Power Spectral Pythagorean Numbers

Walter A. Kehowski, PhD, Rtd.
wkehowski@gmail.com

March 9, 2026

Plimpton 322 is a cuneiform tablet dated to be from the Old Babylonian Period (19th–17th century AD) and displays a method of generating Pythagorean triples [6].

Abstract

The spectral basis of $\mathbf{Z}_n$, where $n = p_1^{e_1} \cdots p_k^{e_k}$ has at least two prime factors, implements the isomorphism $\mathbf{Z}_n \cong \mathbf{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbf{Z}_{p_k^{e_k}}$. An interesting possibility is when the spectral basis consists entirely of primes or powers, and we call such a number *power spectral*. The spectral sum, the sum of all elements of the spectral basis, is of the form $an + 1$, and another interesting possibility is if $an + 1$ is also a power. A search yields only five examples, with two of them being $3^2 + 4^2 = 5^2$, where $\{9, 16\}$ is the spectral basis of 24, and $15^2 + 8^2 = 17^2$, where $\{15^2, 8^2\}$ is the spectral basis of 288. It is the purpose of this note to show that these two are the only power spectral Pythagorean triples. Some observations are made about the other three examples.

1 Statement of the problem

1.1 Pythagorean triples

Recall that a *Pythagorean triple* is a triple a, b, c of positive integers such that $a^2 + b^2 = c^2$. If x and y are positive integers of opposite parity, with $x > y$, then $a = x^2 - y^2$, $b = 2xy$, $c = x^2 + y^2$ is a primitive Pythagorean triple, and it can be shown that all primitive triples arise in this way. A few example are $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $8^2 + 15^2 = 17^2$. Here are several facts we will need about primitive Pythagorean triples, stated without proof.

1.1 Lemma (Pythagorean triples [7]).
Let a, b, c be a primitive Pythagorean triple. Then

1. *Exactly one of a, b is divisible by 3, but never c.*
2. *Exactly one of a, b is divisible by 4, but never c.*
3. *c is of the form $4k + 1$.* ♥

1.2 The spectral basis

Let us recall the spectral basis [1] of $\mathbf{Z}_n$, where n has at least two prime factors. The case when n is a prime power will not concern us, but if $n = p^e$, then any element x of $\mathbf{Z}_n$ has a unique decomposition $x = \sum_{i=0}^{e-1} a_i p^i$, where $0 \le a_i < p$, $0 \le i < e$. Suppose $n = p_1^{e_1} \cdots p_k^{e_k}$, where $k \ge 2$. Define

$$\begin{aligned} u_i &= n/p_i^{e_i}, \\ \bar{u}_i &= u_i^{-1} \mod p_i^{e_i}, \\ \pi_i &= \bar{u}_i u_i, \end{aligned}$$

then we have

$$\begin{aligned} \pi_i^2 &\equiv \pi_i \pmod{n}, \\ \pi_i\pi_j &\equiv 0 \pmod{n}, \quad i \ne j, \\ \pi_1 + \cdots + \pi_k &\equiv 1 \pmod{n}. \end{aligned}$$

Consequently, if x is any element of $\mathbf{Z}_n$, then we have the unique decomposition

$$\begin{aligned} x^r &= \bar{x}_1^r \pi_1 + \cdots + \bar{x}_k^r \pi_k, \\ \bar{x}_i^r &= x^r \mod p_i^{e_i}, \end{aligned}$$

where r is any nonnegative integer. If x is relatively prime to n, then r can also be negative. The spectral basis implements the isomorphism

$$\mathbf{Z}_n \cong \mathbf{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbf{Z}_{p_k^{e_k}}.$$

We shall often say "the spectral basis of n" rather than "the spectral basis of $\mathbf{Z}_n$."

1.3 Power spectral Pythagorean numbers

Definition 1.2 (Power spectral number, [3]). A power spectral number is a positive integer whose spectral basis consists entirely of primes or powers. ◆

1.3 Examples.
Observe that the spectral basis of $\mathbf{Z}_{12}$ is $\{3^2, 2^2\}$, the spectral basis of $\mathbf{Z}_{24}$ is $\{3^2, 4^2\}$, and the spectral basis of $\mathbf{Z}_{288}$ is $\{15^2, 8^2\}$.

