Pure Numbers and Number Line

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Abstract:
In mathematics, real numbers can be represented by points on a straight line called the number line, which includes a point called the origin, the direction of number growth, and a unit length. It is generally assumed that there is a one-to-one correspondence between real numbers and points on the number line, with the position of a point determining the size and order of the numbers. This essentially assumes that all real numbers have a definite position on the number line, and that there is a definite order between any two real numbers.

This paper shows that there are real numbers with uncertain positions, and that all real numbers do not lie on the number line of the same dimension. The number line is composed of discrete points, which are "pure numbers"—that is, only pure numbers exist on the number line, while non-pure numbers exist in "empty space." Therefore, there is a logical contradiction between the continuity of real numbers and the real number line; the real number line is an incomplete and imperfect conception for representing real numbers. This paper gives the definition of a pure number and the relationship between its cardinality and the natural cardinality.

These results verify the viewpoint of quantum theory in physics, namely that the straight line on the "macroscopic" number line is composed of "microscopic" discrete and discontinuous points.

Keywords:
Continuum hypothesis, Discrete, Number line, Pure number, Quantum theory

Quantum theory holds that the microscopic world is discrete and discontinuous. In mathematics, a similar problem is the continuity of real numbers, the real number line, and straight lines. These problems involve the most core and fundamental concepts in mathematics, which are equivalent to the continuum hypothesis, the first of David Hilbert's 23 problems [1]. In 1940, Kurt Gödel proved the compatibility of the continuum hypothesis with the ZFC axiomatic set theory system, i.e., it cannot be falsified [2]. In 1963, Paul Cohen proved the independence of the continuum hypothesis from the ZFC axiomatic set theory system, i.e., it cannot be proved [3]. Therefore, the continuum hypothesis is logically independent of the ZFC axiomatic set theory system, thus concluding that the continuum hypothesis is undecidable under the existing mathematical system, and that the mathematical foundation has limitations.

In mathematics, real numbers can be represented by points on a straight line called the number line, which contains a point called the origin, the direction of number growth, and the unit length [4]. The position of a point on the number line determines the size and order of the numbers. This actually assumes that all real numbers have a definite position on the number line, and that there is a definite order of size between any two real numbers.

What kinds of numbers exist on the number line is a very important and meaningful question. The history of research on this question is long, with different theories emerging in different periods [5]. For example, the real number theory, which was gradually perfected in the 19th and 20th centuries, holds that the number line is a graphical representation of real numbers, that real numbers do not include infinitesimals and infinity, that there is a one-to-one correspondence between real numbers and points on the number line, and that the number line and real numbers are continuous [6].

This paper shows that there are real numbers with uncertain positions, that all real numbers are not on a number line "straight" of the same dimension; that the number line is composed of discrete points, which are "pure numbers"; and that the real number line is an incomplete and imperfect concept for representing real numbers.

1. The problem of discontinuity of both real numbers and the real number line

In 1960, American mathematical logician A. Robinson established the theory of hyperreal numbers, which includes real numbers and infinity and infinitesimals[7]. Because infinitesimals are not on the real number line, numbers composed of real numbers and infinitesimals, such as 3+ε, where ε is an infinitesimal, are also not on the real number line.

Recent studies have discovered the existence of quantum numbers or inaccurate numbers, which can be seen as extensions of real numbers, and their properties are similar to those of real numbers[8][9]. Their existence indicates that even within the framework of the real number system, such numbers still cannot be one-to-one corresponded to points on the number line.

1.1 Numerical Examples

Assume: The real number $\sqrt{231}$ lies on the number line and corresponds to a unique point on the number line. Based on real number theory, calculate m, p, and v using the following assignments:

$$m = \sqrt[3]{-\frac{20971520}{81} + \frac{4194304}{243}\sqrt{231}} - \sqrt[3]{\frac{20971520}{81} + \frac{4194304}{243}\sqrt{231}} - \frac{448}{9}$$

$$p = \sqrt{4855431168 + 32514048m + (93585408 + 626688m)\sqrt{2688 + 18m} - 217728m^2 - 1458m^3}$$

$$v = \frac{1}{18(448 + 3m)}((1344 + 9m)\sqrt{2688 + 18m} - p) + \frac{40}{3}$$

Then u and t, as defined by it, are both on the number line and correspond to a unique point on the number line.

$$u = \sqrt[18]{98304 + \sqrt{9663676416 - v^9}}$$

$$t = \sqrt[18]{98304 - \sqrt{9663676416 - v^9}}$$

Now, construct the two legs of a right triangle using the values of u and t. The value of the hypotenuse should be definite and accurate.

According to the Pythagorean theorem, the length w of the hypotenuse of the triangle is

$$w = \sqrt{\sqrt[9]{98304 + \sqrt{9663676416 - v^9}} + \sqrt[9]{98304 - \sqrt{9663676416 - v^9}}}$$

The number w should be on the axis and correspond to a unique point on the number line.

