}}{k-1} \tag{12}$$

Now substitute (12) into (11) and set $-k$ back into k.

$$a_{-k}=(1-\frac{1}{2^{k-1}})\frac{B_{1-k}}{k-1}$$
$$a_k=B_{k+1}\frac{2^{k+1}-1}{k+1} \tag{13}$$

Now we have defined a_k in terms of the Bernoulli numbers. Substituting (13) into equation (9), we obtain:

$$\sum_{n=1}^{x}(-1)^n n^k=(-1)^x\sum_{j=1}^{k}\binom{k}{j}\frac{2^{j+1}-1}{j+1}B_{j+1}x^{k-j}-\frac{2^{k+1}-1}{k+1}B_{k+1} \tag{14}$$

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