If one investigates whether or not the spectral sum $\sum \pi_i$ of a power spectral number can also be a power, then one finds only the five examples listed in Table 1. Observe that the first two entries are Pythagorean triples and that the last two entries in Table 1 are an example of an *isospectral pair* [3], namely numbers n_1 and n_2 such that $n_1 = 2n_2$, with the same spectral basis.

Definition 1.4 (Power spectral Pythagorean triple). A **power spectral Pythagorean triple** is a primitive Pythagorean triple a, b, c such that $\langle a^2, b^2 \rangle$ is the spectral basis of $n = c^2 - 1$. Note that this implies that n has only two primary parts. ◆

1.5 Theorem (Fermat's Last Theorem, [9]).
Let $k > 2$ be an integer. Then the equation $x^k + y^k = z^k$ has no positive integers solutions.

Thus, only $k = 2$ is possible, the case of Pythagorean triples. What if the powers are different? The question is still unresolved, for we have the famous Beal Conjecture.

1.6 Conjecture (Beal, [11]).
Let A, B, C be pairwise coprime integers and suppose $x, y, z > 2$. Then the equation $A^x + B^y = C^z$ has no positive integer solutions.

n	Factorization	Spectral basis						
24	$(2)^3(3)$			3^2	+	4^2	=	5^2
288	$(2)^5(3)^2$			15^2	+	8^2	=	17^2
2400	$(2)^5(3)(5)^2$	15^2	+	40^2	+	24^2	=	49^2
4704	$(2)^5(3)(7)^2$	63^2	+	56^2	+	48^2	=	97^2
9408	$(2)^6(3)(7)^2$	63^2	+	56^2	+	48^2	=	97^2
Base 12								
20	$(2)^3(3)$	3^2	+	4^2			=	5^2
200	$(2)^5(3)^2$	13^2	+	8^2			=	15^2
1480	$(2)^5(3)(5)^2$	13^2	+	34^2	+	20^2	=	41^2
2880	$(2)^5(3)(7)^2$	53^2	+	48^2	+	40^2	=	81^2
5540	$(2)^6(3)(7)^2$	53^2	+	48^2	+	40^2	=	81^2

Table 1: Power spectral Pythagorean numbers.

1.7 Remark.
The motivation for the Beal conjecture is that the only solutions for $A^x + B^y = C^z$ that occur in practice always have A, B, C with a common factor. For example, $3^3 + 6^3 = 3^5$.

1.8 Remark.
The Beal Conjecture applies only to a two-term sum. Examples [15] like

$$353^4 = 30^4 + 120^4 + 272^4 + 315^4,$$
$$72^5 = 19^5 + 43^5 + 46^5 + 47^5 + 67^5,$$

show that it is an open problem whether or not there are power spectral Pythagorean numbers with four or more factors, possibly with higher powers.

1.9 Remark ([13]).
The *abc conjecture* would imply that there are at most finitely many counterexamples to Beal's conjecture.

Furthermore, we also have

1.10 Conjecture (Fermat–Catalan, [10]).
The equation $A^x + B^y = C^z$ has only finitely many solutions A, B, C in positive integers with no common prime factor and x, y, z being positive integers satisfying $1/x + 1/y + 1/z < 1$.

Consequently, in the case of two prime factors, we will restrict our attention to Pythagorean triples, and prove the following:

1.11 Theorem (Power spectral Pythagorean triples).
The only two power spectral Pythagorean triples are 3,4,5 and 8,15,17, corresponding to the spectral basis $\{3^2, 4^2\}$ of $n = 24 = 2^3 \cdot 3$, and the spectral basis $\{15^2, 8^2\}$ of $n = 288 = 2^5 \cdot 3^2$, respectively.

See Section 3 for a discussion of power spectral Pythagorean numbers with three factors.

2 Two prime factors

We discuss when n is powerful, Subsection 2.1, and when it is not, Subsection 2.2.

