Merry-go-round
and
time-dependent symplectic forms

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Abstract

In the merry-go-round fictitious forces are acting like centrifugal force and Coriolis force. Like the Lorentz force Coriolis force is velocity dependent and, following Arnol′d [Arn61], can be modeled by twisting the symplectic form. If the merry-go-round is accelerated an additional fictitious force shows up, the Euler force.

In this article we explain how one deals symplectically with the Euler force by considering time-dependent symplectic forms. It will turn out that to treat the Euler force one also needs time-dependent primitives of the time-dependent symplectic forms.

Time-dependent symplectic forms are motivated by the elliptic restricted three-body-problem and its relation to periodic orbits around Mars.

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1 Introduction

1.1 Motivation and general perspective

On a merry-go-round fictitious forces appear. One of these fictitious forces is the centrifugal force. The centrifugal force only depends on position and is a conservative force, i.e. it is the gradient of a potential. If one moves on a merry-go-round an additional fictitious force is the Coriolis force. In contrast to the centrifugal force, the Coriolis force depends linearly on the velocity. There is a third fictitious force, the Euler force. The Euler force only appears if the merry-go-round is accelerated or decelerated. As the centrifugal force the Euler force only depends on position, but different from the centrifugal force the Euler force is not a conservative force, i.e. it is not a gradient vector field.

Coriolis force – Twisted symplectic form. It was observed by Arnol′d [Arn61] that velocity dependent forces, as the Lorentz force of a magnetic field or the Coriolis force, can be modeled symplectically by twisting the standard symplectic form on the cotangent bundle. In this article we address the question how to model the Euler force symplectically. In an accelerated merry-go-round the Coriolis force itself is time-dependent and therefore modeling it as Arnol′d did, the symplectic form on the cotangent bundle gets time-dependent.

Euler force – Time-dependent primitive. We explain in this paper that the Euler force can be modeled with the help of a time-dependent primitive of the time-dependent symplectic form. In particular, in contrast to the Coriolis force the choice of the time-dependent primitive matters for the Euler force. This is vaguely reminiscent of the Aharonov-Bohm effect which not only depends on the magnetic field, but also on the choice of the vector potential. On the other hand, we do not see a direct connection between the Aharonov-Bohm effect and the Euler force. The Aharonov-Bohm effect is a quantum mechanical phenomenon, while what we discuss here is a classical phenomenon where time-dependence of the magnetic field is crucial.

Gateway around Mars – Time-periodic symplectic form. The study of accelerated merry-go-rounds is motivated by applying symplectic methods to find a gateway around Mars. This general perspective is explained in our current work [FW26a, FW26b]. In fact, the orbit of Mars is rather eccentric. Therefore

the dynamics around Mars has to be modeled by the elliptic restricted three-body-problem and not just the circular one. In the elliptic restricted three-body-problem the rotational velocity is not constant, like in accelerating and decelerating merry-go-rounds. However, time dependence of acceleration and deceleration is periodic with period 1 Martian year.
With this motivation in mind we are interested in detecting periodic orbits in periodically accelerating and decelerating merry-go-rounds. For that purpose we are looking at symplectic forms depending periodically on time, together with a choice of time-dependent primitives. The time dependence of the primitive itself does not need to be periodic, but twisted-periodic in a sense specified in this paper in Definition 5.1.

Collisions – Delay equation. Due to collisions, the Hamiltonians in the restricted three-body-problems are singular. However, two-body collisions can be regularized. If the Hamiltonian is time-dependent, as in the elliptic restricted three-body-problem, then there is no preserved energy so that instead of blowing up the energy hypersurface one has to blow up the loop space, as first discovered by Barutello, Ortega, and Verzini [BOV21]. After regularizing using a blow-up of the loop space, the symplectic form as well as the equations become non-local. This more general perspective, how to deal with non-local symplectic forms, is the topic of our current work [FW26a, FW26b].

1.2 Outline and main results

In Section 2 we treat a merry-go-round from the Hamiltonian point of view and derive its Hamilton equation in phase space and the force equation in configuration space. In the force equation we will discover the three fictitious forces, namely the centrifugal force, the Coriolis force, and the Euler force, as illustrated by Figure 1. This force equation fits into a more general class of equations which we refer to as the $(\boldsymbol{A}, \phi)$-equation. Here $\boldsymbol{A}$ is a time-dependent vector potential of a time-dependent magnetic field while ϕ is a time-dependent scalar potential.

In Section 3 we are considering periodic solutions of the $(\boldsymbol{A}, \phi)$-equation. In order to find periodic solutions to the $(\boldsymbol{A}, \phi)$-equation one needs some periodicity assumptions on our data. We do not need that the vector potential $\boldsymbol{A}$ itself is periodic in time, but only its differentials with respect to time as well as with respect to space. It is possible to trade-off the potential ϕ with a twist in the periodicity by defining a new vector potential $\boldsymbol{A}^{\phi}$ which is not periodic in time, but twisted-periodic.
We are then discussing how periodic solutions can be detected variationally with the help of an action functional. Since we do not assume that the vector potential itself is periodic, an additional term arises in the action functional which we refer to as the *twist term*. The twist term vanishes in the periodic case.

While in Section 3 we discuss a Lagrangian approach to periodic solutions of the $(\boldsymbol{A}, \phi)$-equation, we study in Section 4 a Hamiltonian approach. This Hamiltonian approach fits onto a more general class of functionals which we discuss in more detail in Section 5.

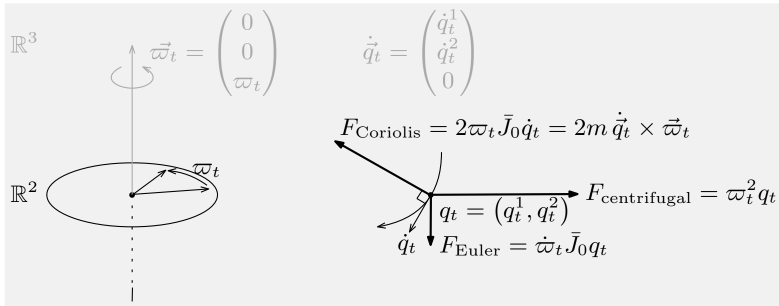

Figure 1: The three fictitious forces in the merry-go-round

In Section 5 we are considering a time-dependent family of primitives of time-dependent symplectic forms. Again this family does not need to depend periodically on time, but twisted-periodically whose precise meaning is explained in (5.10). For such a twisted-periodic family of primitives we show how to associate a well defined action functional. We derive the critical point equation of this functional in two ways. In the proof of Theorem 5.3 we use Cartan's formula on the product of the underlying manifold with the circle. In Section 5.5 we give a different derivation of the critical point equation using Cartan's formula on the loop space. Since the loop space is infinite dimensional, strictly speaking there is no mathematical theory yet to use Cartan's formula there.
It turns out that a new vector field Y is entering the critical point equation which, similarly as the Hamiltonian vector field X, is implicitly defined with the help of the time-derivative $\dot{\lambda}_t$ of the primitive. We refer to this vector field Y as the *Euler vector field* in view of its close relation with the Euler force. In the second interpretation using the Cartan formula on the loop space this derivative appears as the Lie derivative with respect to the vector field on the free loop space which generates the rotation.

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2 Merry-go-round and fictitious forces

We consider a free particle in the plane. In an inertial system the free particle moves with constant velocity along a straight line. In particular, the Hamiltonian is just given by kinetic energy T. There are no accelerations, hence no forces acting. This changes as soon as we rotate the coordinate system with time-dependent angular velocity $\varpi \colon \mathbb{R} \to \mathbb{R}$. In this case apart from kinetic energy T the particle has angular momentum and three pseudo-forces appear, as illustrated by Figure 1, namely

- **centrifugal force** depending on the locus q and the angular velocity ϖ_t is a gradient, hence conservative;
- **Coriolis force** depending on the particle velocity $\dot{q}$ and ϖ_t;

- **Euler force** depending on the locus q and the angular acceleration $\dot\varpi_t$.

