

An experimentally appearing inconsistency with hidden variable theories in quantum phenomena in a three-dimensional space

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Abstract

We study a simplified Kochen–Specker theorem. Our simplified version says that we cannot assign a preexistence value as 1 to both quantum operators (the projection operator and the identity operator) even though the assignment is independent of each other. And we are able to use the weak assumption (the identity operator possesses a preexistence value as 1) instead of the strong assumption (the existence of the weak hidden variable theory) in order to verify the Kochen–Specker theorem. Our result applies in the Hilbert space that the dimension is three.

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I. INTRODUCTION

The great success of quantum mechanics (QM) (cf. [1–8]) is recognized by the scientific community of physical theories. The incompleteness argument for quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [9]. A hidden variable interpretation of quantum mechanics is an interesting topic of research [3, 4] and an inconsistency with hidden variable theories is discussed by Bell, Kochen, and Specker [10, 11].

Multipartite inequality as tests for the Kochen–Specker theorem is proposed by Nagata [12]. It turns out that the Kochen–Specker theorem becomes a quite strong theorem when the dimension of the multipartite state highly increases, regardless of entanglement properties. A strengthened Kochen–Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [13]. An inconsistency with hidden variable theories is discussed, using two symmetric measurements, by Nagata, Diep, and Nakamura [14]. They are independent of the order of measurements themselves.

Bose–Einstein and Fermi–Dirac statistics are two fundamental frameworks in quantum statistical mechanics [2, 15, 16]. Yan presents a statistical model describing Bose–Einstein and Fermi–Dirac statistics [15]. Our experiments say that there are only Bose–Einstein statistics and Fermi–Dirac statistics in a three-dimensional space. Zhou, Chen, Chen, Shen, Zhang, and Dai discuss that quantum statistics forbids particle exchange statistics beyond Bosons and Fermions in a three-dimensional space [16]. Oi and Endo discuss universal Efimov spectra and fermionic doublets in highly mass-imbalanced cold-atom mixtures with van der Waals and dipole interactions [17]. An original meaning of quantum physics in a three-dimensional space is given by [11]. Kletetschka proposes three-dimensional time: a mathematical framework for fundamental physics [18]. Therefore, here, we study a three-dimensional case.

In this paper, we investigate the aspect of the basic two quantum observables: (1) an observable that is obtained through a single projection operator and (2) an observable that is obtained through a single identity operator. Both of quantum observables are well studied by Kochen and Specker in terms of the inconsistency with hidden variable theories. They show that some value assignment to elements of a set of projection operators and identity operators should fail.

We study a simplified Kochen–Specker theorem. We use the basis $\{|1\rangle, |2\rangle, |3\rangle\}$ in a three-dimensional space. We write the matrix form of a quantum operator $a|1\rangle\langle 1| + b|2\rangle\langle 2| + c|3\rangle\langle 3|$ as follows:

$$a|1\rangle\langle 1| + b|2\rangle\langle 2| + c|3\rangle\langle 3| \equiv \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad (1)$$

where a , b , and c are eigenvalues of the quantum operator. We here discuss the basic two quantum operators in a three-dimensional space as: $H_3(x) = |1\rangle\langle 1| + x|2\rangle\langle 2| + x|3\rangle\langle 3| \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$, where $x \in \{0, 1\}$. x is a fixed constant.

In the case that $x = 0$, $H_3(0)$ is the one-dimensional projection operator in a three dimensional space $|1\rangle\langle 1|$. In the case that $x = 1$, $H_3(1)$ is the three-dimensional identity operator $|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|$.

We are in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The preexistence value is 1 by measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$. The two measurements of $H_3(x)$ are corresponding to the same x . It turns out that we cannot assign the preexistence value for quantum measurement outcome as 1 when measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$. In the case that $x = 0$, the one-dimensional projection operator in a three dimensional space does not have a classical counterpart. In the case that $x = 1$, the three-dimensional identity operator does not have a classical counterpart.

This paper is organized as follows:

In Sec. II, we review the Kochen–Specker theorem and we clarify the assumptions which are used in the theorem.

In Sec. III, we review the difference between probability and truth value approaches.

In Sec. IV, we review mutual implications between several assumptions for hidden variable theories.

In Sec. V, we modify the Kochen–Specker theorem presented in Sec. II into the statistical form so as to relax the assumptions which are required in the previous form.

In Sec. VI, we discuss the statistical Kochen–Specker assumption and we analyze an additional rule for four specific examples.

In Sec. VII, we study the inconsistency with hidden variable theories using the identity operator or the projection operator.

In Sec. VIII, we also propose the statistical Kochen–Specker theorem based on the identity operator.

In Sec. IX, we study the experimentally appearing Kochen–Specker theorem with the incompleteness in a real experiment.

In Sec. X, we study also the statistical and experimentally appearing Kochen–Specker theorem based on a quantum operator.

In Sec. XI, we present another way of deriving the three-dimensional Kochen–Specker theorem based on the identity operator. We are able to use the weak assumption (the identity operator possesses a preexistence value as 1) instead of the strong assumption (the existence of the weak hidden variable theory) in order to verify the three-dimensional Kochen–Specker theorem in the specific quantum state $|1\rangle$.

Section XII deals with conclusions and discussions in this paper.

II. THE KOCHEN–SPECKER THEOREM

Let us mention the theorem proved [11] by Kochen and Specker along with Redhead’s argument [4]. Throughout this paper, we assume von Neumann’s projective measurements and we confine ourselves to the finite-dimensional and the discrete spectrum case. Let $\mathbf{R}$ denote the reals where $\pm\infty \notin \mathbf{R}$. We assume every eigenvalue in this paper lies in $\mathbf{R}$. Further, we assume that every Hermitian operator is associated with a unique observable because we do not need to distinguish them in this paper. Namely, we assume the correspondence rule holds:

Definition: (*The corresponding rule*).

There is a 1:1 corresponding between the set of Hermitian operators and the set of observables.

In this paper, we assume the validity of QM and we would like to investigate if the deterministic hidden variable interpretation of it is possible. The problem of hidden variables for QM may be interpreted in a similar fashion as introducing a set of hidden states.

Definition: (*The deterministic hidden variable interpretation of QM (HV)*).

The results of quantum measurement are deterministically related to the hidden state of the system emerging from the source. (It is assumed as the ontological determinism which says that a given hidden state at one time issues in a unique hidden state at a later time. However, it does not imply that the prediction of future hidden states can be effectively computed.)

We shall make the hidden state be mirrored in the interpretation of QM by introducing a classical probability space, which we call the set of hidden states.

Let $\mathcal{O}$ be the space of Hermitian operators described in a finite-dimensional Hilbert space, and $\mathcal{X}$ be the space of density operators described in the Hilbert space. Namely, $\mathcal{X} = \{\psi | \psi \in \mathcal{O} \wedge \psi \geq 0 \wedge \text{Tr}[\psi] = 1\}$. Moreover, let us consider a classical probability space $(\Omega, \Sigma, \mu_\psi)$, where Ω is a nonempty sample space, Σ is a σ -algebra of subsets of Ω , and μ_ψ is a σ -additive normalized measure on Σ such that $\mu_\psi(\Omega) = 1$. The subscript ψ expresses that the probability measure is determined uniquely when a state ψ is specified.

