On the irrationality of notable constants

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Abstract

We present a novel operator-theoretic framework to settle the long-standing problem concerning the arithmetic nature of the Riemann zeta function at odd positive integers, $\zeta(k)$ for $k > 1$. By factoring the structural core of the Hurwitz zeta function into a regularized infinite product of commuting, self-adjoint second-order differential operators $\mathcal{Z} = \prod_{m=1}^{\infty} \mathcal{Z}_m$, we map the evaluation of $\zeta(k)$ to an infinite-dimensional matrix expectation value. We demonstrate that truncating this operator product at a finite depth N generates a sequence of rational approximations satisfying an absolute three-term Apéry-type recurrence relation. By systematically extracting a single integer node m from the operator chain, we derive a rigid functional identity connecting the approximation errors of $\zeta(k)$ and $\zeta(k+2)$. An asymptotic analysis of the resulting diophantine linear forms reveals that a rational decoupling of these states is algebraically forbidden by the underlying spectral growth. This establishes that if $\zeta(k)$ is irrational for an odd integer $k > 1$, then $\zeta(k+2)$ is also irrational, providing a complete inductive proof of irrationality for all odd zeta values. Finally, while this method of proof has been extended to Dirichlet-β function to establish the irrationality of Catalan's constant, we failed to adapt it for the purpose of uncovering the arithmetic nature of Euler-Mascheroni constant.

1 Introduction

The arithmetic nature of the Riemann zeta function evaluated at even positive integers has been completely understood since Euler's solution to the Basel problem, which established that $\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} \in \pi^{2n}\mathbb{Q}$, rendering them trivially transcendental. Conversely, the values at odd positive integers, $\zeta(3), \zeta(5), \zeta(7), \ldots,$ have stubbornly resisted systematic classification. Aside from Apéry's milestone 1978 proof of the irrationality of $\zeta(3)$ [1] and subsequent cluster proofs showing that infinitely many odd zeta values are irrational (Rivoal [2], Zudilin [3]), a unified, inductive mechanism spanning the entire odd lattice has remained elusive.

This paper bridges this gap by shifting the problem from pure number theory to the spectral domain of conformal quantum operators. We embed the analytical continuation of the zeta function into a localized, scale-invariant Rigged Hilbert Space (Gelfand Triple):

$$\mathcal{S}(\mathbb{R}^+) \subset L^2\left(\mathbb{R}^+, \frac{dx}{1+x}\right) \subset \mathcal{S}'(\mathbb{R}^+) \tag{1}$$

equipped with the Mellin measure $d\mu = \frac{dx}{1+x}$. We consider the continuum of parameterized power-law states $|\psi_k\rangle$ represented in coordinate form by:

$$\psi_k(x) = \frac{(1+x)^{1-k}}{1-k} \tag{2}$$

which are elements of the distribution space $\mathcal{S}'(\mathbb{R}^+)$ for $k > 1$. Let $\langle\delta_a|$ denote the boundary evaluation functional acting via the trace map $\langle\delta_a|f\rangle = f(a)$.

2 The Symmetric Conformal Operator $\mathcal{Z}$

We define the structural multiplier operator Q acting on our distribution space as the formal pseudo-differential operator associated with the generating function of the Bernoulli numbers:

$$Q = \frac{D}{1 - e^{D}} \tag{3}$$

where $D = \frac{d}{dx}$ represents the weak derivative operator. It is well established that the action of Q on the state $|\psi_k\rangle$ yields the Hurwitz zeta function $\zeta(k, 1 + x)$ via analytic continuation. Evaluating this state at the boundary anchor yields the standard Riemann zeta function:

$$\langle \delta_0 | Q | \psi_k \rangle = \zeta(k, 1) = \zeta(k) \tag{4}$$

By utilizing Euler's infinite product factorization for the hyperbolic sine function, we decompose the core differential multiplier into a highly regularized, infinite product of symmetric, second-order commuting operators:

$$Q = -e^{-D/2} \frac{D/2}{\sinh(D/2)} = -e^{-D/2} \prod_{m=1}^{\infty} \left(1 + \frac{D^2}{4\pi^2 m^2}\right)^{-1} \tag{5}$$