2.1 n is powerful

In this section we show that if n has two prime factors, is powerful, and power spectral Pythagorean, then $n = 288$. Recall that a positive integer is *powerful* if and only if all the exponents in its prime factorization are greater than one. We shall say that a positive integer is *nonpowerful* if and only if it is not powerful.

Suppose $a^2 + b^2 = c^2 = n + 1$ is the spectral basis of $n = p^\alpha q^\beta$, $\alpha, \beta > 1$. Define $u = n/p^\alpha = q^\beta$ and $v = n/q^\beta = p^\alpha$. Further, define $\bar{u} = u^{-1} \mod p^\alpha$ and $\bar{v} = v^{-1} \mod q^\beta$ so that we have $\bar{u}u + \bar{v}v = n+1$ as the spectral basis for n. The conditions are

$$\bar{u}q^\beta + \bar{v}p^\alpha = p^\alpha q^\beta + 1, \tag{1}$$

$$a^2 = \bar{u}q^\beta, \tag{2}$$

$$b^2 = \bar{v}p^\alpha, \tag{3}$$

$$c^2 = p^\alpha q^\beta + 1. \tag{4}$$

Since c is of the form $4k + 1$, we have

$$\begin{aligned}
(4k+1)^2 &= p^\alpha q^\beta + 1 \\
16k^2 + 8k + 1 &= p^\alpha q^\beta + 1 \\
16k^2 + 8k &= p^\alpha q^\beta \\
8k(2k+1) &= p^\alpha q^\beta.
\end{aligned}$$

We may now assume that $p = 2$ and that q is an odd prime. Let $k = 2^\kappa$ so that we have

$$2^{\kappa+3}(2^{\kappa+1} + 1) = 2^\alpha q^\beta.$$

Thus, $\alpha = \kappa + 3$. If $\kappa + 1$ has an odd factor, then $2^{\kappa+1} + 1$ would have more than one prime factor, so $\kappa + 1$ must be a power of 2, that is, $\kappa = 2^\lambda - 1$ and, consequently, $\alpha = 2^\lambda + 2$. Thus,

$$2^{\kappa+1} + 1 = q^\beta$$

$$q^\beta - 2^{\kappa+1} = 1. \tag{5}$$

Let us now recall *Mihailescu's theorem*.

2.12 Theorem (Mihailescu [5]).
Let x, y, a, b be positive integers greater than one. Then the only solution to $x^a - y^b = 1$ occurs for $3^2 - 2^3 = 1$.

2.13 Remark.
Catalan's conjecture [8] was conjectured by Eugene Charles Catalan in 1844 and proven by Preda Mihailescu [12] in 2002, published in 2004. Consequently, we rename the conjecture to *Mihailescu's theorem*.

The only solution to (5) is $q = 3$, $\beta = 2$, and $\kappa + 1 = 3$. Thus, $\alpha = 2 + 3 = 5$ and $n = 2^5 \cdot 3^2 = 288$.

2.14 Corollary.
The only powerful power spectral Pythagorean number with two prime factors is $n = 288 = 2^5 \cdot 3^2$, with spectral basis $\{15^2, 8^2\}$ and spectral sum $15^2 + 8^2 = 17^2$. See Table 1.

2.2 n is nonpowerful

In this section we show that if n is nonpowerful, then $n = 24$. Either $\beta \le 1$ or $\kappa + 1 \le 1$. If $\beta = 1$, then $q = 2^{\kappa+1} + 1$, and q must be a Fermat prime with $\kappa = 2^\lambda - 1$, $\alpha = 2^\lambda + 2$. If $\kappa + 1 = 1$, then $\kappa = 0$, $q^\beta = 3$, so $\beta = 1$, $q = 3$, $\alpha = 3$, $n = 24$. From now on, we assume that $\beta = 1$ and $\kappa > 0$.