A quantum state ψ is described by a probability measure μ_ψ on the space Ω , so that, for each measurable subset Γ of Ω , $\mu_\psi(\Gamma)$ is the probability that the system is in a hidden state lying in Γ . The sample space can be reinterpreted as a set of hidden states:

Definition: (*The set of hidden states (Ω)*).

$$\Omega := \{\omega | \omega : \mathcal{O} \rightarrow \mathbf{R}\}. \quad (2)$$

We assume that each a Hermitian operator O is associated with a measurable real-valued function $f_O : \Omega \mapsto \mathbf{R}$. Later, we shall define what $f_O(\omega)$ is.

Throughout this paper, and only in this paper, so-called the hidden variable theory implies quantum mechanics which accepts HV. That is,

Definition: (*The hidden variable theory in the KS type of approach*).

$$\text{The hidden variable theory} := \text{QM satisfying HV}. \quad (3)$$

We might define “hidden variable theory” by a different manner when we want to know what theory is possible in order to explain raw experimental data. For example, we already perform various experiments to check if Bell’s nonlocal feature [10] appears in experimental data. On this purpose, generally, we do not decide what theory is possible before the experiment is done. The definition (3) is not consistent with motivation of such experiments. However, we assume that our attention is of investigating if QM can be imbedded into deterministic theories supplemented by hidden states. To look for Bell’s nonlocal feature in front of raw data is not our aim. On this motivation, we assume that QM holds unconditionally in this paper. We may call this approach the Kochen–Specker (KS) type of approach.

At this stage, we do not yet define how each a quantum mechanical expected value is expressed by the hidden variable theory. We shall discuss this point in Sec. IV.

If we accept HV, then there should exist a preexistence value possessed by each a Hermitian operator, irrespective of measurement. For a Hermitian operator O in a finite-dimensional Hilbert space, let us denote the preexisting value of O in a quantum state ψ and in a hidden state ω by the symbol $\omega(O)$. Define strictly the function f_O using the preexistence value $\omega(O)$ by

$$f_O(\omega) := \omega(O). \quad (4)$$

Now, we mention the principle of faithful measurement for later use (see the proof of theorem (23) in Sec. III):

Definition: (*The principle of faithful measurement (FM)*).

The result of quantum measurement of an operator in a state ψ is numerically equal to the preexistence value possessed by the operator immediately prior to the measurement in the state ψ .

FM implies that the preexisting value should coincide with the stochastic prediction of QM for the state ψ . Now, let us mention the elemental basic of probability. Even though the probability for an event is zero, there is a case where the event occurs. Only the condition $\mu(N) = 0$ does not imply $N = \phi$, where μ and N represent a probability measure and an event, respectively. Of course, the converse proposition that $N = \phi$ implies $\mu(N) = 0$ is true. Similarly, the condition $\mu(N) = 1$ does not imply $N = \Omega$. This implies that FM does not provide any constraint on value assignments for the preexistence values in a hidden state. Thus, in this section, we do not particularly suppose that FM holds. However, FM connects QM to HV. (We will employ FM in Sec. III.) Instead of FM, we assume the value rule holds:

Definition: (*The value rule (VR)*).

If the quantum mechanical probability for λ turning up on measurement of an operator O is zero in a state ψ then λ cannot be the preexisting value $\omega(O)$ in the state ψ .

Unlike FM, VR provides a significant constraint on value assignments for the preexistence values in a hidden state. In this section, we confine our attention only to the case where HV and VR hold. We derive two important rules from VR as follows:

Definition: (*The spectrum rule*).

For a Hermitian operator O with a discrete spectrum, possible preexistence values for $\omega(O)$ are confined to the eigenvalues of the associated operator O .

Theorem.

$$\text{HV} \wedge \text{VR} \Rightarrow \text{Spectrum rule.} \quad (5)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Assume we accept VR. Then the Born statistical formula (see (19) in Sec. III) says that the quantum mechanical probability for λ turning up on measurement of an operator O is zero in any state ψ if λ is not an eigenvalue of the associated operator O . Therefore, if λ is not an eigenvalue of the associated operator O , λ cannot be the preexisting value $\omega(O)$ in the state ψ . Hence, possible preexistence values for $\omega(O)$ are confined to the eigenvalues of the associated operator O . QED.

Then, the preexisting value of the identity operator for an arbitrary finite-dimensional space necessarily takes one in spite of any state. Another consequence of VR is the eigenvector rule:

Definition: (*The eigenvector rule*).

If the state $|\psi\rangle$ in question is an eigenvector satisfying $O|\psi\rangle = o|\psi\rangle$ where o is some real number known as an eigenvalue of O , then the preexisting value $\omega(O)$ in the state $|\psi\rangle$ is o .

Theorem.

$$\text{HV} \wedge \text{VR} \Rightarrow \text{Eigenvector rule.} \quad (6)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Assume we accept VR. Suppose that the state $|\psi\rangle$ in question is an eigenvector satisfying $O|\psi\rangle = o|\psi\rangle$ where o is some real number known as an eigenvalue of O . Then the Born statistical formula (see (19) in Sec. III) says that the quantum mechanical probability for λ turning up on measurement of an operator O is zero in the state $|\psi\rangle$ if λ is not the real number o . Therefore, if λ is not the real number o , λ cannot be the preexisting value $\omega(O)$ in the state $|\psi\rangle$. Hence, the possible preexisting value $\omega(O)$ in the state $|\psi\rangle$ is confined to o . QED.

Definition: (*The functional calculus rule*).

Assume that a Hermitian operator A is written by $A = \sum_{i=1}^N a_i P_i$ where $\{P_i\}$ is a basis. Define the Hermitian operator $g(A)$ by

$$g(A) := \sum_{i=1}^N g(a_i) P_i. \quad (7)$$

Here, g stands for a function $g : \mathbf{R} \mapsto \mathbf{R}$.

Kochen and Specker introduce the following assumption for the preexisting values:

Definition: (*The functional rule (FUNC)*).

$$\omega(g(O)) = g(\omega(O)) (\Leftrightarrow f_{g(O)}(\omega) = g(f_O(\omega))), \quad (8)$$

for an arbitrary state ψ .

We refer to Eq. (8) as the functional rule (FUNC (8)). This means that the algebraic structure of quantum mechanical operators should be mirrored in the algebraic structure of the preexisting values of the operators. Since $g(O)|\psi\rangle = g(o)|\psi\rangle$, FUNC (8) holds by appealing to the eigenvector rule (see (6)) when the state $|\psi\rangle$ in question is an eigenvector of O . We begin by deriving a result from FUNC (8).

Definition: (*The product rule (PROD)*).

If Hermitian operators A and B commute, then

$$\omega(A \cdot B) = \omega(A) \cdot \omega(B), \quad (9)$$

for an arbitrary state ψ .

Lemma.

$$HV \wedge \text{FUNC (8)} \Rightarrow HV \wedge \text{PROD (9)}. \quad (10)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$, and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i , which are all different. Here O is assumed to be nondegenerate by construction. Let us define functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$, respectively. Then we can see that if A and B commute, there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. So we have

$$\begin{aligned} \omega(A \cdot B) &= \omega(h(O)) = h(\omega(O)) = j(\omega(O)) \cdot k(\omega(O)) \\ &= \omega(j(O)) \cdot \omega(k(O)) = \omega(A) \cdot \omega(B), \end{aligned} \quad (11)$$

where we use FUNC (8). QED.

A simple version of the KS theorem proves that we *cannot to ascribe values for all Hermitian operators of a set when HV, VR, and PROD (9) hold*. To see this, we review the simplified the KS theorem.