Let $\varpi\colon \mathbb{R}\to\mathbb{R}$ be a smooth function. The function whose Hamiltonian vector field generates rotation is angular momentum. Hence in a rotating coordinate system with time-dependent angular speed ϖ_t, say the xy-plane rotating anti-clockwise in space, the Hamiltonian of a free particle, say of mass $m=1$, consists of kinetic energy minus angular momentum, in symbols

$$
\begin{aligned}
H^\varpi\colon \mathbb{R}^5\to\mathbb{R},\quad (t,q,\boldsymbol{p}) &\mapsto \tfrac{1}{2}\left|\boldsymbol{p}\right|^2 - \left\langle \begin{pmatrix}0\\0\\\varpi_t\end{pmatrix}, \underbrace{\begin{pmatrix}q_1\\q_2\\0\end{pmatrix}\times\begin{pmatrix}p_1\\p_2\\0\end{pmatrix}}_{\text{angular momentum}}\right\rangle\\
&= \tfrac{p_1^2+p_2^2}{2} - \varpi_t(q_1p_2-q_2p_1) =: H^{\varpi_t}(q,\boldsymbol{p})
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^3$ and $|\cdot|$ is the induced norm. After completing the squares H^ϖ is of the form

$$
H^{\varpi_t}(q,\boldsymbol{p}) = \underbrace{\tfrac{1}{2}\left(\left(p_1+\varpi_t q_2\right)^2+\left(p_2-\varpi_t q_1\right)^2\right)}_{T(\boldsymbol{p}-\boldsymbol{A}_t|_q)} \underbrace{-\tfrac{1}{2}\varpi_t^2\left|q\right|^2}_{+\phi_t(q)}
$$

where the time-dependent vector potential and the potential are

$$
\boldsymbol{A}_t|_q = \boldsymbol{A}_{\varpi_t}|_q = \begin{pmatrix}-\varpi_t q_2\\ \varpi_t q_1\end{pmatrix} = \varpi_t J_0 q,\qquad \phi_t|_q = -\tfrac{1}{2}\varpi_t^2\left|q\right|^2. \tag{2.1}
$$

The anti-/clockwise quarter rotation in the plane is encoded by the matrices

$$
J_0 := \begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad \bar J_0 := -J_0 = \begin{pmatrix}0&1\\-1&0\end{pmatrix},
$$

where J_0 is the canonical complex structure on $\mathbb{R}^2$. The **Hamilton equations** for H^ϖ with respect to the **canonical symplectic form** $\omega_{\rm can}=d\lambda_{\rm can}$ are

$$
\begin{cases}
\dot q = \partial_p H^{\varpi_t} &= \boldsymbol{p}-\boldsymbol{A}_t|_q\\
\dot{\boldsymbol{p}} = -\partial_q H^{\varpi_t} &= \sum_{j=1}^{2}(p_j-A_t^j)\nabla \boldsymbol{A}_t^j|_q - \nabla\phi_t|_q.
\end{cases} \tag{2.2}
$$

This is a first order ODE for smooth maps $(q,\boldsymbol{p})\colon\mathbb{R}\to\mathbb{R}^4$. We eliminate $\boldsymbol{p}$ to get a second order ODE in q. For the following calculation we simplify notation $A_1:=A_t^1$ and $A_2:=A_t^2$ as well as $\partial_1:=\partial_{q_1}$ and $\partial_2:=\partial_{q_2}$. Differentiating the first Hamilton equation with respect to t and then using the second one for $\dot{\boldsymbol{p}}$

we obtain for q the second order ODE

$$
\begin{aligned}
\ddot q &= \dot{\boldsymbol p} - \tfrac{d}{dt}\boldsymbol A_t|_q \\
&= -\partial_q H^{\varpi_t}(q,\boldsymbol p) - \dot{\boldsymbol A}_t|_q - d\boldsymbol A_t|_q\, \dot q \\
&= -\partial_q \left(\tfrac12 \left|\boldsymbol p - \boldsymbol A_t(q)\right|^2\right) - \nabla\phi_t|_q - \dot{\boldsymbol A}_t|_q - d\boldsymbol A_t|_q\, \dot q \\
&= -\begin{pmatrix} \partial_1 \frac{(p_1-A_1)^2+(p_2-A_2)^2}{2} \\ \partial_2 \frac{(p_1-A_1)^2+(p_2-A_2)^2}{2} \end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol A}_t - \begin{pmatrix} \partial_1 A_1 & \partial_2 A_1 \\ \partial_1 A_2 & \partial_2 A_2 \end{pmatrix} \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} \\
&= \begin{pmatrix} \underline{(p_1-A_1)\partial_1 A_1} + (p_2-A_2)\partial_1 A_2 \\ (p_1-A_1)\partial_2 A_1 + \underline{\underline{(p_2-A_2)\partial_2 A_2}} \end{pmatrix} - \begin{pmatrix} \underline{(\partial_1 A_1)\dot q_1} + (\partial_2 A_1)\dot q_2 \\ (\partial_1 A_2)\dot q_1 + \underline{\underline{(\partial_2 A_2)\dot q_2}} \end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol A}_t \\
&= \begin{pmatrix} (\partial_1 A_2 - \partial_2 A_1)\dot q_2 \\ (\partial_2 A_1 - \partial_1 A_2)\dot q_1 \end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol A}_t \qquad , \partial_1 A_2 - \partial_2 A_1 =: \operatorname{rot} \boldsymbol A_t \\
&= (\operatorname{rot} \boldsymbol A_t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol A}_t \\
&= -\left(\operatorname{rot} \boldsymbol A_t|_q\right) J_0 \dot q - \dot{\boldsymbol A}_t|_q - \nabla\phi_t|_q = \ddot q.
\end{aligned}
$$

Underlined terms cancel due to the first Hamilton equation. Thus (2.2) ⇒ (2.3).

Lemma 2.1. *The first order ODE (2.2) is equivalent to the second order ODE*

$$
\ddot q = -\left(\operatorname{rot} \boldsymbol A_t|_q\right) J_0 \dot q - \dot{\boldsymbol A}_t|_q - \nabla\phi_t|_q. \tag{2.3}
$$

We call (2.3) the $(\boldsymbol A, \boldsymbol\phi)$***-equation****. It makes sense for smooth* $\phi\colon \mathbb R \times \mathbb R^2 \to \mathbb R$.

Proof. We already showed that (2.2) ⇒ (2.3). Vice versa, let q solve (2.3) and define $\boldsymbol p := \dot q + \boldsymbol A_t|_q$. Hence $\dot{\boldsymbol p} = \ddot q + \dot{\boldsymbol A}_t|_q + d\boldsymbol A_t|_q \dot q$. Now read the earlier displayed calculation backwards to see that $\ddot q = -\partial_q H^{\varpi_t}(q,\boldsymbol p) - \dot{\boldsymbol A}_t|_q - d\boldsymbol A_t|_q\, \dot q$. Hence $\dot{\boldsymbol p} = -\partial_q H^{\varpi_t}(q,\boldsymbol p)$ and this proves (2.2) and Lemma 2.1. □

Corollary 2.2. *In the merry-go-round case ($\boldsymbol A$, ϕ given by (2.1)) we have*

$$
\operatorname{rot} \boldsymbol A_t|_q := \partial_1 A_t^2|_q - \partial_2 A_t^1|_q = 2\varpi_t, \qquad \dot{\boldsymbol A}_t|_q = \dot\varpi_t J_0 q, \qquad \nabla\phi|_q = -\varpi_t^2 q,
$$

so that the $(\boldsymbol A, \phi)$*-equation becomes*

$$
\begin{aligned}
\ddot q &= -2\varpi_t J_0 \dot q - \dot\varpi_t J_0 q + \varpi_t^2 q \\
&= \underbrace{2\varpi_t \bar J_0 \dot q}_{\text{Coriolis}} + \underbrace{\dot\varpi_t \bar J_0 q}_{\text{Euler}} + \underbrace{\varpi_t^2 q}_{\text{centrifug.}} .
\end{aligned}
$$

3 Lagrangian variational approach to periodic solutions of the $(\boldsymbol{A}, \phi)$-equation

In this section we are looking at periodic solutions of the equation (2.3), namely

$$\ddot{q} = -\left(\operatorname{rot} \boldsymbol{A}_t|_q\right) J_0 \dot{q} - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q.$$

In this article **periodic** means 1-periodic and we identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. To make sense of periodicity of solutions of the $(\boldsymbol{A}, \phi)$-equation we need that all three $(\operatorname{rot} \boldsymbol{A}_t)$, $\dot{\boldsymbol{A}}_t$, and ϕ_t are periodic in time t. We do not need that $\boldsymbol{A}_t$ itself is periodic in t, but we need some twisted-periodicity as explained next.

Definition 3.1 (Twisted-periodic 1-form)**.** Let $\mathfrak{Q} \subset \mathbb{R}^2$ be an open subset. A **twisted-periodic 1-form** $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is a smooth real family of 1-forms on $\mathfrak{Q}$

$$\theta_t = A_t^1\, dq_1 + A_t^2\, dq_2, \qquad \boldsymbol{A}_t = \left(A_t^1, A_t^2\right) : \mathfrak{Q} \to \mathbb{R}^2,$$

such that θ is **twisted-periodic** in the sense that a) the t-derivative is periodic

$$\text{a) } \dot{\theta}_{t+1} = \dot{\theta}_t, \qquad \text{b) } \theta_{t+1} = \theta_t + df_t, \tag{3.4}$$

and b) for every time t there is a smooth so-called **twist function** $f_t\colon \mathfrak{Q} \to \mathbb{R}$. A **time-constant** choice, e.g. $f := f_0$, is a twist function for θ, too; cf. (5.10).

Consequently the exterior derivative is periodic, in symbols $d\theta_{t+1} = d\theta_t$. Furthermore, taking the time derivative of $(3.4)_{\mathrm{b}}$ we get $d\dot{f}_t = \dot{\theta}_{t+1} - \dot{\theta}_t = 0$. Thus $\dot{f}_t(q)$ is **locally constant** in q, not necessarily in t. If $\theta_{t+1} = \theta_t$ is periodic, then the zero function is a twist function, indeed $d\theta_{t+1} = d\theta_t + d0$. Viewing the plane as a subspace of space $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ we use the coefficients of the twisted 1-form θ on the plane to define a vector field in space by

$$(\boldsymbol{A}_t, 0)\,|_{(q_1,q_2,q_3)} := \left(A_t^1|_{(q_1,q_2)}, A_t^2|_{(q_1,q_2)}, 0\right)$$

called a **vector potential**.