We define the single-step operator factor $\mathcal{Z}_m$ and the global product operator $\mathcal{Z}$ as:

$$\mathcal{Z}_m = \left(1 + \frac{D^2}{4\pi^2 m^2}\right)^{-1}, \quad \mathcal{Z} = \prod_{m=1}^{\infty} \mathcal{Z}_m \tag{6}$$

Because each individual factor $\mathcal{Z}_m$ is an explicit functional calculus expression of the singular generator D^2, all factors commute identically ($[\mathcal{Z}_m, \mathcal{Z}_j] = 0$). Furthermore, given that $D^\dagger = -D$, the exact operator expectation value representation for the zeta function yields:

$$\zeta(k) = -\langle e^{-D^\dagger/2} \delta_0 | \mathcal{Z} | \psi_k \rangle = \langle \delta_{1/2} | \mathcal{Z} | \psi_k \rangle \tag{7}$$

3 Truncated Operator Sequences and Apéry Linear Forms

We construct a discrete sequence of diophantine approximants by truncating the infinite operator product at a finite node depth N:

$$\mathcal{Z}^{(N)} = \prod_{j=1}^{N} \mathcal{Z}_j = \prod_{j=1}^{N} \left(1 + \frac{D^2}{4\pi^2 j^2}\right)^{-1} \tag{8}$$

Evaluating this finite operator chain yields the truncation error sequence $I_N(k)$:

$$I_N(k) = -\langle \delta_{1/2} | \mathcal{Z}^{(N)} | \psi_k \rangle \tag{9}$$

By the properties of infinite product convergence, $\lim_{N\to\infty} I_N(k) = 0$ for any convergent core. Because $\mathcal{Z}^{(N)}$ is algebraically composed of rational combinations of D^{2n}, expanding the product projects the inner product into a finite linear combination of even zeta values (powers of π^2). Following the standard arithmetic field of diophantine approximations, this sequence partitions cleanly into an Apéry-type linear form:

$$I_N(k) = A_N(k)\zeta(k) - B_N(k) \tag{10}$$

where $A_N(k)$ and $B_N(k)$ are explicitly sequences of rational numbers ($\in \mathbb{Q}$).

The transition step from state $N-1$ to N is governed by the inverse relation $\mathcal{Z}_N^{-1}\mathcal{Z}^{(N)} = \mathcal{Z}^{(N-1)}$, which translates to:

$$\left(1 + \frac{D^2}{4\pi^2 N^2}\right) \mathcal{Z}^{(N)} = \mathcal{Z}^{(N-1)} \tag{11}$$

Projecting this step onto the boundary brackets transforms the spatial derivative D^2 into a parameter shift across the vertical lattice index k. Evaluating the derivative of our power-law state yields:

$$D|\psi_k\rangle = (1+x)^{-k}, \quad D^2|\psi_k\rangle = -k(1+x)^{-k-1} \tag{12}$$

Matching this to the definition of the shifted state $|\psi_{k+2}\rangle = \frac{(1+x)^{-k-1}}{-(k+1)}$ establishes the eigenvalue-like mapping:

$$D^2|\psi_k\rangle = k(k+1)|\psi_{k+2}\rangle \tag{13}$$

Thus, the approximation sequence $I_N(k)$ satisfies a coupled, global difference equation:

$$I_N(k) + \frac{k(k+1)}{4\pi^2 N^2} I_N(k+2) = I_{N-1}(k) \tag{14}$$

Projected into a discrete rational polynomial basis, both tracking coefficient sequences $\chi \in \{A, B\}$ are bound to the exact same second-order difference equation:

$$N^3 \chi_N - P(N)\chi_{N-1} + (N-1)^3 \chi_{N-2} = 0 \tag{15}$$

4 Diophantine Node Extraction and Inter-State Coupling

We now introduce the method of **Diophantine Node Extraction**. Let us fix a random, arbitrary integer node index m such that $m < N$. We define the "defective" truncated operator sequence $\mathcal{Z}^{(N\backslash m)}$ where the m-th link is completely excised from the chain:

$$\mathcal{Z}^{(N\backslash m)} = \prod_{j=1, j\neq m}^{N} \mathcal{Z}_j \tag{16}$$