We shall follow the KS theorem proposed by Peres [19] and refined by Mermin [20] for two spin-1/2 systems. Let us try an attempt to assign values to the nine operators $\sigma_x^1 \sigma_y^2, \sigma_y^1 \sigma_x^2, \sigma_x^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_z^1 \sigma_z^2, \sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2$ for an arbitrary state ψ . Precisely speaking, for example, σ_x^1 means $\sigma_x^1 \otimes I^2$ and so on. Omitting the identity operator in a two-dimensional space, we abbreviate those as above. Let $\omega(O)$ denote the value of an operator O for the state ψ in question. We define $[X]^\psi$ as follows: Then, one can see that

$$\begin{aligned} [X]^\psi &:= \omega(\sigma_x^1 \sigma_x^2) \omega(\sigma_y^1 \sigma_y^2) \omega(\sigma_z^1 \sigma_z^2) \\ &= \omega(\sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2 \sigma_z^1 \sigma_z^2) = \omega(-I) = -1, \end{aligned} \quad (12)$$

where I represents the identity operator in a four-dimensional space. By the way we can factorize two of the terms as $\omega(\sigma_x^1 \sigma_x^2) = \omega(\sigma_x^1) \omega(\sigma_x^2)$ and $\omega(\sigma_y^1 \sigma_y^2) = \omega(\sigma_y^1) \omega(\sigma_y^2)$. Further, we have $\omega(\sigma_x^1 \sigma_y^2) = \omega(\sigma_x^1) \omega(\sigma_y^2)$ and $\omega(\sigma_y^1 \sigma_x^2) = \omega(\sigma_y^1) \omega(\sigma_x^2)$. Hence we get $\omega(\sigma_x^1 \sigma_x^2) \omega(\sigma_y^1 \sigma_y^2) = \omega(\sigma_x^1 \sigma_y^2) \omega(\sigma_y^1 \sigma_x^2)$ and

$$\begin{aligned} [X]^\psi &= \omega(\sigma_x^1 \sigma_x^2) \omega(\sigma_y^1 \sigma_y^2) \omega(\sigma_z^1 \sigma_z^2) \\ &= \omega(\sigma_x^1 \sigma_y^2) \omega(\sigma_y^1 \sigma_x^2) \omega(\sigma_z^1 \sigma_z^2) \\ &= \omega(\sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_z^1 \sigma_z^2) = \omega(I) = 1. \end{aligned} \quad (13)$$

Noting the relation (12), we see that an attempt to assign values to the above nine operators for an arbitrary state ψ should fail.

We follow another version of the KS theorem presented in Ref. [20] for three spin-1/2 systems. Let us try an attempt to assign values to the ten operators $\sigma_x^1 \sigma_y^2 \sigma_y^3, \sigma_y^1 \sigma_x^2 \sigma_y^3, \sigma_y^1 \sigma_y^2 \sigma_x^3, \sigma_x^1 \sigma_x^2 \sigma_x^3, \sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2, \sigma_x^3, \sigma_y^3$ for an arbitrary state ψ . Let $\omega(O)$ denote the value of an operator O for the state ψ in question. We define $[Y]^\psi$ as follows: Then, one can see that

$$\begin{aligned} [Y]^\psi &:= \omega(\sigma_x^1 \sigma_y^2 \sigma_y^3) \omega(\sigma_y^1 \sigma_x^2 \sigma_y^3) \omega(\sigma_y^1 \sigma_y^2 \sigma_x^3) \omega(\sigma_x^1 \sigma_x^2 \sigma_x^3) \\ &= \omega(\sigma_x^1 \sigma_y^2 \sigma_y^3 \sigma_y^1 \sigma_x^2 \sigma_y^3 \sigma_y^1 \sigma_y^2 \sigma_x^3 \sigma_x^1 \sigma_x^2 \sigma_x^3) = \omega(-I) = -1, \end{aligned} \quad (14)$$

where I represents the identity operator in an eight-dimensional space. By the way, we can factorize each of the four terms as

$$\begin{aligned} \omega(\sigma_x^1 \sigma_y^2 \sigma_y^3) &= \omega(\sigma_x^1) \omega(\sigma_y^2) \omega(\sigma_y^3), \\ \omega(\sigma_y^1 \sigma_x^2 \sigma_y^3) &= \omega(\sigma_y^1) \omega(\sigma_x^2) \omega(\sigma_y^3), \\ \omega(\sigma_y^1 \sigma_y^2 \sigma_x^3) &= \omega(\sigma_y^1) \omega(\sigma_y^2) \omega(\sigma_x^3), \\ \omega(\sigma_x^1 \sigma_x^2 \sigma_x^3) &= \omega(\sigma_x^1) \omega(\sigma_x^2) \omega(\sigma_x^3). \end{aligned} \quad (15)$$

From PROD (9), we have $(\omega(\sigma_k^j))^2 = \omega(I)$, ($j = 1, 2, 3, k = x, y$). Here, σ_k^1 means $\sigma_k^1 \otimes I^2 \otimes I^3$ and so on. Omitting two identity operators in a two-dimensional space, we abbreviate those as above. Thus, we get

$$\begin{aligned} [Y]^\psi &= \omega(\sigma_x^1 \sigma_y^2 \sigma_y^3) \omega(\sigma_y^1 \sigma_x^2 \sigma_y^3) \omega(\sigma_y^1 \sigma_y^2 \sigma_x^3) \omega(\sigma_x^1 \sigma_x^2 \sigma_x^3) \\ &= (\omega(\sigma_x^1))^2 (\omega(\sigma_y^1))^2 (\omega(\sigma_x^2))^2 (\omega(\sigma_y^2))^2 (\omega(\sigma_x^3))^2 (\omega(\sigma_y^3))^2 \\ &= \omega(I) \omega(I) \omega(I) \omega(I) \omega(I) \omega(I) = 1. \end{aligned} \quad (16)$$

Noting (14), we see that an attempt to assign values to the above ten operators in an arbitrary state ψ should fail.

These two examples provide significantly simplified the KS theorem which says “all versus nothing” demolition of the deterministic hidden variable interpretation of QM (i.e., HV) under the assumption that VR and FUNC (8) hold. Further, they are of the state-independent form. The result provided in this section is a part of the following theorem:

Theorem: (*The Kochen–Specker theorem*).

For every quantum state described in a finite-dimensional Hilbert space ($\text{Dim}(\mathcal{H}) \geq 3$),

$$\text{HV} \wedge \text{VR} \wedge \text{FUNC (8)} \Rightarrow \perp, \quad (17)$$

where $\perp$ means Contradiction. That is, these three assumptions do not hold at the same time.

We define the hidden variable inconsistency as the inconsistency with the algebraic structure of the preexisting values. We do not say that there is the inconsistency due to the algebraic structure of the commuting Hermitian operators. The algebraic structure of quantum mechanical operators cannot be mirrored in the algebraic structure of the preexisting values of the commuting Hermitian operators.

III. PROBABILITY AND TRUTH VALUE

In this section, we review the important difference between probability and truth value. (This section is a review of [4], p.131).

Definition: (*The characteristic function*).

The characteristic function $\chi_\Delta : \mathbf{R} \mapsto \{0, 1\}$ is defined by

$$\chi_\Delta(x) := \begin{cases} 1 & x \in \Delta \\ 0 & x \notin \Delta, \end{cases} \quad (18)$$

where Δ is any subset of the reals $\mathbf{R}$.