Suppose $a^2 + b^2 = c^2 = n + 1$ is the spectral basis of $n = 2^\alpha q$, where $q = 2^{2^\lambda} + 1$ is a Fermat prime, $\alpha = 2^\lambda + 2$. Define $u = n/2^\alpha = q$ and $v = n/q = 2^\alpha$. Further, define $\bar{u} = q^{-1} \mod 2^\alpha$ and $\bar{v} = 2^{-\alpha} \mod q$ so that we have $\bar{u}u + \bar{v}v = n + 1$ as the spectral basis for n.

2.15 Lemma.
Suppose $q = 2^{2^\lambda} + 1$ is a Fermat prime, and let $\bar{u}$ and $\bar{v}$ be defined as above. Then

1. *If $\lambda = 0$, then $\bar{u} = 3$ and $\bar{v} = 2$.*
2. *If $\lambda > 0$, then $\bar{u} = 3q - 2$ and $\bar{v} = 2^\omega$, $\omega = 2^\lambda - 2$.* ♥

Proof: The case $\lambda = 0$ is easy to compute. Assume $\lambda > 0$. Let us show that $\bar{v} = 2^\omega$, where $\omega = 2^\lambda - 2$. Observe that

$$\begin{aligned} 2^{\alpha+\omega} &= 2^{(2^\lambda+2)+(2^\lambda-2)} \\ &= 2^{2\cdot 2^\lambda} \\ &= (2^{2^\lambda})^2 \\ &= (q-1)^2 \\ &= q^2 - 2q + 1 \quad (q > 3). \end{aligned}$$

Consequently,

$$2^{\alpha+\omega} \equiv 1 \pmod q,$$

and, therefore, $\bar{v} = 2^\omega$. Furthermore, we have

$$\begin{aligned} \bar{u}q + (q-1)^2 &= 2^\alpha q + 1 \\ \bar{u}q + q^2 - 2q + 1 &= 2^\alpha q + 1 \\ \bar{u}q + q^2 - 2q &= 2^\alpha q \\ \bar{u} + q - 2 &= 2^\alpha \\ \bar{u} &= 2^\alpha - q + 2 \\ &= 2^{2^\lambda+2} - q + 2 \\ &= 4 \cdot 2^{2^\lambda} - q + 2 \\ &= 4(q-1) - q + 2 \\ \bar{u} &= 3q - 2 \quad (q > 3). \end{aligned}$$

■

2.16 Remark.
Observe that

$$\begin{aligned} (3q-2)q + (q-1)^2 &= 2^\alpha q + 1 \\ 3q^2 - 2q + q^2 - 2q + 1 &= 2^\alpha q + 1 \\ 4q^2 - 4q &= 2^\alpha q \\ q - 1 &= 2^{\alpha-2} \\ q &= 2^{\alpha-2} + 1, \end{aligned}$$

so q must be a Fermat prime, with $\alpha = 2^\lambda + 2$, $\lambda > 0$.

2.17 Remark.
Observe that $2^\alpha q = 4(q-1)q$ so that we have in fact the polynomial identity

$$(3q-2)q + (q-1)^2 = 4(q-1)q + 1 = (2q-1)^2.$$

2.18 Theorem.
The spectral basis for $n = 2^{\alpha}q$, where $q = 2^{2^{\lambda}} + 1$ is a Fermat prime, $\alpha = 2^{\lambda} + 2$, is given by

$$(3q-2)q + (q-1)^2 = 2^{\alpha}q + 1 \quad (q > 3),$$
$$3^2 + 4^2 = 2^3 \cdot 3 + 1 \quad (q = 3).$$

2.19 Corollary.
The only nonpowerful power spectral Pythagorean number with two prime factors is $n = 24 = 2^3 \cdot 3$, with spectral basis $\{3^2, 4^2\}$ and spectral sum $3^2 + 4^2 = 5^2$.

