

Guessing that the Riemann Hypothesis's proof is very difficult using Non-Standard Analysis

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 164 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is $1/2$. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I conjecture about the independence (difficulty of proof) of propositions equivalent to the Riemann hypothesis. My position is to discuss the difficulty of proof purely as an intuitive argument.

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First, note that nonstandard analysis is a conservative extension of standard analysis, and anything that holds in standard analysis also holds in nonstandard analysis. Here, we will limit our discussion to the difficulty of proving this intuitive argument. Incidentally, this argument has been recognized as correct by chatgpt, copilot, grok, and gemini.

I think in Non-standard analysis and it is delicate at infinity, I think only finite area.

The Riemann hypothesis states that the real part of a nontrivial zero of the ζ function is $1/2$. (Ivić[2]p44)

Proposition 1. *The only nontrivial zero of the Riemann zeta function is on $Re(s) = 1/2$.*

If $\zeta(s) = 0, Re(s) = 1/2$, then for infinitesimal ϵ_0 , we can set $\zeta(s + \epsilon_0) = 0, Re(s + \epsilon_0) \neq 1/2$. In nonstandard analysis, this is the definition of $s + \epsilon_0$, where the Riemann zeta function is both regular and continuous. This condition alone is sufficient. Intuitively, the difficulty of proving the Riemann

hypothesis can be demonstrated if we consider infinitesimal objects and treat them as if they were infinitesimal. While it can be said that the proof is difficult, this difficulty of proof includes a scenario that is hard to imagine. It is difficult to prove, but disproven. This means that while it is possible to find counterexamples to zero points from a small number of numbers, proving the Riemann hypothesis in this direction would be impossible.

Just to summarize, the Riemann hypothesis is either true or false, but intuitively, it can be said that a model in which the Riemann hypothesis is false can be created within a broader axiom system. In other words, While it is difficult to prove that the Riemann hypothesis is true, it is still possible to prove that it is false.

2 On the Equivalence Proposition

Möbius Function Version

Theorem 1. (*Ivić[2]p48, Titchmarsh[5]p370, Theorem 14.25*)

$$\text{the Riemann Hypothesis} \Leftrightarrow \sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$$

The following is a known short proof:
Define $M(x)$ as follows:

$$M(x) := \sum_{n=1}^{[x]} \mu(n)$$

Then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \int_{x=0.1}^{\infty} x^{-s} d(M(x))$$

$d(M(x))$ is the Stieltjes integral of $M(x)$.

$$= [M(x)x^{-s}]_{0.1}^{\infty} + s \int_{x=0.1}^{\infty} M(x)x^{-s-1} dx = s \int_{x=0.1}^{\infty} M(x)x^{-s-1} dx$$

Here, we analytically continue $\frac{1}{\zeta(s)}$ until this value is finite. When $\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$ holds, this integral is finite for $Re(s) > 1/2$. We obtain that there are no zeros of the zeta function for $Re(s) > 1/2$. Furthermore, by combining this with a discussion of functional inequalities, we obtain the Riemann

hypothesis.

Conversely, if we assume the Riemann hypothesis,

$$s \int_{x=0.1}^{\infty} M(x)x^{-s-1}dx$$

$Re(s) > 1/2 + \epsilon > 1/2$, and is not infinite.

$$|M(x)| < Km^{\frac{1}{2}+\epsilon}$$

We obtain. ϵ can be arbitrarily small.

Let s be a nontrivial zero with real part $1/2$, and let the point $s + \epsilon_0$ to the right of the infinitesimal ϵ_0 be a zero of the Riemann zeta function. Even if there is no clear precedent within a finite range, taking into account the above proof, $\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon})$ and $\sum_{n=1}^m \mu(n) = O(m^{\frac{1}{2}+\epsilon_0+\epsilon})$ hold simultaneously and consistently. These two equations are consistent with the equations mentioned above. As we will explain later, this is an "intuitive" manifestation of the difficulty of proving the Riemann hypothesis.

Another one

$$\text{If the Riemann Hypothesis is true} \Leftrightarrow \sum_{\rho} \frac{Re(\rho)}{|\rho|^2} = \sum_{\rho} \frac{\frac{1}{2}}{|\rho|^2}$$

Assuming the Riemann Hypothesis is true, Since ρ and $1 - \rho$ correspond one-to-one,

$$\frac{x}{x^2 + y^2} + \frac{(1-x)}{(1-x)^2 + y^2} = \frac{\frac{1}{2}}{x^2 + y^2} + \frac{\frac{1}{2}}{(1-x)^2 + y^2}$$

This is obtained from If the Riemann hypothesis is false, then $x < 1 - x$, and

$$(1/2 - x)\left(\frac{1}{x^2 + y^2} - \frac{1}{(1-x)^2 + y^2}\right) > 0$$

From this,

$$\frac{x}{x^2 + y^2} + \frac{(1-x)}{(1-x)^2 + y^2} < \frac{\frac{1}{2}}{x^2 + y^2} + \frac{\frac{1}{2}}{(1-x)^2 + y^2}$$

This leads to the question.

Let $\rho_0 = 1/2 + it_0$ be (one of) the zeros of the Riemann zeta function, and take the infinitesimal right-hand side of it, $\rho_0 + \epsilon_0$. Formally,

$$\sum_{\rho \neq \rho_0} \frac{Re(\rho)}{|\rho|^2} + \frac{Re(\rho_0 + \epsilon_0)}{|\rho_0 + \epsilon_0|} = \sum_{\rho \neq \rho_0} \frac{\frac{1}{2}}{|\rho|^2} + \frac{\frac{1}{2}}{|\rho_0 + \epsilon_0|}$$

The counterexample "by intuitive argument" in nonstandard analysis holds true, and in this sense, the difficulty of proof still appears. Furthermore, counterexamples that approach the line with a real part of $1/2$ "by intuitive argument" also present a barrier to proof, but as this example shows, the difficulty is hidden by the equations and is not visible.

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References

- [1] "An Introduction to Nonstandard Real Analysis", A. E. Hard, P. A. Loeb. Pure and Applied Mathematics, A Series of Monographs and Textbooks, Columbia University, New York.
- [2] "The Riemann Hypothesis", Ivić. A, Dover Press
- [3] "Set Theory An Introduction To Independence Proofs", Kunen. K. Studies in Logic and the Foundations of Mathematics, Volume 102.
- [4] "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", Riemann Bernhard, Monatsberichte der Berliner Akademie. In Gesammelte Werke, Teubner, Leipzig (1892)
- [5] "The Theory of the Riemann Zeta function", Titchmarsh, E, C., Clarendon Press Oxford (1996)