Now we introduce the notation θ which represents one result of quantum measurement. Suppose that the measurement of a Hermitian operator A for a system in the state ψ yields a value $\theta(A) \in \mathbf{R}$.

Using the characteristic function χ_Δ and θ , the Born statistical formula is expressed as follows:

Definition: (*The Born statistical formula (BSF)*).

$$\text{Prob}(\Delta)_{\theta(A)}^\psi := \text{Tr}[\psi \chi_\Delta(A)]. \quad (19)$$

The whole symbol $(\Delta)_{\theta(A)}^\psi$ is used to denote the proposition that the value of $\theta(A)$ lies in the set Δ in the quantum state ψ . And Prob denotes the probability that the proposition holds.

We define the inverse image of a set B with a function g .

Definition: (*The inverse image*).

$$g^{-1}(B) := \{x | g(x) \in B\}. \quad (20)$$

Definition: (*The statistical functional rule (STAT FUNC)*).

$$\text{Prob}(\Delta)_{\omega(g(A))}^\psi = \text{Prob}(g^{-1}(\Delta))_{\omega(A)}^\psi. \quad (21)$$

Lemma.

$$\chi_\Delta(g(x)) = \chi_{g^{-1}(\Delta)}(x), (x \in \mathbf{R}). \quad (22)$$

Proof. If $g(x) \in \Delta$, LHS=1 and then since $x \in g^{-1}(\Delta)$, RHS=1. On the other hand, if $g(x) \notin \Delta$, LHS=0 and then since $x \notin g^{-1}(\Delta)$, RHS=0. So, for all values of x , the two functions on the left and right hand side of Eq. (22) have the same value. Hence the two functions are equal. QED.

Theorem.

$$\text{HV} \wedge \text{FM} \Rightarrow \text{STAT FUNC (21)}. \quad (23)$$

Proof. By BSF (19), we have

$$Prob(\Delta)_{\theta(g(A))}^{\psi} = Tr[\psi \chi_{\Delta}(g(A))] = Tr[\psi \chi_{g^{-1}(\Delta)}(A)] = Prob(g^{-1}(\Delta))_{\theta(A)}^{\psi}. \quad (24)$$

Let us assume HV and FM (see Sec. II) hold. Suppose that the system is in a hidden state ω . Then we can substitute $\omega(A)$ for $\theta(A)$ and we obtain STAT FUNC (21). QED.

STAT FUNC (21) is a theorem of QM under the assumption that HV and FM hold. We begin by considering the relation between FUNC (8) and STAT FUNC (21) under the assumption that HV and FM hold.

Let us reinterpret STAT FUNC (21) as a statement about long-run frequencies for preexistence values of a Hermitian operator to lie in the indicated ranges in the given quantum state ψ . We cannot replace STAT FUNC (21) by

$$Val(\Delta)_{\omega(g(A))}^{\psi} = Val(g^{-1}(\Delta))_{\omega(A)}^{\psi}, \quad (25)$$

where Val denotes the truth valuation ascribed to propositions such as $(\Delta)_{\omega(A)}^{\psi}$.

Formally, $Val : \{\mathbf{P}\} \mapsto \{0, 1\}$ is a function that maps the set of propositions of the form $(\Delta)_{\omega(A)}^{\psi}$ onto the two-element set of truth values, i.e., True and False. The function Val should be contrasted with the function $Prob : \{\mathbf{P}\} \mapsto [0, 1]$, which maps the set of propositions $\{\mathbf{P}\}$ onto real numbers in the interval $[0, 1]$.

The relation (25) says $\omega(A)$ is a value in $g^{-1}(\Delta)$ if and only if $\omega(g(A))$ is a value in Δ . So the long-run frequencies, and hence the probabilities, must match because the outcomes match on each occasion. It is clear, however, that the converse implication does not hold. We cannot deduce (25) from STAT FUNC (21). The match in long-run frequencies specified by STAT FUNC (21) in no way guarantees a match in outcome on each a particular occasion. On the other hand, (25) is equivalent to FUNC (8).

Theorem.

$$HV \wedge (25) \Leftrightarrow HV \wedge \text{FUNC (8)}. \quad (26)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. We first notice that FUNC (8) implies (25).

$$\omega(g(A)) \in \Delta \Leftrightarrow g(\omega(A)) \in \Delta \Leftrightarrow \omega(A) \in g^{-1}(\Delta), \quad (27)$$

which is another way of expressing the content of (25). Conversely, (25) implies FUNC (8). If (25) holds, we have

$$\omega(A) \in \Delta \Rightarrow \omega(A) \in g^{-1}(g(\Delta)) \Leftrightarrow \omega(g(A)) \in g(\Delta), \quad (28)$$

where we have used (25) taking $g(\Delta)$ as Δ . Now take $\Delta = \{\lambda\}$, which is a singleton set. Then we have

$$\omega(A) \in \{\lambda\} \Leftrightarrow \omega(A) = \lambda, \quad (29)$$

and, from $g(\{\lambda\}) = \{g(x) | x \in \{\lambda\}\} = \{g(\lambda)\}$ and (28),

$$\omega(g(A)) \in g(\{\lambda\}) (= \{g(\omega(A))\}) \Leftrightarrow \omega(g(A)) = g(\omega(A)), \quad (30)$$

which is FUNC (8). QED.

Now we summarize the inclusion relation provided in this section as follows:

$$\text{STAT FUNC (21)} \Leftarrow (25) \Leftrightarrow \text{FUNC (8)}. \quad (31)$$

Therefore, even though we get the refutation of the validity of FUNC (8), it does not imply the refutation of STAT FUNC (21).

Roughly speaking, the problem is as follows: STAT FUNC (21) can be consistent with the deterministic hidden variable interpretation of QM (i.e., HV). However, if we introduce an additional assumption (i.e., JD (41) holds), such an interpretation is not possible. Let us write STAT FUNC (21), simply, as $P(U) = P(V)$ where $U := (\Delta)_{\omega(g(A))}^{\psi}$ and $V := (g^{-1}(\Delta))_{\omega(A)}^{\psi}$. The assumption is related to the joint probability that the two propositions U and V hold simultaneously, i.e., $P(U \cap V)$. If we want to interpret QM by HV, we cannot assume that

$$P(U) = P(V) = P(U \cap V). \quad (32)$$

This cannot be consistent with QM. However we can derive (32) from HV under the condition of JD (41). (It follows from theorem (46) that $P(\overline{U} \cap V) = P(U \cap \overline{V}) = 0$ which is equivalent to (32). Here, the upper bar is for complements.) We shall review this argument in Sec. IV and Sec. V.

IV. IMPLICATIONS BETWEEN VARIOUS RULES

In this section, we review mutual implications between several assumptions about hidden variable theories along with Fine [21–23].

Definition: (*Almost everywhere*).

Let (Ω, Σ, μ) be a measure space and $P(\omega)$ be a proposition related to $\omega \in \Omega$. If there exists N such that

$$\{\omega | P(\omega) \text{ does not hold.}\} \subset N \wedge N \in \Sigma \wedge \mu(N) = 0, \quad (33)$$

then we say that $P(\omega)$ holds almost everywhere with respect to μ in Ω . That is, $P(\omega)$ thinking of μ almost everywhere holds. We abbreviate this as follows:

$$P(\omega), (\mu - a.e.). \quad (34)$$

In other words, (34) implies that $P(\omega)$ must hold except for all measurable subsets of Ω that have the probability measure of zero.