Numerical calculations show that the number w has no definite and accurate value because it contains multiple layers of square root operators. Whether the numerical calculation result is greater than or less than the value 2 depends entirely on the choice of calculation precision, as detailed in Table 1.

Table 1. Numerical calculation results for number w with different floating-point bit depths.

Floating-point bit depths	Calculated values
20	1.9999999998382627442
25	2.000000000000000600586153
30	1.99999999999999999998946014833
32	2.0000000000000000000000635914900
35	1.9999999999999999999999999708310861

1.2 Results and Analysis

The above results show that the number w cannot correspond to a single point on the number line, contradicting the assumption.

This result conversely proves that the real number $\sqrt{231}$ does not lie on the number line, meaning that within the framework of the real number system, real numbers and the real number line are discontinuous.

2. Types of numbers on the number line

The real number system divides real numbers into rational and irrational numbers. The hyperreal number system adds infinitesimals and infinity to the real number system. The newly established super-number system includes real numbers, infinity, infinitesimals, and quantum numbers or inaccurate numbers [10].

The numerical calculation examples in this paper show that real numbers and the real number line are discontinuous; real numbers are not all on the number line. Therefore, what kinds of numbers constitute the number line is a very important and interesting question worthy of in-depth exploration.

Through meticulous logical reasoning, it is proven that only "pure numbers" exist on the number line, defined as follows:

Pure numbers: numbers with a finite number of digits consisting only of digits and a decimal point, without any operational symbols.

The reasons why the real number line does not actually exist and only pure numbers exist on the number line are as follows:

The numerical calculation examples in this article demonstrate that irrational numbers formed by square roots do not exist on the number line; real numbers include non-pure numbers, such as infinite repeating decimals and irrational numbers such as square roots, logarithms, trigonometric functions, and transcendental numbers; the number line can be considered a virtual one-dimensional static object, while non-pure numbers contain operation symbols, have an infinite number of digits, and do not have fixed values. Non-pure numbers are always in operation, similar to a moving object that does not have static position data and cannot be correlated with points on the number line.

3. Conclusions and Discussions

In summary, this study demonstrates that there exist real numbers with uncertain positions, and that all real numbers do not lie on a single-dimensional number line. The number line is composed of discrete points, which are "pure numbers," meaning that only pure numbers exist on the number line, while non-pure numbers exist in "emptiness." Therefore, there is a logical contradiction between the continuity of real numbers and the real number line; the real number line is an incomplete and imperfect concept representing real numbers.

Pure numbers include natural numbers, zero, negative integers, and decimals formed by dividing nine-tenth of all integer by $10^{\aleph_0}$. Within each precision of the integer unit on the number line, there are only nine decimals, such as 0.1, 0.2, 0.3, ..., 0.9. Therefore, each nine-tenth of natural numbers corresponds to an infinite number of decimals, and the decimals corresponding to one-tenth of a natural number are repeated, plus a zero. Therefore, the quantitative relationship between the cardinality $\aleph_p$ of pure numbers and the cardinality $\aleph_0$ of natural numbers is $\aleph_p = 9/5*\aleph_0^2 + 1/5*\aleph_0 + 1$, and the interval between pure numbers on the number line is $10/(9\aleph_0 + 1)$.

These results verify the viewpoint of quantum theory in physics, namely that a straight line on a "macroscopic" number line is composed of "microscopic" discrete and discontinuous points.

In practical applications, selecting a certain precision in calculation is equivalent to performing a measurement. At this point, all non-pure numbers are transformed into pure numbers and lie on the number line.

From this, it can be deduced that the intersection of two straight lines in a plane is most likely just an illusion caused by limitations in measurement precision.

4 References

[1] Hilbert's problems. Wikipedia.
https://en.wikipedia.org/wiki/Hilbert%27s_problems
[2] Zermelo-Fraenkel Set Theory. Wikipedia.
https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory
[3] Paul Cohen. Wikipedia. https://en.wikipedia.org/wiki/Paul_Cohen
[4] James B.Stewart, Lothar Redlin amd Saleem Watson. College Algebra (5th ed.). Cengage Learning, 2009
[5] Real number. Wikipedia.
https://en.wikipedia.org/w/index.php?title=Real_number&oldid=1333181925
[6] Peter Koellner. The Continuum Hypothesis. In Zalta, Edward N. (ed.). The Stanford Encyclopedia of Philosophy, Stanford University, 2013
[7] Hyperreal number. Wikipedia. https://en.wikipedia.org/wiki/Hyperreal_number
[8] Zhi Li and Hua Li. Uncertainty in Multi-layer Radical Calculations and Inaccurate Numbers. https://vixra.org/abs/2201.0055
[9] Zhi Li and Hua Li. Quantum Numbers, Quantum Superposition Numbers and Non-rational Numbers and Their Applications.
https://vixra.org/abs/2504.0084
[10] Zhi Li and Hua Li. Supernumber Theory and Quantum Superposition Phenomena.
https://vixra.org/abs/2512.0065, 2025