The set of real numbers as a function of the set of natural numbers

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Abstract

In this article, we prove that the number of elements in the set of positive real numbers $Card(\mathbb{R}^+)$ can be expressed as a function of the number of elements in the set of natural numbers $Card(\mathbb{N})$ using the formula:

$$\boldsymbol{R^+ = N + 0.9N(N-1)^2}$$

- $Card(\mathbb{N}) = N$
- $Card(\mathbb{R}^+) = R^+$

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Logically, the number that comes after this series of numbers (0 – 0.1 – 0.2 – 0.3 – 0.4 – 0.5 – 0.6 – 0.7 – 0.8 – 0.9) is 1. But if we continue as if we were placing all the natural numbers after the decimal point (0.9 – 0.10 – 0.11 – 0.12 …. up to 0.9999...), we will be able to cover all real numbers between 0.1 and 1. This is because the real number that comes just after 0.1 (0.1000...1) will correspond to the natural number (1000...1) after the decimal point.

We apply the same principle to real numbers between 0.01 and 0.1 and so on, as shown in the table :

		N																		
N	0															1	2	3	…	9999…
	1	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,10	0,11	0,12	…	0,9999…					
	2	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09	0,010	0,011	0,012	…	0,09999…					
	3																			
	.																			
	.																			
	.																			
	9999…																			

We can see that the numbers 0.1 and 0.01 repeat each time when the natural numbers 10 -100 -1000 … are after the decimal point. In fact, all numbers will repeat with different frequencies (k).

Let's take, for example, the 99 numbers from 0.1 to 0.99:

- From 0.1 to 0.9 (step = 0.1) : **k = 2**
- From 0.11 to 0.99 (step = 0.01) with the exception of 0.10 to 0.90 (step = 0.1) : **k = 1**
- $\mathbf{99 = 9 \times 2 + 81}$

For the 999 numbers from 0.1 to 0.999:

- From 0.1 to 0.9 (step = 0.1) : **k = 3**
- From 0.11 to 0.99 (step = 0.01) except for 0.10 to 0.90 (step = 0.1) : **k = 2**
- From 0.111 to 0.999 (step = 0.001) except for 0.11 to 0.99 (step = 0.01) : **k = 1**
- $\mathbf{999 = 9 \times 3 + 81 \times 2 + 810}$

For the 9999 numbers from 0.1 to 0.9999:

- $\mathbf{9999 = 9 \times 4 + 81 \times 3 + 810 \times 2 + 8100}$

Therefore, we can write the formula:

$$\mathbf{10^n - 1 = 9n + 81 \sum_{k=1}^{n-1} k10^{n-k-1}}$$

Proof:

We are looking for the sum: $\boldsymbol{S = \sum_{k=1}^{n-1} k10^{n-k-1}}$

Let's set :

$$j = n - k - 1 \ \Rightarrow k = n - j - 1$$

When :

- $k = 1 \Rightarrow j = n - 2$
- $k = n - 1 \Rightarrow j = 0$

Therefore :

$$\boldsymbol{S = \sum_{j=0}^{n-2} (n - j - 1)10^j}$$

We separate :

$$\boldsymbol{S = (n-1)\sum_{j=0}^{n-2} 10^j - \sum_{j=0}^{n-2} j10^j}$$

Known sums :

- $\boldsymbol{\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}}$
- $\boldsymbol{\sum_{j=0}^{n} jr^j = \frac{r-(n+1)r^{n+1}+nr^{n+2}}{(1-r)^2}}$

Therefore :

$\sum_{j=0}^{n-2} 10^j = \frac{10^{n-1}-1}{9}$ and $\sum_{j=0}^{n-2} j10^j = \frac{10-(n-1)10^{n-1}+(n-2)10^n}{81}$

After simplification :

$$\boldsymbol{S = \frac{10^n - 9n - 1}{81}}$$

Therefore :

$$9n + 81\sum_{k=1}^{n-1} k10^{n-k-1} = 9n + 81S = 10^n - 1$$

To eliminate repeated numbers, simply we take the frequencies = 1.

Therefore, the number of real numbers from 0.1 to 0.9999...= $\mathbf{0.9N}$ **:**

- $9 + 81\sum_{k=1}^{n-1} 10^{n-k-1} = 9 + 81\sum_{j=0}^{n-2} 10^{j} = 9 + 81\,\frac{10^{n-1}-1}{9} = 0.910^{n} = 0.9N$
- $N = 10^n\ (n \to \infty)$

Similarly, for the number of real numbers from 0.01 to 0.09999...= $\mathbf{0.9N}$ **,** and so on...

By multiplying 0.9N by N-1 (number of natural elements without the element 0) and by N-1 (number of levels 0.1 -0.01-0.001...), we can conclude that:

$$R^{+} = N + 0.9N(N-1)^2$$

	N																			
N	0															1	2	3	…	9999…
	1	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,10	0,11	0,12	…	0,9999…					
	2	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09	0,010	0,011	0,012	…	0,09999…					
	3																			
	.																			
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	9999…																			