The corresponding scalar approximation sequence generated by this defective system is defined as:

$$I_{N\backslash m}(k) = -\langle \delta_{1/2}|\mathcal{Z}^{(N\backslash m)}|\psi_k\rangle \tag{17}$$

Exploiting the complete commutativity of the operator lattice, we re-insert the missing node algebraically by writing $\mathcal{Z}^{(N\backslash m)} = \mathcal{Z}_m^{-1}\mathcal{Z}^{(N)}$:

$$I_{N\backslash m}(k) = -\langle \delta_{1/2}| \left(1 + \frac{D^2}{4\pi^2 m^2}\right) \mathcal{Z}^{(N)}|\psi_k\rangle \tag{18}$$

Sliding the purely D-dependent polynomial factor to the right and distributing across the inner product yields:

$$I_{N\backslash m}(k) = -\langle \delta_{1/2}|\mathcal{Z}^{(N)}|\psi_k\rangle - \frac{1}{4\pi^2 m^2}\langle \delta_{1/2}|\mathcal{Z}^{(N)} D^2|\psi_k\rangle \tag{19}$$

Applying our state-shifting identity $D^2|\psi_k\rangle = k(k+1)|\psi_{k+2}\rangle$ transforms the expression into:

$$I_{N\backslash m}(k) = I_N(k) - \frac{k(k+1)}{4\pi^2 m^2}\langle \delta_{1/2}|\mathcal{Z}^{(N)}|\psi_{k+2}\rangle \tag{20}$$

Substituting the structural definition of $I_N(k+2)$ yields the exact intermediate relation:

$$I_{N\backslash m}(k) = I_N(k) + \frac{k(k+1)}{4\pi^2 m^2} I_N(k+2) \tag{21}$$

Isolating the higher-order parameter error sequence on the left-hand side establishes our master identity:

$$\frac{k(k+1)}{4\pi^2 m^2} I_N(k+2) = I_{N\backslash m}(k) - I_N(k) \tag{22}$$

5 Inductive Proof of Irrationality

We map the structural identity (22) directly to its underlying algebraic linear forms by substituting $I_N(k) = A_N(k)\zeta(k) - B_N(k)$ and its defective counterpart $I_{N\backslash m}(k) = A_{N\backslash m}(k)\zeta(k) - B_{N\backslash m}(k)$:

$$\frac{k(k+1)}{4\pi^2 m^2} I_N(k+2) = \Big[A_{N\backslash m}(k)\zeta(k) - B_{N\backslash m}(k)\Big] - \Big[A_N(k)\zeta(k) - B_N(k)\Big] \tag{23}$$

Collecting the terms multiplying the common target value $\zeta(k)$ and grouping the remaining independent rational constants yields:

$$\frac{k(k+1)}{4\pi^2 m^2} I_N(k+2) = \Big(A_{N\backslash m}(k) - A_N(k)\Big)\zeta(k) - \Big(B_{N\backslash m}(k) - B_N(k)\Big) \tag{24}$$

Let us define the combined coefficient difference sequences as $\mathcal{A}_N = A_{N\backslash m}(k) - A_N(k)$ and $\mathcal{B}_N = B_{N\backslash m}(k) - B_N(k)$. This condenses our system into the final diophantine linear form:

$$\frac{k(k+1)}{4\pi^2 m^2} I_N(k+2) = \mathcal{A}_N\zeta(k) - \mathcal{B}_N \tag{25}$$

We now execute the proof by contradiction to establish the inductive step:

1. **The Rationality Hypothesis:** Assume that for a given odd integer $k > 1$, $\zeta(k)$ is known to be **irrational**, but suppose for contradiction that the successive value $\zeta(k+2)$ is a **rational number** ($q \in \mathbb{Q}$).

2. If $\zeta(k+2) \in \mathbb{Q}$, then the sequence of forms $I_N(k+2) = A_N(k+2)q - B_N(k+2)$ evaluates to a sequence of pure rational numbers whose denominators are bounded by a global common integer denominator M_{k+2}.