Remark 3.2. Magnetic fields are described by closed 2-forms σ, where closedness $d\sigma = 0$ encodes the fact that there are no magnetic charges; see e.g. [Web17, § 2.4.1]. For a twisted-periodic 1-form θ the family of 2-forms

$$\sigma_t := d\theta_t = d\left(A_t^1\, dq_1 + A_t^2\, dq_2\right) = \underbrace{\left(\partial_1 A_t^2 - \partial_2 A_t^1\right)}_{=:\operatorname{rot} \boldsymbol{A}_t} dq_1 \wedge dq_2 \tag{3.5}$$

is time-periodic $\sigma_t := d\theta_t = d\theta_{t+1} = \sigma_{t+1}$ and closed $d\sigma_t = dd\theta_t = 0$. The magnetic vector field corresponding to the magnetic 2-form is $\boldsymbol{B}_t := (*\sigma)^\#$, i.e.

$$\boldsymbol{B}_t := \underbrace{\nabla\times}_{=\operatorname{rot}} \begin{pmatrix} \boldsymbol{A}_t \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_{q_1} \\ \partial_{q_2} \\ \partial_{q_3} \end{pmatrix} \times \begin{pmatrix} A_t^1 \\ A_t^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial_{q_3} A_t^2 \\ \partial_{q_3} A_t^1 \\ \partial_1 A_t^2 - \partial_2 A_t^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{rot} \boldsymbol{A}_t \end{pmatrix}.$$

This shows that the magnetic field is perpendicular to the q_1-q_2-plane in $\mathbb{R}^3$.

3.1 Classical action functional

Definition 3.3. Let $\{\theta_t\}$ be a twisted-periodic 1-form on $\mathfrak{Q} \subset \mathbb{R}^2$ open. Let $\phi\colon \mathbb{S}^1 \times \mathfrak{Q} \to \mathbb{R}$ be a smooth map. Define the **classical action functional** by

$$\begin{aligned}\mathcal{S} = \mathcal{S}_{L^\theta} \colon \Lambda\mathfrak{Q} := W^{1,2}(\mathbb{S}^1, \mathfrak{Q}) &\to \mathbb{R}\\ q &\mapsto \int_0^1 \underbrace{\tfrac{1}{2}\left|\dot q_t\right|^2 + \theta_t|_{q_t}\dot q_t - \phi_t|_{q_t}}_{=L_t^\theta(q_t,\dot q_t)} dt - f_0(q_0).\end{aligned}$$

For f_0 see (3.4). The function $L_t^\theta\colon T\mathfrak{Q} \to \mathbb{R}$, called **Lagrangian**, is defined by

$$L_t^\theta(q,v) := \tfrac{1}{2}\left|v\right|^2 + \theta_t|_q v - \phi_t(q).$$

Remark 3.4 (twist term and integration interval)**.** (i) Given the integration interval $[\underline{0},1]$, it is important to subtract the **twist term $\boldsymbol{f_{\underline{0}}(q_{\underline{0}})}$** to get as critical points the periodic solutions of the $(\boldsymbol{A},\phi)$-equation (Propositions 3.6–3.7).

(ii) If θ_t is twisted-periodic, and not periodic, then L_t^θ is not periodic in time. Therefore the integral defining $\mathcal{S}(q)$ is not over the circle $\mathbb{S}^1$, but defined over the interval $[0,1]$. Given $r \in \mathbb{R}$, using $\int_r^{r+1}$ in the definition of $\mathcal{S}(q)$ causes that $\mathcal{S}(q)$ depends on r modulo 1: indeed shift t by 1 to obtain equality one

$$\begin{aligned}\int_{r+1}^{r+2} \theta_t|_{q_t}\dot q_t\, dt - \int_r^{r+1} \theta_t|_{q_t}\dot q_t\, dt &\stackrel{1}{=} \int_r^{r+1} \underbrace{\theta_{t+1}}_{\theta_t + df_t} \underbrace{|_{q_{t+1}}}_{q_t} \underbrace{\dot q_{t+1}}_{\dot q_t} dt - \int_r^{r+1} \theta_t|_{q_t}\dot q_t\, dt\\ &\stackrel{2}{=} \int_r^{r+1} \left(\tfrac{d}{dt}\big|_0 f_t(q_t) - \dot f_t(q_t)\right) dt\\ &\stackrel{3}{=} f_{r+1}(q_{r+1}) - f_r(q_r) - \int_r^{r+1} \underbrace{\dot f_t(q_t)}_{\dot f_t(q_r)} dt = 0.\end{aligned}$$

After equality 1 we use periodicity $q_{r+1} = q_r$ and twisted-periodicity. Equality 2 cancels the two θ_t summands with opposite signs and rewrites $df_t|_{q_t}\dot q_t$ by the chain rule. Equalities 3 and four use the fundamental theorem of calculus. After equality 3 we use that $\dot f_t$ is locally constant, so we may replace q_t by q_r.

(iii) If $\theta_{t+1} = \theta_t$ is periodic, then there is no twist term ($f \equiv 0$) and the integrals $\int_r^{r+1} L_t^\theta(q_t,\dot q_t)dt = \int_0^1 L_t^\theta(q_t,\dot q_t)dt$ are equal for any $r \in \mathbb{R}$.[1]

Definition 3.5 (2-form $\Sigma := D\Theta$ on $\mathbb{S}^1 \times \mathfrak{Q}$)**.** A twisted-periodic 1-form $\{\theta_t\}_{t\in\mathbb{R}}$ on $\mathfrak{Q} \subset \mathbb{R}^2$ induces a 1-form Θ on $\mathbb{R} \times \mathfrak{Q}$ only, not $\mathbb{S}^1 \times \mathfrak{Q}$, namely

$$\Theta|_{(t,x)} \begin{pmatrix}\tau\\ \xi\end{pmatrix} := \theta_t|_x \xi \qquad \forall (t,x) \in \mathbb{R} \times \mathfrak{Q} \quad \forall (\tau,\xi) \in \mathbb{R} \times T_x\mathfrak{Q}.$$

The exterior derivative on $\mathbb{R} \times \mathfrak{Q}$, notation D, is given by

$$\begin{aligned}D\Theta|_{(t,x)}\left(\begin{pmatrix}\tau\\ \xi\end{pmatrix}, \begin{pmatrix}\sigma\\ \eta\end{pmatrix}\right) &= d\theta_t|_x(\xi,\eta) + \left(dt \wedge \dot\theta_t|_x\right)\left(\begin{pmatrix}\tau\\ \xi\end{pmatrix}, \begin{pmatrix}\sigma\\ \eta\end{pmatrix}\right)\\ &= d\theta_t|_x(\xi,\eta) + \dot\theta_t|_x\eta \cdot \tau - \dot\theta_t|_x\xi \cdot \sigma.\end{aligned}$$

[1] schematically $\int_r^{r+1} \theta_t = \int_r^1 \theta_t + \int_1^{r+1} \theta_t = \int_r^1 \theta_t + \int_0^r \theta_{t+1} = \int_0^1 \theta_t$ as $\theta_{t+1} = \theta_t$ is periodic

By twisted periodicity (3.4) this is a 2-form on $\mathbb{S}^1 \times \mathfrak{Q}$ notation $\Sigma := D\Theta$.

The following proposition gives a different description of the magnetic part of the functional which is useful to compute the critical points in Section 3.2.

Proposition 3.6 (Magnetic part – alternative description)**.** *Let θ and Σ be as in Definition 3.5. Fix a base loop $b \in \Lambda\mathfrak{Q}$. Then every loop q homotopic to b in $\Lambda\mathfrak{Q}$ through a smooth homotopy, say h from b to q, satisfies the identity*

$$\int_{[0,1]} q^*\theta - f_0(q_0) = \int_R H^*\Sigma + \int_{[0,1]} b^*\theta - f_0(b_0)$$

with $H\colon R = [0,1] \times [0,1] \to \mathbb{S}^1 \times \mathfrak{Q}$, $(r,t) \mapsto (t, h(r,t))$ and f_0 from (3.4).