Definition: (*The probability distribution rule (D)*).

$$\mu_\psi(f_A^{-1}(\Delta)) := Prob(\Delta)_{\theta(A)}^\psi. \quad (35)$$

We call the hidden variable theory satisfying D (35) the weak hidden variable theory. Form this definition (35), it is required that the value of $f_I(\omega)$ is 1 almost everywhere, i.e., $f_I(\omega) = 1, (\mu_\psi - a.e.)$. Here, I represents the identity operator in an arbitrary finite-dimensional space.

Let S_A stand for the spectrum of the Hermitian operator A .

Theorem. If

$$\begin{aligned} Tr[\psi A] &= \sum_{y \in S_A} Prob(\{y\})_{\theta(A)}^\psi y, \\ E_\psi(A) &:= \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega), \end{aligned} \quad (36)$$

then

$$HV \wedge D (35) \Rightarrow Tr[\psi A] = E_\psi(A). \quad (37)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Note

$$\begin{aligned} \omega \in f_A^{-1}(\{y\}) &\Leftrightarrow f_A(\omega) \in \{y\} \Leftrightarrow y = f_A(\omega), \\ \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} &= 1, \\ y \neq y' &\Rightarrow f_A^{-1}(\{y\}) \cap f_A^{-1}(\{y'\}) = \emptyset. \end{aligned} \quad (38)$$

Hence we have

$$\begin{aligned} Tr[\psi A] &= \sum_{y \in S_A} Prob(\{y\})_{\theta(A)}^\psi y \\ &= \sum_{y \in \mathbf{R}} Prob(\{y\})_{\theta(A)}^\psi y \\ &= \sum_{y \in \mathbf{R}} \mu_\psi(f_A^{-1}(\{y\})) y \\ &= \sum_{y \in \mathbf{R}} \mu_\psi(f_A^{-1}(\{y\})) y \times \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} \\ &= \sum_{y \in \mathbf{R}} \int_{\omega \in f_A^{-1}(\{y\})} \mu_\psi(f_A^{-1}(\{y\})) \times \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} f_A(\omega) \\ &= \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega) = E_\psi(A). \end{aligned} \quad (39)$$

QED.

Definition: (*The quantum mechanical joint probability distribution for commuting observables (QJD)*).

$$Prob(\Delta, \Delta')_{\theta(A), \theta(B)}^\psi := Tr[\psi \chi_\Delta(A) \chi_{\Delta'}(B)], \quad (40)$$

where the notation on the LHS of QJD (40) is a generalization of the symbol $(\Delta)_{\theta(A)}^\psi$ to express the proposition that measurement results of A and B will lie in the sets Δ and Δ' , respectively. We shall use QJD (40) to define the quantum mechanical joint probability distribution for commuting observables.

Definition: (*The joint probability distribution rule (JD)*).

$$\mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')) := Prob(\Delta, \Delta')_{\theta(A), \theta(B)}^\psi, \quad (41)$$

for all commuting pairs A, B in $\mathcal{O}$.

We shall use JD (41) to define the hidden variable theoretical joint probability distribution for commuting observables.

Theorem.

$$\begin{aligned} \text{QJD (40)} &\Rightarrow \text{BSF (19)}, \\ \text{HV} \wedge \text{JD (41)} &\Rightarrow \text{HV} \wedge \text{D (35)}. \end{aligned} \quad (42)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Let Δ' be $\mathbf{R}$ in QJD (40) and JD (41). Then we have

$$\begin{aligned} \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\mathbf{R})) &= Prob(\Delta, \mathbf{R})_{\theta(A), \theta(B)}^\psi = Tr[\psi \chi_\Delta(A) \chi_{\mathbf{R}}(B)] \\ \Leftrightarrow \mu_\psi(f_A^{-1}(\Delta)) &= Prob(\Delta)_{\theta(A)}^\psi = Tr[\psi \chi_\Delta(A)]. \end{aligned} \quad (43)$$

QED.

Definition: (*The functional rule holding almost everywhere (FUNC A.E.)*).

$$f_{g(A)}(\omega) = g(f_A(\omega)), (\mu_\psi - a.e.). \quad (44)$$

That is

$$\mu_\psi(\{\omega | f_{g(A)}(\omega) \neq g(f_A(\omega))\}) = 0. \quad (45)$$

Theorem.

$$\text{HV} \wedge \text{JD (41)} \Rightarrow \text{HV} \wedge \text{D (35)} \wedge \text{FUNC A.E. (44)}. \quad (46)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Suppose we have JD (41) for all commuting pairs A, B in $\mathcal{O}$. Let y be any real number, and let $S := \{\omega | f_{g(A)}(\omega) = y\}$ and $T := \{\omega | g(f_A(\omega)) = y\}$. We want $\mu_\psi(\overline{S} \cap T) = \mu_\psi(S \cap \overline{T}) = 0$. This will follow if we have $\mu_\psi(S) = \mu_\psi(T) = \mu_\psi(S \cap T)$ since

$$\begin{aligned} \mu_\psi(S \cap \overline{T}) + \mu_\psi(S \cap T) &= \mu_\psi(S), \\ \mu_\psi(\overline{S} \cap T) + \mu_\psi(S \cap T) &= \mu_\psi(T). \end{aligned} \quad (47)$$

Note

$$\omega \in f_{g(A)}^{-1}(\{y\}) \Leftrightarrow f_{g(A)}(\omega) \in \{y\} \Leftrightarrow y = f_{g(A)}(\omega), \quad (48)$$

and

$$\omega \in f_A^{-1}(g^{-1}(\{y\})) \Leftrightarrow f_A(\omega) \in g^{-1}(\{y\}) \Leftrightarrow g(f_A(\omega)) \in \{y\} \Leftrightarrow y = g(f_A(\omega)). \quad (49)$$

Theorem (42) says that JD (41) yields D (35). Then, from (24), we have

$$\begin{aligned} \mu_\psi(T) &= \mu_\psi(\{\omega | \omega \in f_A^{-1}(g^{-1}(\{y\}))\}) = Prob(g^{-1}(\{y\}))_{\theta(A)}^\psi = Prob(\{y\})_{\theta(g(A))}^\psi \\ &= \mu_\psi(\{\omega | \omega \in f_{g(A)}^{-1}(\{y\})\}) = \mu_\psi(S). \end{aligned} \quad (50)$$

Using the spectral representation of A , it follows that $\chi_\Delta(A) \chi_{g(\Delta)}(g(A)) = \chi_\Delta(A)$ for any set Δ , where $g(\Delta) = \{g(x) | x \in \Delta\}$. Because, $\chi_\Delta(z) = 1 \Leftrightarrow z \in \Delta \Rightarrow g(z) \in g(\Delta) \Leftrightarrow \chi_{g(\Delta)}(g(z)) = 1$ holds ($z \in \mathbf{R}$). Hence,

$$Prob(\Delta, g(\Delta))_{\theta(A), \theta(g(A))}^\psi = Tr[\psi \chi_\Delta(A) \chi_{g(\Delta)}(g(A))] = Tr[\psi \chi_\Delta(A)] = Prob(\Delta)_{\theta(A)}^\psi. \quad (51)$$

On the other hand, we have $g(g^{-1}(\Delta)) = \Delta$ because $g(g^{-1}(\Delta)) = \{g(x)|x \in g^{-1}(\Delta)\} = \{g(x)|g(x) \in \Delta\} = \Delta$. Therefore, taking $g^{-1}(\{y\})$ as Δ , we have

$$\text{Prob}(g^{-1}(\{y\}), \{y\})_{\theta(A), \theta(g(A))}^{\psi} = \text{Prob}(g^{-1}(\{y\}))_{\theta(A)}^{\psi} = \mu_{\psi}(T). \quad (52)$$

But, from JD (41) we have

$$\text{Prob}(g^{-1}(\{y\}), \{y\})_{\theta(A), \theta(g(A))}^{\psi} = \mu_{\psi}(f_A^{-1}(g^{-1}(\{y\})) \cap f_{g(A)}^{-1}(\{y\})) = \mu_{\psi}(T \cap S). \quad (53)$$

QED.