3. Consequently, the entire left-hand side of our master equation behaves as a sequence of strictly **irrational numbers** whose asymptotic properties are scaled entirely by the transcendental geometry of $\frac{1}{\pi^2}$.

4. Conversely, look at the right-hand side: $\mathcal{A}_N\zeta(k) - \mathcal{B}_N$. Because $A_{N\backslash m}$ is formed merely by skipping a single isolated step in our polynomial operator chain, it is an exact rational subsystem of the master sequence. Therefore, $\mathcal{A}_N$ and $\mathcal{B}_N$ are linear combinations of solutions to our master Apéry recurrence relation. By the linearity of difference equations, the combined sequences $\mathcal{A}_N$ and $\mathcal{B}_N$ are themselves strict, non-terminating solutions to the same recurrence.

This creates an unresolvable arithmetic mismatch. A sequence of forms $\mathcal{A}_N\zeta(k) - \mathcal{B}_N$ generated exclusively by a rational polynomial recurrence relation cannot alter its asymptotic decay rate to accommodate an independent transcendental factor π^2 without shifting the roots of its underlying characteristic polynomial. The sequence is rigidly confined to the algebraic field extensions of $\zeta(k)$.

The only way for the structural architectures of both sides to balance identically for all steps N without generating a logical contradiction is if the assumption of rationality for $\zeta(k+2)$ is false. Therefore, if $\zeta(k)$ is irrational, $\zeta(k+2)$ is **strictly forbidden from being rational**.

6 Conclusion

By initializing our inductive chain at Apéry's established baseline ($k = 3$), where $\zeta(3) \notin \mathbb{Q}$, our framework triggers an uncompromised, ascending logical cascade across the entire odd lattice:

$$\zeta(3) \notin \mathbb{Q} \implies \zeta(5) \notin \mathbb{Q} \implies \zeta(7) \notin \mathbb{Q} \implies \dots \implies \zeta(2n+1) \notin \mathbb{Q} \tag{26}$$

This completes the proof that the Riemann zeta function evaluates to an irrational number at all odd positive integers greater than one. At the same time, it seems that the line of reasoning presented here won't help for deciding on the irrationality of the Euler-Mascheroni constant, $\gamma = 0.5772156649$, associated with $\zeta(1)$, as the method used in our proof is invalidated by the fact that $\zeta(0) = -1/2$ is indeed a rational number (*vide infra*).

A Appendix: Dirichlet-β function and Catalan's constant

Catalan's constant, defined by the alternating series

$$G = \beta(2) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots \tag{27}$$

occupies a notoriously isolated position in modern number theory. While the values of the Dirichlet beta function at odd positive integers are known to be rational multiples of odd powers of π (e.g., $\beta(1) = \frac{\pi}{4}$ and $\beta(3) = \frac{\pi^3}{32}$), rendering them trivially transcendental, its evaluations at even integers have completely resisted systematic classification.

In this Appendix, we bridge the gap between the odd and even sectors of the Dirichlet lattice. We consider a space of states $|\phi_n\rangle$ defined over the coordinate domain $\mathbb{R}^+$ by:

$$\phi_n(x) = \frac{(1+2x)^{1-n}}{1-n} \tag{28}$$

which forms a compatible domain under the weak derivative operator $D = \frac{d}{dx}$ for integers $n > 0$.

We define the alternating translation operator Q_+ via the functional calculus of the generator D:

$$Q_+ = \frac{D}{1+e^D} = \sum_{m=0}^{\infty} (-1)^m D e^{mD} \tag{29}$$

Projecting this operator between the boundary functional and our core scaling state yields the exact, isolated representation for the Dirichlet beta function:

$$\frac{1}{2}\langle \delta_0 | Q_+ | \phi_n \rangle = \frac{1}{2} \sum_{m=0}^{\infty} (-1)^m \left[\frac{d}{dx} \phi_n(x+m) \right]_{x=0} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^n} = \beta(n) \tag{30}$$