Proof. Let $h\colon Z = [0,1] \times \mathbb{S}^1 \to \mathfrak{Q}$, $(r,t) \mapsto h(r,t)$, be a smooth homotopy of loops from $b = h(0,\cdot)$ to $q = h(1,\cdot)$. Consider the induced homotopy H between loops B and Q in $\mathbb{S}^1 \times \mathfrak{Q}$ defined by

$$H(r,t) := \begin{pmatrix} t \\ h(r,t) \end{pmatrix}, \quad B_t := \begin{pmatrix} t \\ b_t \end{pmatrix} = H(0,t), \quad Q_t := \begin{pmatrix} t \\ q_t \end{pmatrix} = H(1,t). \tag{3.6}$$

We use Definition 3.5 of Θ in each of the following two calculations

$$\int_{[0,1]} Q^*\Theta = \int_0^1 \Theta|_{(t,q_t)} \begin{pmatrix} 1 \\ \dot q_t \end{pmatrix} dt = \int_0^1 \theta_t|_{q_t} \dot q_t \, dt = \int_{[0,1]} q^*\theta \tag{3.7}$$

and

$$\begin{aligned} \int_{[0,1]} H(\cdot,1)^*\Theta = \int_0^1 \Theta|_{(1,h(r,1))} \begin{pmatrix} 0 \\ \partial_1 h(r,1) \end{pmatrix} dr &= \int_0^1 \theta_1|_{h(r,1)} \partial_1 h(r,1)\, dr \\ &= \int_{[0,1]} h(\cdot,1)^*\theta_1. \end{aligned}$$

Similarly for $B^*\Theta$ and $H(\cdot,0)^*\Theta$. The rectangle boundary ∂R has four segments

$$\Gamma_1 = \{1\} \times [0,1]_t, \;\; \Gamma_2 = [0,1]_r \times \{1\}, \;\; \Gamma_3 = \{0\} \times [0,1]_t, \;\; \Gamma_4 = [0,1]_r \times \{0\},$$

endowed with the induced (anti-clockwise) boundary orientation. Exactness $\Sigma = d\Theta$ and as exterior differentiation D and pull-back commute, Stokes yields

$$\begin{aligned} \int_R H^*\Sigma &= \int_{\partial R = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} H^*\Theta \\ &= \int_{\Gamma_1} Q^*\Theta + \int_{\Gamma_2} H(\cdot,1)^*\Theta + \int_{\Gamma_3} B^*\Theta + \int_{\Gamma_4} H(\cdot,0)^*\Theta \\ &\stackrel{3}{=} \int_{t=0}^1 q^*\theta - \int_{r=0}^1 \underbrace{(h(\cdot,1))}_{=h(\cdot,0)}{}^*\theta_1 - \int_{t=0}^1 b^*\theta + \int_{r=0}^1 h(\cdot,0)^*\theta_0 \\ &\stackrel{4}{=} \int_{t=0}^1 q^*\theta - \int_{t=0}^1 b^*\theta - \int_{r=0}^1 \underbrace{h(\cdot,0)^*\left(\theta_1 - \theta_{\underline{0}}\right)}_{=h(\cdot,0)^* df_{\underline{0}} = dh(\cdot,0)^* f_{\underline{0}}} \end{aligned}$$

$$\overset{5}{=} \int_{t=0}^{1} q^*\theta - \int_{t=0}^{1} b^*\theta - f_{\underline{0}} \circ \underbrace{h(1,0)}_{q_0} + f_{\underline{0}} \circ \underbrace{h(0,0)}_{b_0}.$$

Equality 3 parametrizes the four segments $\Gamma_1, \dots, \Gamma_4$ by $[0,1]$ taking care of orientations, then the previously prepared formulas such as for $\int Q^*\Theta$ are used. Note that $h(\cdot,1) \equiv h(\cdot,0)$ since $t \in \mathbb{S}^1$. Equality 4 is by linearity of integral and pull-back. Now twisted-periodicity (3.4) is used and d and pull-back are interchanged. Equality 5 is by Stokes' theorem. This proves Proposition 3.6. □

3.2 Critical points

Proposition 3.7. *The critical points of the classical action functional $\mathcal{S}\colon \Lambda\mathfrak{Q} \to \mathbb{R}$, Definition 3.3, are the solutions of the $(\boldsymbol{A}, \phi)$-equation, namely*

$$\ddot{q} = -\left(\operatorname{rot} \boldsymbol{A}_t|_q\right) J_0 \dot{q} - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q$$

for smooth loops $q\colon \mathbb{S}^1 \to \mathfrak{Q}$.

Proof. Pick $q \in \Lambda\mathfrak{Q}$. Fix a base loop $b \in \Lambda\mathfrak{Q}$ smoothly homotopic to q in $\Lambda\mathfrak{Q}$. We move from $\mathfrak{Q}$ to $\mathbb{S}^1 \times \mathfrak{Q}$ and define Θ and $D\Theta$ as in Definition 3.5. Loops b and q in $\mathfrak{Q}$ induce loops B and Q in $\mathbb{S}^1 \times \mathfrak{Q}$, and a vector field ξ along q induces a vector field Ξ along Q, namely

$$B_t = \begin{pmatrix} t \\ b_t \end{pmatrix}, \qquad Q_t = \begin{pmatrix} t \\ q_t \end{pmatrix}, \qquad \dot{Q}_t = \begin{pmatrix} 1 \\ \dot{q}_t \end{pmatrix}, \qquad \Xi = \begin{pmatrix} 0 \\ \xi \end{pmatrix}.$$

Let $h\colon Z = [0,1] \times \mathbb{S}^1 \to \mathfrak{Q}$, $(r,t) \mapsto h(r,t)$, be a smooth homotopy of loops from $b = h(0,\cdot)$ to $q = h(1,\cdot)$. Consider the induced homotopy $H\colon [0,1] \times [0,1] \to \mathbb{S}^1 \times \mathfrak{Q}$ between loops B and Q in $\mathbb{S}^1 \times \mathfrak{Q}$ defined by (3.6).

Now we construct variations with real parameter τ. We vary the point q by $q_{(\tau)} := q + \tau\xi$ and we add to the homotopy h from b to q the linear homotopy from q to $q_{(\tau)}$ extending the parameter interval to $r \in [0, 1+\tau]$, in symbols

$$h_{(\tau)}(r,\cdot) := \begin{cases} h(r,\cdot) & , r \in [0,1], \\ q + (r-1)\xi & , r \in [1, 1+\tau]. \end{cases} \tag{3.8}$$

We lift $q_{(\tau)}$ to $Q_{(\tau)} = Q + \tau\Xi = (\cdot, q + \tau\xi)$ and $h_{(\tau)}$ to $H_{(\tau)}(r,t) := (t, h_{(\tau)}(r,t))$.

For $\Sigma = D\Theta$ from Definition 3.5 we determine the derivative of the magnetic part in the alternative description of Proposition 3.6. We compute, using in equality 1 that only the first summand depends on τ, the following τ-derivative

$$\begin{aligned}
&\left.\frac{d}{d\tau}\right|_0 \left(\int_{[0,1+\tau]\times\mathbb{S}^1} H_{(\tau)}^*\Sigma + \int_{[0,1]} b^*\theta - f_0(b_0) \right) \\
&\overset{1}{=} \lim_{\tau\to 0} \frac{1}{\tau} \left(\int_{[0,1+\tau]\times\mathbb{S}^1} H_{(\tau)}^*\Sigma - \int_{[0,1]} H_{(\tau)}^*\Sigma \right) \\
&= \lim_{\tau\to 0} \frac{1}{\tau} \int_{[1,1+\tau]\times\mathbb{S}^1} H_{(\tau)}^*\Sigma \\
&= \lim_{\tau\to 0} \frac{1}{\tau} \int_{r=1}^{1+\tau} \int_{t=0}^{1} \Sigma_{H_{(\tau)}(r,t)} \left(\partial_r H_{(\tau)}(r,t), \partial_t H_{(\tau)}(r,t) \right) dt\, dr
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\tau\to 0} \frac{1}{\tau} \int_{r=1}^{1+\tau} \int_{t=0}^{1} D\Theta_{(t,q_t+(r-1)\xi_t)} \left(\begin{pmatrix} 0 \\ \xi_t \end{pmatrix}, \begin{pmatrix} 1 \\ \dot q_t + (r-1)\dot\xi_t \end{pmatrix} \right) dt\, dr \\
&\overset{5}{=} \lim_{\tau\to 0} \frac{1}{\tau} \int_{r=1}^{1+\tau} \int_{0}^{1} \Big(d\theta_t|_{q_t+(r-1)\xi_t} \big(\xi_t, \dot q_t + (r-1)\dot\xi_t\big) - \dot\theta_t|_{q_t+(r-1)\xi_t}\xi_t \Big)\, dt\, dr \\
&\overset{6}{=} \lim_{\tau\to 0} \frac{1}{\tau} \int_{\rho=0}^{1} \int_{t=0}^{1} \Big(d\theta_t|_{q_t+\rho\tau\xi_t} \big(\xi_t, \dot q_t + \rho\tau\dot\xi_t\big) - \dot\theta_t|_{q_t+\rho\tau\xi_t}\xi_t \Big)\, \tau\, dt\, d\rho \\
&\overset{7}{=} \int_{\rho=0}^{1} \int_{t=0}^{1} \Big(d\theta_t|_{q_t} \big(\xi_t, \dot q_t\big) - \dot\theta_t|_{q_t}\xi_t \Big)\, dt\, d\rho \\
&\overset{8}{=} \int_{t=0}^{1} \Big(d\theta_t|_{q_t} \big(\xi_t, \dot q_t\big) - \dot\theta_t|_{q_t}\xi_t \Big)\, dt.
\end{aligned}
$$

Equality 5 uses the formula for $D\Theta$ in Definition 3.5. Equality 6 is by variable substitution $\rho(r) := \frac{r-1}{\tau}$, hence $dr = \tau\, d\rho$. This extra factor τ and $\frac{1}{\tau}$ cancel, so in equality 7 we can set $\tau = 0$. Equality 8 uses that the integrand does not depend on ρ and that $\int_0^1 d\rho = 1$.