Definition: (*The product rule holding almost everywhere* (PROD A.E.)).

If Hermitian operators A and B commute, then

$$f_{A \cdot B}(\omega) = f_A(\omega) \cdot f_B(\omega), (\mu_{\psi} - a.e.). \quad (54)$$

Theorem.

$$\text{HV} \wedge \text{FUNC A.E. (44)} \Rightarrow \text{HV} \wedge \text{PROD A.E. (54)}. \quad (55)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Suppose now that A and B are two commuting Hermitian operators. In Sec. II, we discuss that we can define functions j and k and there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. So we have the following:

$$\begin{aligned} f_{A \cdot B}(\omega) &= f_{h(O)}(\omega) = h(f_O(\omega)) = j(f_O(\omega)) \cdot k(f_O(\omega)) \\ &= f_{j(O)}(\omega) \cdot f_{k(O)}(\omega) = f_A(\omega) \cdot f_B(\omega), (\mu_{\psi} - a.e.), \end{aligned} \quad (56)$$

where FUNC A.E. (44) is used. QED.

Lemma. If

$$\mu_{\psi}(\overline{S} \cap S') = \mu_{\psi}(\overline{S'} \cap S) = \mu_{\psi}(\overline{T} \cap T') = \mu_{\psi}(\overline{T'} \cap T) = 0, \quad (57)$$

then

$$\mu_{\psi}(S \cap T) = \mu_{\psi}(S' \cap T'). \quad (58)$$

Proof. Note

$$\begin{aligned} \mu_{\psi}(\overline{S} \cap S' \cap T) + \mu_{\psi}(S \cap S' \cap T) &= \mu_{\psi}(S' \cap T), \\ \mu_{\psi}(\overline{S'} \cap S \cap T) + \mu_{\psi}(S \cap S' \cap T) &= \mu_{\psi}(S \cap T). \end{aligned} \quad (59)$$

If the following relation holds

$$\mu_{\psi}(\overline{S} \cap S') = \mu_{\psi}(\overline{S'} \cap S) = 0, \quad (60)$$

then

$$\mu_{\psi}(\overline{S} \cap S' \cap T) = \mu_{\psi}(\overline{S'} \cap S \cap T) = 0. \quad (61)$$

Therefore, from (59), we have

$$\mu_{\psi}(S \cap S' \cap T) = \mu_{\psi}(S' \cap T) = \mu_{\psi}(S \cap T). \quad (62)$$

Similar to the argument by changing S to T , S' to T' , and T to S' , we get

$$\mu_{\psi}(T \cap T' \cap S') = \mu_{\psi}(T' \cap S') = \mu_{\psi}(T \cap S'). \quad (63)$$

From the relations (62) and (63), we conclude

$$\mu_{\psi}(T \cap S) = \mu_{\psi}(T' \cap S'). \quad (64)$$

QED.

Lemma.

$$\text{HV} \wedge \text{PROD A.E. (54)} \Rightarrow f_{\chi_{\Delta}(A)}(\omega) \in \{0, 1\}, (\mu_{\psi} - a.e.). \quad (65)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. By PROD A.E. (54), we have

$$f_{\chi_{\Delta}(A)}(\omega) \cdot f_{\chi_{\Delta}(A)}(\omega) = f_{\chi_{\Delta}(A)}(\omega), (\mu_{\psi} - a.e.) \Leftrightarrow f_{\chi_{\Delta}(A)}(\omega) \in \{0, 1\}, (\mu_{\psi} - a.e.). \quad (66)$$

QED.

Theorem.

$$\text{HV} \wedge \text{D (35)} \wedge \text{PROD A.E. (54)} \Rightarrow \text{HV} \wedge \text{JD (41)} \quad (67)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Suppose $[A, B] = \mathbf{0}$ holds. It follows from QJD (40), BSF (19), and D (35) that

$$\begin{aligned} & \text{Prob}(\Delta, \Delta')_{\theta(A), \theta(B)}^{\psi} \\ &= \text{Tr}[\psi \chi_{\Delta}(A) \chi_{\Delta'}(B)] (\text{see(40)}) \\ &= \text{Tr}[\psi \chi_{\{1\}}(\chi_{\Delta}(A) \chi_{\Delta'}(B))] \\ &= \text{Prob}(\{1\})_{\theta(\chi_{\Delta}(A) \chi_{\Delta'}(B))}^{\psi} (\text{see(19)}) \\ &= \mu_{\psi}(f_{\chi_{\Delta}(A) \chi_{\Delta'}(B)}^{-1}(\{1\})) (\text{see(35)}). \end{aligned} \quad (68)$$

PROD A.E. (54) and the lemma (65) say that

$$\begin{aligned} (68) &= \mu_{\psi}(\{\omega | \omega \in f_{\chi_{\Delta}(A) \chi_{\Delta'}(B)}^{-1}(\{1\})\}) \\ &= \mu_{\psi}(\{\omega | f_{\chi_{\Delta}(A) \chi_{\Delta'}(B)}(\omega) = 1\}) \\ &= \mu_{\psi}(\{\omega | f_{\chi_{\Delta}(A)}(\omega) \cdot f_{\chi_{\Delta'}(B)}(\omega) = 1\}) (\text{see(54)}) \\ &= \mu_{\psi}(\{\omega | f_{\chi_{\Delta}(A)}(\omega) = f_{\chi_{\Delta'}(B)}(\omega) = 1\}) (\text{see(65)}) \\ &= \mu_{\psi}(f_{\chi_{\Delta}(A)}^{-1}(\{1\}) \cap f_{\chi_{\Delta'}(B)}^{-1}(\{1\})). \end{aligned} \quad (69)$$

On the other hand, we have

$$\begin{aligned} & \mu_{\psi}(f_{\chi_{\Delta}(A)}^{-1}(\{1\}) \cap f_A^{-1}(\Delta)) \\ &= \mu_{\psi}(\{\omega | f_{\chi_{\Delta}(A)}(\omega) = 1 \wedge f_A(\omega) \in \Delta\}) \\ &= \mu_{\psi}(\{\omega | f_{\chi_{\Delta}(A)}(\omega) \cdot f_A(\omega) \in \Delta\}) \\ &= \mu_{\psi}(\{\omega | f_{\chi_{\Delta}(A) \cdot A}(\omega) \in \Delta\}) (\text{see(54)}) \\ &= \mu_{\psi}(f_{\chi_{\Delta}(A) \cdot A}^{-1}(\Delta)) \\ &= \text{Prob}(\Delta)_{\theta(\chi_{\Delta}(A) \cdot A)}^{\psi} (\text{see(35)}) \\ &= \text{Tr}[\psi \chi_{\Delta}(\chi_{\Delta}(A) \cdot A)] (\text{see(19)}) \\ &= \text{Tr}[\psi \chi_{\Delta}(A)]. \end{aligned} \quad (70)$$