PROOF: The spectral basis of $n = 2^{\alpha}q$, where $q = 2^{2^{\lambda}} + 1$ is a Fermat prime, $\alpha = 2^{\lambda} + 2$, $\lambda > 0$, is given by

$$(3q-2)q + 2^{2\cdot 2^{\lambda}} = 2^{\alpha}q + 1, \quad (q > 3).$$

Clearly, 3 does not divide the second term on the left, so it must divide the first term. Thus, either $3|q$ or $3|(3q-2)$, but both are impossible. Therefore, the only Pythagorean power spectral number occurs for $\lambda = 0$, that is, when $q = 3$, $\alpha = 3$, so $n = 2^3 \cdot 3 = 24$. ■

Combining Corollaries 2.14 and 2.19 we have

2.20 Theorem (Power spectral Pythagorean triples).
The only two power spectral Pythagorean triples are $3, 4, 5$ *and* $8, 15, 17$*, corresponding to the spectral basis* $\{3^2, 4^2\}$ *of* $n = 24 = 2^3 \cdot 3$*, and the spectral basis* $\{15^2, 8^2\}$ *of* $n = 288 = 2^5 \cdot 3^2$*, respectively.*

3 Three prime factors

Let d be a nonsquare positive integer, $d > 1$, and consider the Diophantine equation $x^2 - dy^2 = 1$, called *Pell's equation*. Joseph Louis Lagrange proved that, as long as d is not a perfect square, Pell's equation has infinitely many distinct integer solutions [14]. Recursive solutions to Pell's equation for specific d can be obtained from Dario Alpern's web page [17].

The following two theorems are from [3].

3.21 Theorem (Mersenne).
Let M_p be a Mersenne prime with Mersenne exponent $p > 2$. Then

1. $2^{2p-1} \cdot 3 \cdot M_p^2$ *has power spectral basis*

   $$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}$$

   with index 2.
2. $2^{2p} \cdot 3 \cdot M_p^2$ *has power spectral basis*

   $$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

3. $2^{2p+1} \cdot 3 \cdot M_p^2$ *has power spectral basis*

   $$\left\{M_p^2(M_p+2)^2, 4M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$

Note that numbers 1 and 2 comprise an isospectral pair, that is, numbers n_1 and n_2 such that $n_1 = 2n_2$ that have the same spectral basis.

3.22 Remark.
The fourth and fifth entries in Table 1 are $M_3 = 7$ in entries 1 and 2 of Theorem 3.21, respectively.

3.23 Theorem (Fermat).
Let F_i be a Fermat prime with exponent $f_i = 2^i$, $i > 0$. Then

1. $2^{2f_i - 1} \cdot 3 \cdot F_i^2$ *has power spectral basis*
$$\left\{(F_i - 2)^2 F_i^2, (F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}$$
with index 2.
2. $2^{2f_i} \cdot 3 \cdot F_i^2$ *has power spectral basis*
$$\left\{(F_i - 2)^2 F_i^2, (F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$
3. $2^{2f_i + 1} \cdot 3 \cdot F_i^2$ *has power spectral basis*
$$\left\{(F_i - 2)^2 F_i^2, 4(F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$

Note that numbers 1 and 2 comprise an isospectral pair, that is, numbers n_1 and n_2 such that $n_1 = 2n_2$ that have the same spectral basis.

3.24 Remark.
The third entry in Table 1 is $F_1 = 5$ in entry 3 of Theorem 3.23.

3.1 Mersenne

The power spectral numbers $2^{2p-1} \cdot 3 \cdot M_p^2$ and $2^{2p} \cdot 3 \cdot M_p^2$, where M_p is a Mersenne prime, have the spectral basis [3]
$$\left\{M_p^2 (M_p + 2)^2, M_p^2 (M_p + 1)^2, (M_p^2 - 1)^2\right\}.$$
If we let $q = M_p$, then the spectral sum is equivalent to the polynomial identity
$$q^2(q + 2)^2 + q^2(q + 1)^2 + (q^2 - 1)^2 = 3q^2(q + 1)^2 + 1.$$
If we let $y = q(q+1)$ and $3y^2 + 1 = x^2$, then we have the Pell equation $x^2 - 3y^2 = 1$. A pair of recursive solutions is given by [17]:
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$
Using the first recursive solution, we obtain
$$[[1,0,1],[2,1,2],[3,4,7],[4,15,26],[5,56,97],[6,209,362]].$$
It just so happens that $y = 56 = 7 \cdot 8$, $x = 97$ is a solution to the Pell equation $x^2 - 3y^2 = 1$, and $q = 7 = 2^3 - 1$ is a Mersenne prime.