Next we use Proposition 3.6 for $q_{(\tau)} := q + \tau\xi$ and $q_{(\tau)}$ in equation 3 and the result of the previous calculation in equation 4 of what follows

$$
\begin{aligned}
&d\mathcal{S}|_q\xi \\
&= \tfrac{d}{d\tau}\big|_{\tau=0} \mathcal{S}(q+\tau\xi) \\
&= \int_0^1 \tfrac{d}{d\tau}\big|_0 \Big(\tfrac{1}{2}|\dot q_t + \tau\dot\xi_t|^2 - \phi_t(q_t+\tau\xi_t) \Big)\, dt + \tfrac{d}{d\tau}\big|_0 \Big(\int_{[0,1]} q_{(\tau)}^*\theta - f_0(q_{(\tau)}(0)) \Big) \\
&\overset{3}{=} \int_0^1 \Big(\langle \dot q_t, \dot\xi_t\rangle - d\phi_t|_{q_t}\xi_t \Big)\, dt + \tfrac{d}{d\tau}\big|_0 \Big(\int_R H_{(\tau)}^*\Sigma + \int_{[0,1]} b^*\theta - f_0(b_0) \Big) \\
&\overset{4}{=} \int_0^1 \Big(\langle \dot q_t, \dot\xi_t\rangle - \langle \nabla\phi_t|_{q_t}, \xi_t\rangle + d\theta_t|_{q_t}(\xi, \dot q_t) - \dot\theta_t|_{q_t}\xi \Big)\, dt \\
&\overset{5}{=} \int_0^1 \Big(\langle \dot q_t, \dot\xi_t\rangle - \langle \nabla\phi_t|_{q_t}, \xi_t\rangle + (\operatorname{rot} \boldsymbol{A}_t)\, \underbrace{(dq_1\wedge dq_2)\,(\xi,\dot q)}_{=\xi_1\dot q_2 - \xi_2\dot q_1} - \dot\theta_t|_{q_t}\xi \Big) dt \\
&\overset{6}{=} \int_0^1 \langle \dot q_t, \dot\xi_t\rangle dt + \Big\langle \xi, -\nabla\phi_t|_q - (\operatorname{rot} \boldsymbol{A}_t)\, J_0\dot q - \dot{\boldsymbol{A}}_t \Big\rangle_{L^2(\mathbb{S}^1,\mathbb{R}^2)}.
\end{aligned}
$$

Equality 5 replaces $d\theta_t$ by (3.5). Equality 6 uses $\boldsymbol{A}_t$ from Definition 3.1.

Now suppose that q is a critical point of $\mathcal{S}$, i.e. $d\mathcal{S}|_q = 0$. Then the identity we just proved tells that $\dot q$ has a weak derivative, notation $\ddot q$, and

$$
\ddot q = -\left(\operatorname{rot} \boldsymbol{A}_t|_q\right) J_0\dot q - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q.
$$

Note that the right hand side is in L^2, hence $q \in W^{2,2}(\mathbb{S}^1,\mathbb{R}^2)$. But then the right hand side is in $W^{1,2}$, hence $q \in W^{3,2}(\mathbb{S}^1,\mathbb{R}^2)$ and so on. Therefore $q \in \cap_{\ell\in\mathbb{N}_0} W^{\ell,2}(\mathbb{S}^1,\mathbb{R}^2) = C^\infty(\mathbb{S}^1,\mathbb{R}^2)$. This proves Proposition 3.7. □

3.3 Eliminate periodic scalar potentials

Definition 3.8 (Twisted-periodic vector potential)**.** A smooth map $\boldsymbol{A}\colon \mathbb{R}\times\mathfrak{Q}\to\mathbb{R}^2$, $(t,q)\mapsto \boldsymbol{A}(t,q)=:\boldsymbol{A}_t|_q$, is called a **twisted-periodic vector potential on $\mathfrak{Q}$** if at each time t it holds that

$$\dot{\boldsymbol{A}}_{t+1}=\dot{\boldsymbol{A}}_t,\qquad \operatorname{rot}\boldsymbol{A}_{t+1}=\operatorname{rot}\boldsymbol{A}_t,\qquad \boldsymbol{A}_{t+1}-\boldsymbol{A}_t=\nabla f_t,$$

for a smooth function $f_t\colon\mathfrak{Q}\to\mathbb{R}$, called a **twist function**.

Lemma 3.9. *Let $\phi\colon\mathbb{S}^1\times\mathfrak{Q}\to\mathbb{R}$ be a smooth function and $\boldsymbol{A}$ a twisted-periodic vector potential on $\mathfrak{Q}$ with twist function f. The **$(\boldsymbol{A},\phi)$-equation on $\mathfrak{Q}$***

$$\ddot q=-\left(\operatorname{rot}\boldsymbol{A}_t|_q\right)J_0\dot q-\dot{\boldsymbol{A}}_t|_q-\nabla\phi_t|_q \tag{3.9}$$

is an ODE for smooth maps $q\colon\mathbb{R}\to\mathfrak{Q}$. Then the maps defined by

$$(\boldsymbol{A}^\phi)_t:=\boldsymbol{A}_t+\int_0^t\nabla\phi_s\,ds,\qquad f_t^\phi:=f_t+\int_t^{t+1}\phi_s\,ds,$$

are a twisted-periodic vector potential with twist function. Moreover, a map q is a solution of the $(\boldsymbol{A},\phi)$-equation iff q is a solution of the $(\boldsymbol{A}^\phi,0)$-equation.

Proof. To see that $(\boldsymbol{A}^\phi)_t$ is a twisted-periodic vector potential note that, firstly

$$\dot{(\boldsymbol{A}^\phi)}_{t+1}=\dot{\boldsymbol{A}}_{t+1}+\nabla\phi_{t+1}=\underline{\dot{\boldsymbol{A}}_t+\nabla\phi_t=\dot{(\boldsymbol{A}^\phi)}_t},$$

secondly,

$$\underbrace{\operatorname{rot}(\boldsymbol{A}^\phi)_{t+1}}=\operatorname{rot}\boldsymbol{A}_{t+1}+\int_0^{t+1}\underbrace{\operatorname{rot}(\nabla\phi_s)}_{=0}\,ds\underbrace{=\operatorname{rot}\boldsymbol{A}_{t+1}}=\underline{\operatorname{rot}\boldsymbol{A}_t\overset{4}{=}\operatorname{rot}(\boldsymbol{A}^\phi)_t}.$$

where identity 4 holds true by the underbraced one at time t and, thirdly

$$\boldsymbol{A}^\phi_{t+1}-\boldsymbol{A}^\phi_t=\boldsymbol{A}_{t+1}-\boldsymbol{A}_t+\int_t^{t+1}\nabla\phi_s\,ds=\nabla\left(f_t+\int_t^{t+1}\phi_s\,ds\right).$$

Now suppose that q solves (3.9). Then by the two underlined identities above

$$-\left(\operatorname{rot}(\boldsymbol{A}^\phi)_t\right)J_0\dot q-\dot{(\boldsymbol{A}^\phi)}_t-0=(\operatorname{rot}\boldsymbol{A}_t)\,J_0\dot q-\left(\dot{\boldsymbol{A}}_t+\nabla\phi_t(q)\right)=\ddot q,$$

i.e. q solves the $(\boldsymbol{A}^\phi,0)$-equation. Vice versa, in the $(\boldsymbol{A}^\phi,0)$-equation for q insert $\partial_t(\boldsymbol{A}^\phi)_t=\dot{\boldsymbol{A}}_t+\nabla\phi_t$ and $\operatorname{rot}(\boldsymbol{A}^\phi)_t=\operatorname{rot}\boldsymbol{A}_t$ to get (3.9) proving Lemma 3.9. □

4 Hamiltonian variational approach to periodic solutions

Let $\mathfrak{Q} \subset \mathbb{R}^2$ be an open subset. Let $\phi\colon \mathbb{S}^1 \times \mathfrak{Q} \to \mathbb{R}$ be a smooth map. Consider the Hamiltonian given by **kinetic plus potential energy**

$$H\colon \mathbb{S}^1 \times T^*\mathfrak{Q} \to \mathbb{R}, \quad (t, q, p) \mapsto \tfrac{1}{2} \left|p\right|^2 + \phi_t(q).$$

Let $\{\theta_t\}$ be a twisted-periodic 1-form on $\mathfrak{Q}$, see (3.4), in particular

$$\theta_t = A_t^1\, dq_1 + A_t^2\, dq_2, \qquad \dot\theta_{t+1} = \dot\theta_t, \qquad \theta_{t+1} = \theta_t + df_t.$$

On the cotangent bundle $\pi\colon T^*\mathfrak{Q} \to \mathfrak{Q}$ we consider the time-dependent 1-form

$$\lambda_t := \lambda_{\rm can} + \pi^*\theta_t$$

where $\lambda_{\rm can}$ is the Liouville form, see e.g. [Arn89, §37 B]. Then $\{\lambda_t\}$ satisfies twisted-periodicity (3.4), e.g. $\lambda_{t+1} = \lambda_t + F_t$ for $F_t(q,p) := (\pi^* f_t)(q,p) = f_t(q)$. Furthermore, at each time t the exterior derivative

$$\omega_t := d\lambda_t = d\lambda_{\rm can} + \pi^* d\theta_t, \qquad \omega_{t+1} = \omega_t,$$

is a periodic **twisted symplectic form**, see e.g. [FW26a, appendix].