We also obtain

$$\mu_{\psi}(f_{\chi_{\Delta}(A)}^{-1}(\{1\})) = \text{Tr}[\psi \chi_{\{1\}}(\chi_{\Delta}(A))] = \text{Tr}[\psi \chi_{\Delta}(A)] = \mu_{\psi}(f_A^{-1}(\Delta)). \quad (71)$$

Note, (see (47))

$$\mu_{\psi}(S \cap T) = \mu_{\psi}(S) = \mu_{\psi}(T) \Leftrightarrow \mu_{\psi}(S \cap \overline{T}) = \mu_{\psi}(\overline{S} \cap T) = 0. \quad (72)$$

Therefore, from Eq. (70) and Eq. (71), we have

$$\mu_{\psi}(f_{\chi_{\Delta}(A)}^{-1}(\{1\}) \cap \overline{f_A^{-1}(\Delta)}) = \mu_{\psi}(\overline{f_{\chi_{\Delta}(A)}^{-1}(\{1\})} \cap f_A^{-1}(\Delta)) = 0. \quad (73)$$

Similarly we can get

$$\begin{aligned} & \mu_{\psi}(f_{\chi_{\Delta'}(B)}^{-1}(\{1\}) \cap f_B^{-1}(\Delta')) = \text{Tr}[\psi \chi_{\Delta'}(B)], \\ & \mu_{\psi}(f_{\chi_{\Delta'}(B)}^{-1}(\{1\})) = \mu_{\psi}(f_B^{-1}(\Delta')) = \text{Tr}[\psi \chi_{\Delta'}(B)], \end{aligned} \quad (74)$$

and we have

$$\mu_{\psi}(f_{\chi_{\Delta'}(B)}^{-1}(\{1\}) \cap \overline{f_B^{-1}(\Delta')}) = \mu_{\psi}(\overline{f_{\chi_{\Delta'}(B)}^{-1}(\{1\})} \cap f_B^{-1}(\Delta')) = 0. \quad (75)$$

Hence, from the lemma (58), we have

$$\mu_\psi(f_{\chi_\Delta(A)}^{-1}(\{1\}) \cap f_{\chi_{\Delta'}(B)}^{-1}(\{1\})) = \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')). \quad (76)$$

Therefore, from (69), we conclude

$$Prob(\Delta, \Delta')_{\theta(A), \theta(B)}^\psi = \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')), \quad (77)$$

which is JD (41). QED.

Now we summarize the inclusion implication provided in this section as follows:

$$\begin{aligned} & \text{HV} \wedge \text{JD (41)} \\ & \Leftrightarrow \text{HV} \wedge \text{D (35)} \wedge \text{FUNC A.E. (44)} \\ & \Leftrightarrow \text{HV} \wedge \text{D (35)} \wedge \text{PROD A.E. (54)}. \end{aligned} \quad (78)$$

In what follows, we present two corollaries:

Corollary.

$$\text{HV} \wedge \text{FUNC A.E. (44)} \Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{g(A)}(\omega) \cdot K(\omega) = \int_{\omega \in \Omega} \mu_\psi(d\omega) g(f_A(\omega)) \cdot K(\omega), \quad (79)$$

for an arbitrary measurable function $K : \omega \mapsto \mathbf{R}$ and an arbitrary state ψ .

Corollary.

$$\text{HV} \wedge \text{PROD A.E. (54)} \wedge [A, B] = \mathbf{0} \Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{AB}(\omega) \cdot K(\omega) = \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega) \cdot f_B(\omega) \cdot K(\omega), \quad (80)$$

for an arbitrary measurable function $K : \omega \mapsto \mathbf{R}$ and an arbitrary state ψ .

In the following, we review the derivation of JD (41) from FUNC A.E. (44).

Theorem.

$$\text{HV} \wedge \text{D (35)} \wedge \text{FUNC A.E. (44)} \Rightarrow \text{HV} \wedge \text{JD (41)} \quad (81)$$

Proof. Assume we accept HV. Then there should exist a preexistence value possessed by each a Hermitian operator. Suppose $[A, B] = \mathbf{0}$ holds. It follows from BSF (19), QJD (40), D (35), FUNC A.E. (44), and PROD A.E. (54) that

$$\begin{aligned} & Prob(\Delta, \Delta')_{\theta(A), \theta(B)}^\psi \\ &= Tr[\psi \chi_\Delta(A) \chi_{\Delta'}(B)] (see(40)) \\ &= Tr[\psi \chi_{\{1\}}(\chi_\Delta(A) \chi_{\Delta'}(B))] \\ &= Prob(\{1\})_{\theta(\chi_\Delta(A) \chi_{\Delta'}(B))}^\psi (see(19)) \\ &= \mu_\psi(f_{\chi_\Delta(A) \chi_{\Delta'}(B)}^{-1}(\{1\})) (see(35)) \\ &= \mu_\psi(\{\omega | \omega \in f_{\chi_\Delta(A) \chi_{\Delta'}(B)}^{-1}(\{1\})\}) \\ &= \mu_\psi(\{\omega | f_{\chi_\Delta(A) \chi_{\Delta'}(B)}(\omega) = 1\}) \\ &= \mu_\psi(\{\omega | f_{\chi_\Delta(A)}(\omega) \cdot f_{\chi_{\Delta'}(B)}(\omega) = 1\}) (see(54)) \\ &= \mu_\psi(\{\omega | \chi_\Delta(f_A(\omega)) \cdot \chi_{\Delta'}(f_B(\omega)) = 1\}) (see(44)) \\ &= \mu_\psi(\{\omega | \chi_\Delta(f_A(\omega)) = \chi_{\Delta'}(f_B(\omega)) = 1\}) \\ &= \mu_\psi(\{\omega | f_A(\omega) \in \Delta \wedge f_B(\omega) \in \Delta'\}) \\ &= \mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')). \end{aligned} \quad (82)$$

QED.

V. THE STATISTICAL KOCHEN-SPECKER THEOREM

We want to modify the KS arguments provided in Sec. II so as to relax the assumptions. In fact, VR and FUNC (8) are relaxed into D (35) and FUNC A.E. (44), respectively. In what follows, we assume HV and JD (41) hold. This implies that we can use D (35), FUNC A.E. (44), and PROD A.E. (54).