3.2 Fermat

The power spectral number $2^{2f_i + 1} \cdot 3 \cdot F_i^2$, where F_i is a Fermat prime, $f_i = 2^i$, $i > 0$, has power spectral basis [3]
$$\left\{(F_i - 2)^2 F_i^2, 4(F_i - 1)^2 F_i^2, (F_i^2 - 1)^2\right\}.$$
If we let $q = F_i$, then the spectral sum is equivalent to the polynomial identity
$$q^2(q - 2)^2 + 4q^2(q - 1)^2 + (q^2 - 1)^2 = 6q^2(q - 1)^2 + 1.$$

If we let $y = q(q-1)$ and $6y^2+1 = x^2$, then we have the Pell equation $x^2-6y^2 = 1$. A pair of recursive solutions is given by [17]:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 5 & -12 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Using the first recursive solution, we obtain

$$[[1,0,1],[2,2,5],[3,20,49],[4,198,485],[5,1960,4801],[6,19402,47525]].$$

It just so happens that $y = 20 = 5 \cdot 4$, $x = 49$ is a solution to the Pell equation $x^2 - 6y^2 = 1$, and $q = 5 = 2^2 + 1$ is a Fermat prime.

Only G∗d knows what's going on here!

Let's summarize our results as a theorem.

3.25 Theorem (Summary of results).

1. *Suppose a, b, c is a primitive Pythagorean triple such that $\{a^2, b^2\}$ is the spectral basis of $n = c^2-1$. Then $n = 24$ or $n = 288$, with spectral bases $\{3^2, 4^2\}$ and $\{15^2, 8^2\}$, respectively. Observe that $24 = 20|_{12}$ and $288 = 200|_{12}$, and that the spectral bases are $\{9, 14\}|_{12}$ and $\{13^2, 8^2\}|_{12}$, respectively.*

2. *The power spectral numbers $2^{2p-1} \cdot 3 \cdot M_p^2$ and $2^{2p} \cdot 3 \cdot M_p^2$, where M_p is a Mersenne prime, $p > 2$, have the spectral basis*
$$\left\{M_p^2(M_p+2)^2, M_p^2(M_p+1)^2, (M_p^2-1)^2\right\}.$$
If we let $q = M_p$, then the spectral sum is equivalent to the polynomial identity
$$q^2(q+2)^2 + q^2(q+1)^2 + (q^2-1)^2 = 3q^2(q+1)^2 + 1.$$
If x and y solutions of the Pell equation $x^2 - 3y^2 = 1$, where $y = q(q+1)$, then the spectral sum is x^2. Observe that $y = 7 \cdot 8$, $x = 97$ is a solution, so we have

n	*Factorization*	*Spectral basis*						
4704	$(2)^5(3)(7)^2$	63^2	+	56^2	+	48^2	=	97^2
9408	$(2)^6(3)(7)^2$	63^2	+	56^2	+	48^2	=	97^2
Base 12								
2880	$(2)^5(3)(7)^2$	53^2	+	48^2	+	40^2	=	81^2
5540	$(2)^6(3)(7)^2$	53^2	+	48^2	+	40^2	=	81^2

Furthermore, 4704 and 9408 comprise an isospectral pair.

3. *The power spectral number $2^{2f_i+1} \cdot 3 \cdot F_i^2$, where F_i is a Fermat prime, $f_i = 2^i$, $i > 0$, has power spectral basis*
$$\left\{(F_i-2)^2F_i^2, 4(F_i-1)^2F_i^2, (F_i^2-1)^2\right\}.$$
If we let $q = F_i$, then the spectral sum is equivalent to the polynomial identity
$$q^2(q-2)^2 + 4q^2(q-1)^2 + (q^2-1)^2 = 6q^2(q-1)^2 + 1.$$
If x and y solutions of the Pell equation $x^2 - 6y^2 = 1$, where $y = q(q-1)$, then the spectral sum is x^2. Observe that $y = 5 \cdot 4$, $x = 49$ is a solution, so we have

n	*Factorization*	*Spectral basis*						
2400	$(2)^5(3)(5)^2$	15^2	+	40^2	+	24^2	=	49^2
Base 12								
1480	$(2)^5(3)(7)^2$	13^2	+	34^2	+	20^2	=	41^2