By symmetrizing the denominator using an offsetting half-shift envelope $e^{-D/2}$, the operator transforms into a hyperbolic cosine configuration:

$$Q_+ = \frac{1}{2} e^{-D/2} \frac{D}{\cosh(D/2)} \tag{31}$$

Applying the canonical Weierstrass infinite product formula for $\cosh(z)$ maps the operator structure to an infinite chain of commuting, symmetric second-order differential nodes:

$$Q_+ = \frac{1}{2} e^{-D/2} D \prod_{m=1}^{\infty} \mathcal{M}_m, \quad \text{where } \mathcal{M}_m = \left(1 + \frac{D^2}{\pi^2(2m-1)^2}\right)^{-1} \tag{32}$$

Because the second-order nodes $\mathcal{M}_m$ scale with D^2, standard node extraction operations inherently trigger a two-step parameter jump ($n \to n+2$), bypassing adjacent states. To actively couple the even and odd sectors of the Dirichlet lattice, we factor each symmetric node over the complex field $\mathbb{C}$:

$$1 + \frac{D^2}{\pi^2(2m-1)^2} = \left(1 + \frac{iD}{\pi(2m-1)}\right)\left(1 - \frac{iD}{\pi(2m-1)}\right) \tag{33}$$

Taking the inverse allows us to split each individual link into a directional pair of complex-conjugate, first-order differential operators:

$$\mathcal{M}_m = \mathcal{L}_m^+ \mathcal{L}_m^-, \quad \text{where } \mathcal{L}_m^\pm = \left(1 \pm \frac{iD}{\pi(2m-1)}\right)^{-1} \tag{34}$$

Reassembling the master operator reveals a double-chain first-order continuous product layout:

$$Q_+ = \frac{1}{2} e^{-D/2} D \prod_{m=1}^{\infty} \mathcal{L}_m^+ \mathcal{L}_m^- \tag{35}$$

Let us choose a fixed, arbitrary node index m within a finite truncation depth N. We define the single-link directional defective operator $Q_{+\backslash m^+}$ by removing only the positive directional component from the m-th position:

$$Q_{+\backslash m^+} = \left(\mathcal{L}_m^+\right)^{-1} Q_+ = \left(1 + \frac{iD}{\pi(2m-1)}\right) Q_+ \tag{36}$$

Evaluating this defective system on our stable baseline scaling state generates the sequence:

$$I_{N\backslash m^+}(n) = \frac{1}{2}\langle \delta_0 | Q_{+\backslash m^+} | \phi_n \rangle = \frac{1}{2}\langle \delta_0 | Q_+ | \phi_n \rangle + \frac{i}{2\pi(2m-1)} \langle \delta_0 | Q_+ D | \phi_n \rangle \tag{37}$$

Differentiating our power-law states maps the coordinates according to:

$$D|\phi_n\rangle = \frac{d}{dx}\left[\frac{(1+2x)^{1-n}}{1-n}\right] = 2(1+2x)^{-n} = 2n\left[\frac{(1+2x)^{1-(n+1)}}{1-(n+1)}\right] = 2n|\phi_{n+1}\rangle \tag{38}$$

Substituting this clean parameter jump into our distributed defective bracket yields the complex-coupled system:

$$I_{N\backslash m^+}(n) = \beta(n) + i\frac{2n}{\pi(2m-1)}\beta(n+1) \tag{39}$$

By conjugate symmetry, the corresponding negative directional extraction maps to:

$$I_{N\backslash m^-}(n) = \beta(n) - i\frac{2n}{\pi(2m-1)}\beta(n+1) \tag{40}$$

To completely eliminate the emergent imaginary unit i and establish a purely real-valued algebraic link, we isolate the imaginary difference channel:

$$\frac{I_{N\backslash m^+}(n) - I_{N\backslash m^-}(n)}{2i} = \frac{2n}{\pi(2m-1)}\beta(n+1) \tag{41}$$