Definition 4.1. The **symplectic action functional** is defined by

$$\begin{aligned}
\mathcal{A} = \mathcal{A}_{\lambda,H}\colon \Lambda T^*\mathfrak{Q} &\to \mathbb{R}\\
v = (q,p) &\mapsto \int_0^1 \Big(v^* \left(\lambda_{\rm can} + \pi^*\theta_t\right) - H_t(v_t) \Big)\, dt - F_0(v_0)\\
&= \int_0^1 \Big((p_t + \theta_t|_{q_t}) \dot q_t - \tfrac{1}{2} \left|p_t\right|^2 - \phi_t(q_t) \Big)\, dt - f_0(q_0).
\end{aligned}$$

Again, see Definition 3.3 and Remark 3.4, the integral is over the interval $[0,1]$, and not the circle $\mathbb{S}^1$. The subtraction of the **twist term** $F_0(v_0) = f_0(q_0)$ is important to get as critical points periodic solutions of the $(\boldsymbol{A}, \phi)$-equation. If $\theta_{t+1} = \theta_t$ is periodic, then the twist function $f_t \equiv 0$ vanishes at every time t.

4.1 Critical points

Proposition 4.2. *A loop $v = (q, \boldsymbol{p})$ is a critical point of the symplectic action $\mathcal{A}_{\lambda,H}\colon \Lambda T^*\mathfrak{Q} \to \mathbb{R}$ iff q is a periodic solution of $(\boldsymbol{A}, \phi)$-equation (2.3) and $\boldsymbol{p} = \dot q$.*

Proof. In this proof, in order to avoid repetition, we already use parts of Section 5 below on general symplectic manifolds. By Theorem 5.3 the critical points of $\mathcal{A}_{\lambda,H}$ are smooth solutions $v = (q, \boldsymbol{p})\colon \mathbb{S}^1 \to \mathfrak{Q} \times \mathbb{R}^2$ of the Euler-Hamilton or **$(\boldsymbol{\lambda}, \boldsymbol{H})$-equation** $\dot v = X(v) + Y(v)$, see (5.12).

'⇒' We must calculate the Hamiltonian vector field X and the Euler vector field Y determined by (5.11). For the Euler field Y we need the time derivative

$$\dot\lambda_t = \pi^*\dot\theta_t = \dot A_t^1 dq_1 + \dot A_t^2 dq_2$$

pointwise along $T^*\mathfrak{Q}$ and at time $t \in \mathbb{R}$ and the symplectic form

$$\begin{aligned}\omega_t &= \omega_{\mathrm{can}} + \pi^* d\theta_t \\ &= dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \underbrace{\left(-\partial_{q_2} A_t^1 + \partial_{q_1} A_t^2\right)}_{=\mathrm{rot}\,\boldsymbol{A}_t} dq_1 \wedge dq_2.\end{aligned}$$

Then Y_t, implicitly defined by $\dot\lambda_t = \omega_t(\cdot, Y_t)$, is explicitly given by the formula

$$Y_t = -\dot A_t^1 \tfrac{\partial}{\partial p_1} - \dot A_t^2 \tfrac{\partial}{\partial p_2}.$$

This is a vertical vector field and it only depends on points of the base $\mathfrak{Q}$ of $T^*\mathfrak{Q}$. To compute the Hamilton vector field X observe that

$$dH_t = p_1\, dp_1 + p_2\, dp_2 + (\partial_{q_1}\phi_t)\, dq_1 + (\partial_{q_2}\phi_t)\, dq_2.$$

Then X_t, implicitly defined by $dH_t = \omega_t(\cdot, X_t)$, is explicitly given by the formula

$$X_t = p_1 \tfrac{\partial}{\partial q_1} + p_2 \tfrac{\partial}{\partial q_2} + \left(p_2 \,\mathrm{rot}\, \boldsymbol{A}_t - \tfrac{\partial \phi_t}{\partial q_1}\right) \tfrac{\partial}{\partial p_1} - \left(p_1 \,\mathrm{rot}\, \boldsymbol{A}_t + \tfrac{\partial \phi_t}{\partial q_2}\right) \tfrac{\partial}{\partial p_2}.$$

Summarizing

$$X_t + Y_t = \begin{pmatrix} p_1 \\ p_2 \\ p_2 \,\mathrm{rot}\, \boldsymbol{A}_t - \frac{\partial\phi_t}{\partial q_1} \\ -p_1 \,\mathrm{rot}\, \boldsymbol{A}_t - \frac{\partial\phi_t}{\partial q_2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\dot{\boldsymbol{A}}_t^1 \\ -\dot{\boldsymbol{A}}_t^2 \end{pmatrix}.$$

Therefore the (λ, H)-equation (5.12) is the ODE system

$$\begin{cases} \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ \begin{pmatrix} \dot p_1 \\ \dot p_2 \end{pmatrix} = \begin{pmatrix} p_2 \,\mathrm{rot}\, \boldsymbol{A}_t|_q - \frac{\partial\phi_t}{\partial q_1}|_q - \dot{\boldsymbol{A}}_t^1|_q \\ -p_1 \,\mathrm{rot}\, \boldsymbol{A}_t|_q - \frac{\partial\phi_t}{\partial q_2}|_q - \dot{\boldsymbol{A}}_t^2|_q \end{pmatrix} \end{cases}$$

for smooth loops $(q, \boldsymbol{p})\colon \mathbb{S}^1 \to T^*\mathfrak{Q}$. Equivalently, the (λ, H)-equation is

$$\begin{cases} \dot q = \boldsymbol{p} \\ \dot{\boldsymbol{p}} = -\left(\mathrm{rot}\, \boldsymbol{A}_t|_q\right) J_0 \boldsymbol{p} - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q \end{cases}.$$

Write this 1st order ODE system as a 2nd order system to get the $(\boldsymbol{A}, \phi)$-equation

$$\ddot q = \underbrace{-\left(\mathrm{rot}\, \boldsymbol{A}_t|_q\right) J_0 \dot q}_{\text{Lorentz force}} \underbrace{-\dot{\boldsymbol{A}}_t|_q}_{\text{Euler f.}} \underbrace{-\nabla\phi_t(q)}_{\text{conservative}}$$

for smooth loops $q\colon \mathbb{S}^1 \to \mathfrak{Q}$. This completes the proof of '⇒'.

'⇐' Let q be a periodic solution of (2.3) and define $\boldsymbol{p} := \dot q$. Then $\dot q = \boldsymbol{p}$ and $\dot{\boldsymbol{p}} = \ddot q$ satisfy the (λ, H)-equation displayed above. This proves Proposition 4.2. □

5 Euler vector field

Definition 5.1. Let $(M, d\lambda_t)_{t\in\mathbb{R}}$ be a smooth[2] family of exact symplectic manifolds such that λ_t is **twisted-periodic**, i.e. at each time t it holds that

$$\dot\lambda_{t+1} = \dot\lambda_t, \qquad \lambda_{t+1} = \lambda_t + dF_t, \qquad \omega_t := d\lambda_t = d\lambda_{t+1}, \tag{5.10}$$

for a smooth function $F_t\colon M \to \mathbb{R}$, called a **twist function**. The time-constant function $F := F_0\colon M \to \mathbb{R}$ is a twist function.[3]

Taking the time derivative of identity two in (5.10) tells $d\dot F_t = \dot\lambda_{t+1} - \dot\lambda_t = 0$. Thus the time derivative $\dot F_t(x)$ of a general twist function F_t is locally constant in x, not necessarily in t.

Observe that the symplectic form depends periodically on time $\omega_{t+1} = \omega_t$. Let $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ be a smooth function on M depending periodically on time. At each time t non-degeneracy of the symplectic form ω_t allows for transforming the 1-forms dH_t and $\dot\lambda_t$ into vector fields along M, namely

$$dH_t = d\lambda_t(\cdot, X_t), \qquad \dot\lambda_t = d\lambda_t(\cdot, Y_t). \tag{5.11}$$

Note that both X_t and Y_t are periodic. While $X_t = X_{H_t}^{d\lambda_t}$ is called **Hamiltonian vector field**, we refer to

$$Y_t = Y_{\dot\lambda_t}^{d\lambda_t}$$

as **Euler vector field**, because of its relation to the Euler force as explained in the merry-go-round example in Section 2.

5.1 Hamiltonian action functional and critical points

Definition 5.2. For $(M, d\lambda_t)_{t\in\mathbb{R}}$ twisted-periodic define the **action functional**

$$\begin{aligned} \mathcal{A} = \mathcal{A}_{\lambda,H}\colon \Lambda M := W^{1,2}(\mathbb{S}^1, M) &\to \mathbb{R} \\ v &\mapsto \int_0^1 \left(v^*\lambda - H(t, v_t)\right) dt - F_0(v_0). \end{aligned}$$

Again, as in Definition 3.3, the integral is over the interval $[0,1]$ and not the circle $\mathbb{S}^1$. The subtraction of the **twist term** $F_0(v_0)$, see (5.10), is important to get as critical points periodic solutions of the (λ, H)-equation (5.12). If $\lambda_{t+1} = \lambda_t$ is periodic, then twist function and twist term vanish.

Theorem 5.3 (Euler-Hamilton equation)**.** *Critical points of $\mathcal{A} = \mathcal{A}_{\lambda,H}$ are smooth solutions $v\colon \mathbb{S}^1 \to M$ of the **Euler-Hamilton** or **$(\boldsymbol{\lambda}, \boldsymbol{H})$-equation***

$$\dot v = X_t(v) + Y_t(v) \tag{5.12}$$

where $X_t = X_{H_t}^{d\lambda_t}$ and $Y_t = Y_{\dot\lambda_t}^{d\lambda_t}$ are determined by (5.11).