Equation	Solutions
$x^2 - 3y^2 = \ 1$	Theorem 3.25, Mersenne: The solution is $y = 7 \cdot 8$, $x = 97$.
$x^2 - 6y^2 = \ 1$	Theorem 3.25, Fermat: The solution is $y = 5 \cdot 4$, $x = 49$.
$x^2 - 9y^2 = \ 1$	Theorem 4.27, no nontrivial solutions.
$q^2 - 2p^4 = -1$	Theorem 4.26, [16]: $p = 13$, $q = 239$ are the only solutions, and happen to be prime.
$q^2 - 2p^2 = \ 1$	Theorem 4.26, possibly infinitely many prime solutions,
$=$	although I could find only 4.
$q - 2p^s = \ \epsilon$	Theorem 4.26, possibly infinitely many prime solutions.
$q^t - 2p = \ \epsilon$	Theorem 4.26, possibly infinitely many prime solutions.

Table 2: The upper part shows the solutions to appropriate Pell equations. The lower part shows solutions [4] to $q^t - 2p^s = \epsilon$, $\epsilon = \pm 1$, where $s, t \geq 1$ are integers, and p and q are to be prime solutions.

4 An impossiblity result

We have the following theorem:

4.26 Theorem (Crescenzo [4]).
The Diophantine equation $q^t = 2p^s + \epsilon$, with $\epsilon = \pm 1$ and $s, t > 1$, has prime solutions only if either $s = t = 2$, $\epsilon = 1$, a Pell equation, or $s = 4$, $t = 2$, $\epsilon = -1$, Ljunggren's equation, with $p = 13$, $q = 239$.

4.27 Theorem ([3]).
Suppose $q^t = 2p^s + \epsilon$, $\epsilon = \pm 1$, has prime solutions, $p, q \neq 3$, for some positive integers s and t. Then $n = 9p^{2s}q^{2t}$ is power-spectral with power spectral basis

$$\left\{p^{2s}q^{2t}, \quad (4p^{2s} - 1)^2, \quad 16(p^s + \epsilon)^2 p^{2s}\right\}.$$

Furthermore, there are no power spectral Pythagorean numbers of this form.

PROOF: Consult [3] for the power spectral basis. The spectral sum has the form

$$p^{2s}q^{2t} + (4p^{2s} - 1)^2 + 16(p^s + \epsilon)^2 p^{2s} = 9p^{2s}q^{2t} + 1$$
$$p^{2s}(2p^s + \epsilon)^2 + (4p^{2s} - 1)^2 + 16(p^s + \epsilon)^2 p^{2s} = 9p^{2s}(2p^s + \epsilon)^2 + 1.$$

If we let $y = p^s(2p^s + \epsilon)$, and if x could be a solution to the Pell equation $x^2 - 9y^2 = 1$, then $9p^{2s}q^{2t}$ would be power spectral with spectral sum x^2. However, since 9 is a perfect square, there are only trivial solutions, and so there are no power spectral Pythagorean numbers of this form. ■

See Table 2 for a summary of the Diophantine equations and their solutions that occur in this paper.

4.1 Another Pell equation

Since the previous section had applications of the Pell equations $x^2 - 3y^2 = 1$ and $x^2 - 6y^2 = 1$, let us give honorable mention to the following result.

4.28 Theorem.
Let x and y be positive solutions to the Pell equation $x^2 - 2y^2 = 1$. Then

$$\begin{aligned} a &= \frac{x-1}{2}, \\ b &= a+1 = \frac{x+1}{2}, \\ c &= y, \\ d &= ab = \frac{x^2-1}{4}, \\ e &= d+1 = \frac{x^2+3}{4}, \end{aligned}$$

have the property

$$\begin{aligned} a^2 + b^2 &= c^2, \\ c^2 + d^2 &= e^2. \end{aligned}$$

PROOF: Left to the reader. ∎

4.29 Corollary (Rational Frenet frame).
The Frenet frame of $\mathbf{r}(t) = \langle e^{at}\cos bt, e^{at}\sin bt, be^{at}\rangle$, an instructive assignment in Calc III, is computable without radicals.