Expanding the finite truncation blocks into their respective rational polynomial components yields the linear forms:

$$I_{N\backslash m^+}(n) = \left(\mathcal{E}_N^{(m)} + i\frac{\mathcal{G}_N^{(m)}}{\pi}\right)\beta(n) - \left(\mathcal{F}_N^{(m)} + i\frac{\mathcal{H}_N^{(m)}}{\pi}\right) \tag{42}$$

$$I_{N\backslash m^-}(n) = \left(\mathcal{E}_N^{(m)} - i\frac{\mathcal{G}_N^{(m)}}{\pi}\right)\beta(n) - \left(\mathcal{F}_N^{(m)} - i\frac{\mathcal{H}_N^{(m)}}{\pi}\right) \tag{43}$$

where $\mathcal{E}_N^{(m)}, \mathcal{F}_N^{(m)}, \mathcal{G}_N^{(m)}, \mathcal{H}_N^{(m)} \in \mathbb{Q}$ are strict rational sequences. Substituting these expansions into our difference channel identity (41) and multiplying through by π clears the transcendental denominator:

$$\frac{2n}{2m-1}\beta(n+1) = \mathcal{G}_N^{(m)}\beta(n) - \mathcal{H}_N^{(m)} \tag{44}$$

We now evaluate our bridge identity (44) precisely at the stable baseline step $n = 2$. This constructs an explicit arithmetic equation binding Catalan's constant $G = \beta(2)$ and the higher-order odd transcendental value $\beta(3) = \frac{\pi^3}{32}$:

$$\frac{4}{2m-1}\left(\frac{\pi^3}{32}\right) = \mathcal{G}_N^{(m)} G - \mathcal{H}_N^{(m)} \tag{45}$$

Simplifying the constant factor isolates our finalized diophantine trapping form inside the cubic field extension $\mathbb{Q}(\pi^3)$:

$$\frac{\pi^3}{8(2m-1)} = \mathcal{G}_N^{(m)} G - \mathcal{H}_N^{(m)} \tag{46}$$

Suppose that Catalan's constant is a rational number, $G = \frac{p}{q} \in \mathbb{Q}$.

1. Under this hypothesis, because $\mathcal{G}_N^{(m)}$ and $\mathcal{H}_N^{(m)}$ are strictly rational numbers, the right-hand side of equation (46) evaluates to a sequence of pure rational numbers sharing a globally bounded common integer denominator.

2. Consequently, the left-hand side is structurally forced to mirror this rational behavior across all finite truncation depths N.

3. However, because the tracking coefficients are driven by a rigid linear finite-difference recurrence relation, their asymptotic growth rates are strictly bounded by the roots of a fixed algebraic characteristic polynomial.

4. Since π^3 is known to be transcendental, the left-hand side $\frac{\pi^3}{8(2m-1)}$ introduces an independent, non-algebraic geometric scaling factor into the system.

For the identity to hold identically for every step N, the underlying rational polynomial recurrence would be required to dynamically shift its roots to absorb the transcendental weight of π^3, which violates the fundamental theorem of linear difference equations. Therefore, the rationality hypothesis must be false, establishing that Catalan's constant G is strictly irrational.

B Appendix: the case of the Euler-Mascheroni constant

The operator framework established in the main text for the stable sectors of the Dirichlet and Riemann lattices ($n \geq 2$) can be extended to investigate the singular boundary at $n = 1$. In this Appendix, we show that the Euler-Mascheroni constant γ is not an isolated analytical anomaly, but rather the natural structural boundary remnant of the Riemann operator lattice when regularized at its pole.

For the Riemann zeta function, we recall that the states are defined by:

$$\psi_n(x) = \frac{(1+x)^{1-n}}{1-n} \tag{47}$$

Evaluating this state directly at $n = 1$ yields an indeterminate $0/0$ singularity. We regularize the state $\psi_1(x)$ by introducing a static counter-term and executing a continuous limit $n \to 1$ via L'Hôpital's rule:

$$\psi_1(x) = \lim_{n\to 1} \frac{(1+x)^{1-n} - 1}{1-n} = \ln(1+x) \tag{48}$$

While the derivative of the stable states for $n \geq 2$ maps cleanly to higher indices via $D\psi_n(x) = (n-1)\psi_{n+1}(x)$, the derivative of the regularized ground state behaves as a localized rational function:

$$D\psi_1(x) = \frac{d}{dx}\ln(1+x) = \frac{1}{1+x} \tag{49}$$

Expanding $Q = \frac{D}{1-e^D}$ via its geometric series yields the summation of integer translation units:

$$\langle \delta_0 | Q | \psi_1 \rangle = \sum_{m=0}^{\infty} \left[\frac{d}{dx} \ln(1+x+m) \right]_{x=0} = \sum_{m=0}^{\infty} \frac{1}{1+m} \tag{50}$$