---

[2] here smoothness refers to $\mathbb{R} \times TM \times \ni (t, (x,\xi)) \mapsto \lambda_t|_x \xi \in \mathbb{R}$ being smooth

[3] Indeed $\lambda_{t+1} - \lambda_t - dF = \underline{\lambda_{t+1} - \lambda_t - dF_t} + d(F_t - F_0) = d\int_0^t \dot F_s \, ds = 0$ since $d\dot F_s = 0$.

Definition 5.4 (2-form $\Omega := D\Lambda$ on $\mathbb{S}^1 \times M$)**.** Let us move to $\mathbb{R} \times M$. Let D be the exterior derivative on $\mathbb{R} \times M$. The family of 1-forms λ_t on M induces a 1-form Λ on $\mathbb{R} \times M$, namely

$$\Lambda|_{(t,x)} \begin{pmatrix} r \\ \xi \end{pmatrix} := \lambda_t|_x \xi$$

whenever $(t,x) \in \mathbb{R} \times M$ and $(r,\xi) \in \mathbb{R} \times T_xM$. The exterior derivative is

$$\begin{aligned}\Omega := D\Lambda|_{(t,x)} \left(\begin{pmatrix} r \\ \xi \end{pmatrix}, \begin{pmatrix} s \\ \eta \end{pmatrix} \right) &= d\lambda_t|_x(\xi,\eta) + \left(dt \wedge \dot\lambda_t|_x \right) \left(\begin{pmatrix} r \\ \xi \end{pmatrix}, \begin{pmatrix} s \\ \eta \end{pmatrix} \right) \\ &= \omega_t|_x(\xi,\eta) + \dot\lambda_t|_x \eta \cdot r - \dot\lambda_t|_x \xi \cdot s.\end{aligned}$$

By twisted periodicity (5.10) this is actually a 2-form on $\mathbb{S}^1 \times M$.

The following proposition gives a different description of the non-Hamiltonian part of the functional which is useful to compute the critical points.

Proposition 5.5 (Non-Hamiltonian part – alternative description)**.** *Let λ and Ω be as in Definition 5.4. Fix a base loop $b \in \Lambda M$. Then every loop v homotopic to b in ΛM through a smooth homotopy, say $\mathfrak{h}$ from b to q, satisfies the identity*

$$\int_{[0,1]} v^*\lambda - F_0(v_0) = \int_R \mathfrak{H}^*\Omega + \int_{[0,1]} b^*\lambda - F_0(b_0)$$

with $\mathfrak{H}\colon R = [0,1] \times [0,1] \to \mathbb{S}^1 \times M$, $(r,t) \mapsto (t, \mathfrak{h}(r,t))$ and F_0 from (5.10).

Proof. Literally the same proof as Proposition 3.6. □

Proof of Theorem 5.3. Let $\mathfrak{h}\colon Z = [0,1] \times \mathbb{S}^1 \to M$, $(r,t) \mapsto \mathfrak{h}(r,t)$, be a smooth homotopy of loops from $b = \mathfrak{h}(0,\cdot)$ to $v = \mathfrak{h}(1,\cdot)$.
To construct variations with real parameter τ pick a Riemannian metric on M and let exp be the associated exponential map. We vary v by $v_{(\tau)} := \exp_v \tau\xi$ and we add to the homotopy $\mathfrak{h}$ from b to v the linear homotopy from v to $v_{(\tau)}$ extending the parameter interval to $r \in [0, 1+\tau]$, in symbols

$$\mathfrak{h}_{(\tau)}(r,\cdot) := \begin{cases} \mathfrak{h}(r,\cdot) & , r \in [0,1], \\ \exp_v (r-1)\xi & , r \in [1, 1+\tau]. \end{cases}$$

We lift $\mathfrak{h}_{(\tau)}$ to $\mathfrak{H}_{(\tau)}(r,t) := (t, \mathfrak{h}_{(\tau)}(r,t))$. Set $\omega_t := d\lambda_t$. Using the formula for $\Omega = D\Lambda$ in Definition 5.4, one computes in complete analogy to the lengthy calculation after (3.8) the underlined identity shown in step 4 below. In step 3 below we apply Proposition 5.5 for $v_{(\tau)} := \exp_v \tau\xi$ and $\mathfrak{h}_{(\tau)}$ to obtain

$$\begin{aligned} & d\mathcal{A}|_v \xi \\ &= \tfrac{d}{d\tau}\Big|_{\tau=0} \mathcal{A}(\exp_v \tau\xi) \\ &= -\int_0^1 \tfrac{d}{d\tau}\Big|_0 H_t(\exp_{v_t} \tau\xi_t)\, dt + \tfrac{d}{d\tau}\Big|_0 \left(\int_{[0,1]} v_{(\tau)}^*\lambda - F_0(v_{(\tau)}(0)) \right) \\ &\stackrel{3}{=} -\int_0^1 dH_t|_{v_t}\xi_t\, dt + \underline{\tfrac{d}{d\tau}\Big|_0 \left(\int_R \mathfrak{H}_{(\tau)}^*\Omega + \int_{[0,1]} b^*\lambda - F_0(b_0) \right)} \end{aligned}$$

$$\stackrel{4}{=} -\int_0^1 \omega_t|_{v_t}\big(\xi_t, X_t|_{v_t}\big)\, dt + \underline{\int_0^1 \Big(\omega_t|_{v_t}(\xi_t, \dot v_t) - \dot\lambda_t|_{v_t}\xi_t\Big)\, dt}$$

$$\stackrel{5}{=} \int_0^1 \omega_t|_{v_t}\big(X_t|_{v_t} + Y_t|_{v_t} - \dot v_t, \xi_t\big)\, dt.$$

Equalities 4 and 5 use that $X_t = X^{\omega_t}_{H_t}$ and $Y_t = Y^{\omega_t}_{\dot\lambda_t}$ are determined by (5.11).

Now suppose that v is a critical point of $\mathcal{A}$, i.e. $d\mathcal{A}|_v = 0$. Then the identity we just proved tells that $\dot v = X_t(v) + Y_t(v)$. Note that the right hand side is in $W^{1,2}$, so $v \in W^{2,2}(\mathbb{S}^1, M)$. But then the right hand side is in $W^{2,2}$, so $v \in W^{3,2}$ etc. Hence $v \in \cap_{\ell\in\mathbb{N}_0} W^{\ell,2}(\mathbb{S}^1, M) = C^\infty(\mathbb{S}^1, M)$. This proves Theorem 5.3. □

5.2 Eliminating the Hamiltonian vector field

Proposition 5.6. *Let $(M, d\lambda_t)_{t\in\mathbb{R}}$ be a twisted-periodic exact symplectic manifold and $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ a smooth function. Then the 1-form family defined by*

$$\lambda_t^H := \lambda_t + \int_0^t dH_s\, ds, \qquad t \in \mathbb{R},$$

is twisted-periodic. The critical points of the functions $\mathcal{A}_{\lambda,H}$ and $\mathcal{A}_{\lambda^H,0}$ coincide.

Proof. We verify the three conditions in (5.10): Condition one: We get that

$$\underline{d\lambda_t^H} = d\lambda_t + \int_0^1 ddH_s\, ds \underline{= d\lambda_t} \stackrel{(5.10))}{=} d\lambda_{t+1} = d\lambda_{t+1}^H$$

since $dd = 0$ and the final identity uses the underlined one at time $t+1$. To show condition two use the definition of λ_t^H to get the first and last identity in

$$\dot\lambda_t^H = \dot\lambda_t + dH_t = \dot\lambda_{t+1} + dH_{t+1} = \dot\lambda_{t+1}^H$$

where identity two uses periodicity of $\dot\lambda_t$ and of H_t. Condition three: Note that

$$\lambda_{t+1}^H - \lambda_t^H \stackrel{1}{=} \lambda_{t+1} - \lambda_t + \int_t^{t+1} dH_s\, ds \stackrel{2}{=} d(F_t + \overline{H})$$

where equality 1 is by definition of λ_{t+1}^H and λ_t^H. Equality 2 uses the last hypothesis in (5.10) and the definition of the pointwise time-mean $\overline{H} := \int_t^{t+1} H_s\, ds$. This proves that the family λ_t^H is twisted-periodic.

To show that the critical points coincide it suffices to show that the (λ, H)- and the $(\lambda^H, 0)$-equation are equal, equivalently that the vector field identity

$$Y_t^H = Y_t + X_t, \qquad Y_t^H := Y^{d\lambda_t^H}_{\dot\lambda_t^H}, \quad Y_t := Y^{d\lambda_t}_{\dot\lambda_t}, \quad X_t := X^{d\lambda_t}_{H_t},$$

holds true. By definition (5.11) of the respective vector fields we get $1, 3$ in

$$d\lambda_t^H(\cdot, Y_t^H) \stackrel{1}{=} \dot\lambda_t^H \stackrel{2}{=} \dot\lambda_t + dH_t \stackrel{3}{=} d\lambda_t(\cdot, Y_t) + d\lambda_t(\cdot, X_t) = d\lambda_t(\cdot, Y_t + X_t).$$

Here equality 2 holds by definition of λ_t^H and the fundamental theorem of calculus. But $d\lambda_t^H = d\lambda_t$ since $dd = 0$. Now the equality $Y_t^H = Y_t + X_t$ follows by non-degeneracy of the symplectic form $d\lambda_t$. This proves Proposition 5.6. □

5.3 Periodizing the twisted-periodic symplectic primitive

Proposition 5.7. *Let $(M, d\lambda_t)_{t\in\mathbb{R}}$ be a twisted-periodic exact symplectic manifold (5.10) and $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ smooth. Then the two families defined by*

$$\tilde{\lambda}_t := \lambda_t - t\,dF, \qquad \tilde{H}_t := H_t + F, \qquad F := F_0,$$

are periodic. Furthermore, the critical points of $\mathcal{A}_{\lambda,H}$ and $\mathcal{A}_{\tilde{\lambda},\tilde{H}}$ coincide.