5 Sequences

The descriptions of these sequences have been edited to suit the needs of this document.

A000043: Mersenne exponents: primes p such that $2^p - 1$ is prime. Then $2^p - 1$ is called a Mersenne prime.

A000215: Fermat numbers: $a(n) = 2^{2^n} + 1$.

A000225: Mersenne numbers: $a(n) = 2^n - 1$.

A001075: Solutions x to the Pell equation $x^2 - 3y^2 = 1$.

A001078: Solutions y to the Pell equation $x^2 - 6y^2 = 1$. Numbers n such that $6n^2 + 1$ is a square.

A001079: Solutions x to the Pell equation $x^2 - 6y^2 = 1$.

A001348: Mersenne primes: $2^p - 1$, where p is prime.

A001353: Solutions y to the Pell equation $x^2 - 3y^2 = 1$. Numbers n such that $3n^2 + 1$ is a square.

A019434: Fermat primes: primes of the form $2^{2^k} + 1$, for some $k \geq 0$.

A103606: Primitive Pythagorean triples in nondecreasing order of perimeter, with each triple in increasing order, and if perimeters coincide then increasing order of the even members.

References

[1] Sobczyk, G. (2001). *The Missing Spectral Basis in Algebra and Number Theory.* The American Mathematical Monthly, 108(4), 336–346. https://doi.org/10.1080/00029890.2001.11919757. Downloadable: https://www.researchgate.net/publication/242251192_The_Missing_Spectral_Basis_in_Algebra_and_Number_Theory

[2] Schoof, R., *Catalan's Conjecture*, https://dn710103.ca.archive.org/0/items/springers-collection-of-books/Catalan%27s%20Conjecture%20%28%20PDFDrive%20%29.pdf

[3] Kehowski, W., *Power-spectral Numbers,* https://vixra.org/abs/2409.0031.

[4] P. Crescenzo, *A Diophantine Equation Which Arises in the Theory of Groups*, Advances in Mathematics **17** (1975), 25-29.

[5] Mihăilescu, Preda (2004). "Primary Cyclotomic Units and a Proof of Catalan's Conjecture". Journal für die reine und angewandte Mathematik. 2004 (572): 167–195. Reference pulled from [12].

[6] Columbia University Plimpton 322, *Before Pythagoras: The Culture of Old Babylonian Mathematics*, https://isaw.nyu.edu/exhibitions/before-pythagoras/items/plimpton-322/.

[7] *Pythagorean triple*, Wikipedia, https://en.wikipedia.org/wiki/Pythagorean_triple.

[8] *Catalan's conjecture*, Wikipedia, https://en.wikipedia.org/wiki/Catalan's_conjecture.

[9] *Fermat's Last Theorem*, Wikipedia, https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem.

[10] *The Fermat–Catalan conjecture*, Wikipedia, https://en.wikipedia.org/wiki/Fermat%E2%80%93Catalan_conjecture.

[11] *The Beal conjecture*, Wikipedia, https://en.wikipedia.org/wiki/Beal_conjecture.

[12] *Mihailescu, Preda,* Wikipedia, https://en.wikipedia.org/wiki/Preda_Mih%C4%83ilescu.

[13] *The abc conjecture,* Wikipedia, https://en.wikipedia.org/wiki/Abc_conjecture.

[14] *Pell Equation,* Wikipedia, https://en.wikipedia.org/wiki/Pell%27s_equation.

[15] Euler's sum of powers conjecture, https://en.wikipedia.org/wiki/Euler's_sum_of_powers_conjecture.

[16] Wilhelm Ljunggren, https://en.wikipedia.org/wiki/Wilhelm_Ljunggren.

[17] Alpern, D., Generic two integer variable equation solver, https://www.alpertron.com.ar/QUAD.HTM.