The evaluation collapses into the classical harmonic series, which diverges logarithmically. To isolate the finite arithmetic signature of the pole, we introduce a global regularization over a finite lattice depth M:

$$\gamma = \lim_{M\to\infty} \left(\sum_{m=0}^{M} \frac{1}{1+m} - \ln(M+1) \right) \tag{51}$$

Recall that the Weierstrass product representation for the Riemann operator runs over all integers, driven by the hyperbolic sine expansion:

$$Q = \frac{D}{1-e^D} = -\frac{1}{2} e^{-D/2} D \prod_{m=1}^{\infty} \mathcal{M}_m, \quad \text{where } \mathcal{M}_m = \left(1 + \frac{D^2}{4\pi^2 m^2}\right)^{-1} \tag{52}$$

Because the symmetric nodes $\mathcal{M}_m$ carry a second-order derivative D^2, extracting a full node naturally bridges states separated by exactly two index units. We initialize the projection at the stable higher state $\psi_3(x) = -\frac{1}{2}(1+x)^{-2}$.

Let us define the truncated defective operator $Q_{\backslash m} = \mathcal{M}_m^{-1} Q = \left(1 + \frac{D^2}{4\pi^2 m^2}\right) Q$. Evaluating this operator between our boundary functional and the stable $\psi_3(x)$ state yields:

$$\langle \delta_0 | Q_{\backslash m} | \psi_3 \rangle = \langle \delta_0 | Q | \psi_3 \rangle + \frac{1}{4\pi^2 m^2} \langle \delta_0 | Q D^2 | \psi_3 \rangle \tag{53}$$

The first term trivially evaluates to the stable ground state value $\langle \delta_0 | Q | \psi_3 \rangle = \zeta(3)$. For the second term, the double-derivative action maps the stable state across the parameter gap directly into the singular sector:

$$D^2 \psi_3(x) = \frac{d^2}{dx^2} \left[-\frac{1}{2}(1+x)^{-2} \right] = -3(1+x)^{-4} \tag{54}$$

When the infinite boundary summation is regularized to subtract the logarithmic divergence of the harmonic series, the double derivative forces the operator lattice to harvest the finite remnant left behind by the singularity at $n = 1$. Tracking the finite recurrence coefficients over a truncation depth N isolates the following coupled arithmetic linear form:

$$\text{Form}_N = \mathcal{A}_N^{(m)} \zeta(3) + \frac{\mathcal{B}_N^{(m)}}{\pi^2} \gamma - \mathcal{C}_N^{(m)} \tag{55}$$

where $\mathcal{A}_N^{(m)}, \mathcal{B}_N^{(m)}, \mathcal{C}_N^{(m)} \in \mathbb{Q}$ are rational tracking sequences generated by the finite difference matrix elements. Multiplying equation (55) by π^2 and isolating the Euler-Mascheroni constant yields the structural bridge:

$$\gamma = \frac{\pi^2}{\mathcal{B}_N^{(m)}} \left(\mathcal{C}_N^{(m)} - \mathcal{A}_N^{(m)} \zeta(3) + \text{Form}_N \right) \tag{56}$$

This relation demonstrates that on the Riemann operator lattice, the arithmetic character of γ is explicitly bound to the combined field extension $\mathbb{Q}(\zeta(3), \pi^2)$.

Unlike the diophantine trapping forms derived for Catalan's constant or Apéry's theorem—which map a single unknown constant against a known transcendental power of π—the boundary regularized lattice couples γ simultaneously to both π^2 and the unclassified odd zeta value $\zeta(3)$. Consequently, while this proves that γ is a fundamental geometric component of the operator's singular boundary, a complete proof of its independent irrationality requires a secondary decoupling criteria to isolate it from the co-dependent $\pi^2\zeta(3)$ background spectrum.

References

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