Proof. Since H_t is periodic in t, the family $\tilde{H}_t$ is obviously periodic in t. To check that $\tilde{\lambda}_t$ is periodic in t, we compute

$$\tilde{\lambda}_{t+1} - \tilde{\lambda}_t = \lambda_{t+1} - (t+1)dF - \lambda_t + t\,dF = \lambda_{t+1} - \lambda_t - dF = 0.$$

By (5.11) the Euler vector fields are determined by $\dot{\lambda}_t = d\lambda_t(\cdot, Y_t)$ and $\dot{\tilde{\lambda}}_t = d\tilde{\lambda}_t(\cdot, \tilde{Y}_t)$. Using that $d\tilde{\lambda}_t = d\lambda_t$, we compute

$$d\lambda_t(\cdot, Y_t - \tilde{Y}_t) = \dot{\lambda}_t - \dot{\tilde{\lambda}}_t = \dot{\lambda}_t - (\dot{\lambda}_t - dF) = dF = d\lambda_t(\cdot, X_F^{d\lambda_t}).$$

Since $d\lambda_t$ is symplectic, hence non-degenerate, we obtain that $\tilde{Y}_t = Y_t - X_F^{d\lambda_t}$. By (5.11), now for the Hamilton vector fields, since $\tilde{H}_t - H_t = F$ we get

$$d\lambda_t(\cdot, \tilde{X}_t - X_t) = d\tilde{H}_t - dH_t = dF = d\lambda_t(\cdot, X_F^{d\lambda_t}),$$

thus $\tilde{X}_t = X_t + X_F^{d\lambda_t}$. Consequently $\tilde{X}_t + \tilde{Y}_t = X_t + Y_t$ which means that the $(\tilde{\lambda}, \tilde{H})$- and the (λ, H)-equation are equal and, by Theorem 5.3, so are the critical points of $\mathcal{A}_{\tilde{\lambda},\tilde{H}}$ and $\mathcal{A}_{\lambda,H}$. This proves Proposition 5.7. $\square$

5.4 The Euler flow is symplectic

Let $(M, \lambda_t)_{t\in\mathbb{R}}$ be a twisted-periodic exact symplectic manifold. To simplify notation we assume in the following that the flow φ_Y^t of the Euler vector field Y_t globally exists, i.e. for any $t \in \mathbb{R}$ there is a diffeomorphism $\varphi_Y^t\colon M \to M$ such that $\varphi_Y^0 = \mathrm{id}_M$ and $\frac{d}{dt}\varphi_Y^t = Y_t \circ \varphi_Y^t$.

Proposition 5.8. *The Euler flow is symplectic, in symbols $(\varphi_Y^t)^*\omega_0 = \omega_t$.*

Proof. Since the inverse of φ_Y^t is φ_Y^{-t}, the identity

$$\omega_t = (\varphi_Y^t)^*\omega_0 = ((\varphi_Y^{-t})^{-1})^*\omega_0 = ((\varphi_Y^{-t})^*)^{-1}\omega_0$$

is equivalent to the identity $(\varphi_Y^{-t})^*\omega_t = \omega_0$. To prove this identity it suffices to show that the function $t \mapsto (\varphi_Y^{-t})^*\omega_t$ is constant since at time zero the value is $(\varphi_Y^0)^*\omega_0 = \mathrm{id}_M^*\omega_0 = \omega_0$. Indeed by Cartan's formula

$$L_{Y_t}\omega_t = di_{Y_t}\omega_t + i_{Y_t}d\omega_t = -d\dot{\lambda}_t = -\dot{\omega}_t$$

and then by the Leibniz rule the derivative vanishes

$$\tfrac{d}{dt}(\varphi_Y^{-t})^*\omega_t = (\varphi_Y^{-t})^*\big(-L_{Y_t}\omega_t\big) + (\varphi_Y^{-t})^*\dot{\omega}_t = 0.$$

This proves Proposition 5.8. $\square$

5.5 Applying Cartan's formula on the loop space

In Section 5.5 we restrict to the *periodic* case ($\theta_{t+1} = \theta_t$). We show how Theorem 5.3 is related to application of Cartan's formula on the loop space.
Since the loop space is infinite dimensional, to the best of our knowledge, the Cartan formula is not established in this setup. However, since the formal application of Cartan's formula coincides with the result of Theorem 5.3, this gives evidence that Cartan's formula is valid on the loop space as well.

We first consider the following geometric setup. Assume that N is a manifold, $\Lambda \in \Omega^1(N)$ is a 1-form on N, and $\mathcal{V} \in \Gamma(TN)$ is a vector field on N. If N is finite dimensional the following discussion is completely rigorous. However, we want to apply the discussion below to the case where N is the loop space of a finite dimensional manifold. Plugging in the vector field into the 1-form we obtain a function

$$f := i_{\mathcal{V}}\Lambda \colon N \to \mathbb{R}.$$

By Cartan's formula the differential is given by

$$df = di_{\mathcal{V}}\Lambda = L_{\mathcal{V}}\Lambda - i_{\mathcal{V}}d\Lambda.$$

In the special case where $\Omega := d\Lambda$ is symplectic we can define a further vector field $\mathcal{Y}$ on N, called **Euler vector field**, by the requirement of equal 1-forms

$$i_{\mathcal{Y}}\Omega = L_{\mathcal{V}}\Lambda.$$

In this case the differential can be written as $df = i_{\mathcal{Y}-\mathcal{V}}\Omega$ and therefore critical points of f are points $z \in N$ satisfying the **abstract Euler equation**

$$\mathcal{V}(z) = \mathcal{Y}(z). \tag{5.13}$$

Example: Loop space

In the following we apply this observation to the loop space case $N = \mathcal{L}(M) := C^\infty(\mathbb{S}^1, M)$ with its **canonical vector field** and a 1-form on $\mathcal{L}M$, namely

$$\mathcal{V}(z) = \dot{z} := \partial_t z, \qquad \Lambda := \int_0^1 \lambda_t \, dt,$$

where $\{\lambda_t\}_{t\in\mathbb{S}^1}$ is a periodic 1-form on M such that the periodic family $\omega_t := d\lambda_t$ consists of symplectic forms on M. The flow of $\mathcal{V} = \partial_t$ on $\mathcal{L}M$ is $\Phi^r_{\mathcal{V}} z = r_* z$ where $(r_* z)(t) = z(t+r)$ for every time t. Linearization yields

$$(\Phi^r_{\mathcal{V}} z)_t = z_{t+r}, \qquad (d\Phi^r_{\mathcal{V}}|_z \xi)_t = \xi_{t+r}.$$

Let $z \in \mathcal{L}M$ and $\xi \in T_z\mathcal{L}M$. We compute the pull-back

$$\begin{aligned}
\left({\Phi^r_{\mathcal{V}}}^* \Lambda\right)_z \xi := \Lambda_{\Phi^r_{\mathcal{V}}(z)}\, d\Phi^r_{\mathcal{V}}|_z \xi &= \int_0^1 \lambda_t|_{z_{t+r}} \xi_{t+r} \, dt \\
&\stackrel{2}{=} \int_r^{r+1} \lambda_{t-r}|_{z_t} \xi_t \, dt \\
&\stackrel{3}{=} \int_0^1 \lambda_{t-r}|_{z_t} \xi_t \, dt.
\end{aligned}$$

Equality 2 is by change of variables. Equality 3 is by periodicity $\lambda_{t+1} = \lambda_t$. The Lie derivative of Λ with respect to $\mathcal{V}$ is by definition

$$L_{\mathcal{V}}\Lambda := \tfrac{d}{dr}\big|_{r=0} {\Phi_{\mathcal{V}}^r}^* \Lambda = -\int_0^1 \dot\lambda_t \, dt.$$

The exterior derivative of Λ is symplectic, namely

$$\Omega := d\Lambda = \int_0^1 \omega_t \, dt.$$

Therefore the Euler vector field localizes in the sense that

$$\omega_t|_{z_t} \left(\mathcal{Y}|_z(t), \cdot\right) = -\dot\lambda_t|_{z_t}.$$

Hence the loop space Euler vector field

$$\mathcal{Y}|_z(t) = Y_t|_{z_t}$$

coincides with the Euler vector field Y along M as defined by (5.11).

In particular, in this example the abstract Euler equation (5.13) gives rise to the manifold Euler equation

$$\dot z_t = Y_t|_{z_t}$$

as obtained earlier, see (5.12) with vanishing H, hence vanishing X.

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