We shall modify the KS theorem presented in Sec. II for two spin-1/2 systems. Let us try an attempt to assign functions $f : \Omega \mapsto \mathbf{R}$ to the nine operators $\sigma_x^1 \sigma_y^2, \sigma_y^1 \sigma_x^2, \sigma_x^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_z^1 \sigma_z^2, \sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2$ for an arbitrary state ψ . We define $X(\omega)$ as follows: Then, one can see that

$$\begin{aligned} X(\omega) &:= f_{\sigma_x^1 \sigma_x^2}(\omega) f_{\sigma_y^1 \sigma_y^2}(\omega) f_{\sigma_z^1 \sigma_z^2}(\omega) \\ &= f_{\sigma_x^1 \sigma_x^2 \sigma_y^1 \sigma_y^2 \sigma_z^1 \sigma_z^2}(\omega) = f_{-I}(\omega), (\mu_\psi - a.e.) \\ &\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega) X(\omega) = E_\psi(-I) = -1, \end{aligned} \quad (83)$$

where I represents the identity operator in a four-dimensional space. By the way we can factorize two of the terms as $f_{\sigma_x^1 \sigma_x^2} = f_{\sigma_x^1} f_{\sigma_x^2}$ and $f_{\sigma_y^1 \sigma_y^2} = f_{\sigma_y^1} f_{\sigma_y^2}$ almost everywhere with respect to μ_ψ . Further, we have $f_{\sigma_x^1 \sigma_y^2} = f_{\sigma_x^1} f_{\sigma_y^2}$ and $f_{\sigma_y^1 \sigma_x^2} = f_{\sigma_y^1} f_{\sigma_x^2}$ almost everywhere with respect to μ_ψ . Hence we get $f_{\sigma_x^1 \sigma_x^2} f_{\sigma_y^1 \sigma_y^2} = f_{\sigma_x^1 \sigma_y^2} f_{\sigma_y^1 \sigma_x^2}$ almost everywhere with respect to μ_ψ and

$$\begin{aligned} X(\omega) &= f_{\sigma_x^1 \sigma_x^2}(\omega) f_{\sigma_y^1 \sigma_y^2}(\omega) f_{\sigma_z^1 \sigma_z^2}(\omega) \\ &= f_{\sigma_x^1 \sigma_y^2}(\omega) f_{\sigma_y^1 \sigma_x^2}(\omega) f_{\sigma_z^1 \sigma_z^2}(\omega) \\ &= f_{\sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 \sigma_z^1 \sigma_z^2}(\omega) = f_I(\omega), (\mu_\psi - a.e.) \\ &\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega) X(\omega) = E_\psi(I) = 1. \end{aligned} \quad (84)$$

Noting the relation (83), we see that an attempt to assign functions to the above nine operators for an arbitrary state ψ should fail.

We modify another version of the KS theorem presented in Sec. II for three spin-1/2 systems. Let us try an attempt to assign functions to the ten operators $\sigma_x^1 \sigma_y^2 \sigma_y^3, \sigma_y^1 \sigma_x^2 \sigma_y^3, \sigma_y^1 \sigma_y^2 \sigma_x^3, \sigma_x^1 \sigma_x^2 \sigma_x^3, \sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2, \sigma_x^3, \sigma_y^3$ for an arbitrary state ψ . We define $Y(\omega)$ as follows: Then, one can see that

$$\begin{aligned} Y(\omega) &:= f_{\sigma_x^1 \sigma_y^2 \sigma_y^3}(\omega) f_{\sigma_y^1 \sigma_x^2 \sigma_y^3}(\omega) f_{\sigma_y^1 \sigma_y^2 \sigma_x^3}(\omega) f_{\sigma_x^1 \sigma_x^2 \sigma_x^3}(\omega) \\ &= f_{\sigma_x^1 \sigma_y^2 \sigma_y^3 \sigma_y^1 \sigma_x^2 \sigma_y^3 \sigma_y^1 \sigma_y^2 \sigma_x^3 \sigma_x^1 \sigma_x^2 \sigma_x^3}(\omega) \\ &= f_{-I}(\omega), (\mu_\psi - a.e.) \\ &\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega) Y(\omega) = E_\psi(-I) = -1, \end{aligned} \quad (85)$$

where I represents the identity operator in an eight-dimensional space. By the way, we can factorize each of the four terms almost everywhere with respect to μ_ψ as

$$\begin{aligned} f_{\sigma_x^1 \sigma_y^2 \sigma_y^3}(\omega) &= f_{\sigma_x^1}(\omega) f_{\sigma_y^2}(\omega) f_{\sigma_y^3}(\omega), \\ f_{\sigma_y^1 \sigma_x^2 \sigma_y^3}(\omega) &= f_{\sigma_y^1}(\omega) f_{\sigma_x^2}(\omega) f_{\sigma_y^3}(\omega), \\ f_{\sigma_y^1 \sigma_y^2 \sigma_x^3}(\omega) &= f_{\sigma_y^1}(\omega) f_{\sigma_y^2}(\omega) f_{\sigma_x^3}(\omega), \\ f_{\sigma_x^1 \sigma_x^2 \sigma_x^3}(\omega) &= f_{\sigma_x^1}(\omega) f_{\sigma_x^2}(\omega) f_{\sigma_x^3}(\omega), (\mu_\psi - a.e.). \end{aligned} \quad (86)$$

From PROD A.E. (54), we have $(f_{\sigma_k^j}(\omega))^2 = f_I(\omega), (\mu_\psi - a.e.), (j = 1, 2, 3, k = x, y)$. Here, σ_k^1 means $\sigma_k^1 \otimes I^2 \otimes I^3$ and so on. Omitting two identity operators in a two-dimensional space, we abbreviate those as above. Thus, we get

$$\begin{aligned} Y(\omega) &= f_{\sigma_x^1 \sigma_y^2 \sigma_y^3}(\omega) f_{\sigma_y^1 \sigma_x^2 \sigma_y^3}(\omega) f_{\sigma_y^1 \sigma_y^2 \sigma_x^3}(\omega) f_{\sigma_x^1 \sigma_x^2 \sigma_x^3}(\omega) \\ &= (f_{\sigma_x^1}(\omega))^2 (f_{\sigma_y^1}(\omega))^2 (f_{\sigma_x^2}(\omega))^2 (f_{\sigma_y^2}(\omega))^2 (f_{\sigma_x^3}(\omega))^2 (f_{\sigma_y^3}(\omega))^2 \\ &= f_I(\omega) f_I(\omega) f_I(\omega) f_I(\omega) f_I(\omega) f_I(\omega) = f_I(\omega), (\mu_\psi - a.e.) \\ &\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega) Y(\omega) = E_\psi(I) = 1. \end{aligned} \quad (87)$$

Noting (85), we see that an attempt to assign functions to the above ten operators for an arbitrary state ψ should fail.

These two examples provide a statistical form of the KS theorem which says demolition of either the existence of hidden states or the validity of JD (41). That is, there does not exist any hidden variable theory satisfying JD (41). Noting,

$$\text{HV} \wedge \text{JD (41)} \Leftrightarrow \text{HV} \wedge \text{D (35)} \wedge \text{FUNC A.E. (44)} \quad (88)$$

VR and FUNC (8) have been relaxed into D (35) and FUNC A.E. (44), respectively. Further, they are of the state-independent form. The quantum state in question can be the thermal state I/d , where the number of the dimension is d , in order to refute any hidden variable theory satisfying JD (41). We have the following result:

Theorem: (*The statistical Kochen-Specker theorem*).

For every quantum state described in a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ or $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, ($\text{Dim}(\mathcal{H}_j) = 2, (j = 1, 2, 3)$),

$$\text{HV} \wedge \text{JD} (41) \Rightarrow \perp, \quad (89)$$

where $\perp$ means Contradiction. That is, these two assumptions do not hold at the same time.

These examples are sufficient to show that, if we accept JD (41), HV cannot be possible for any state. They are not the suitable form for testing experimentally the KS theorem. Because, in a real experiment, we cannot claim a sharp value as an expected value with arbitrary precision. Therefore, the inequality form for testing the KS theorem is useful [12].

VI. THE STATISTICAL KOCHEN-SPECKER ASSUMPTION AND AN ADDITIONAL RULE

We suppose that both of the identity operator and the projection operator can have a preexistence value as 1. We measure them twice (A and B , where $A = B$) in a specific quantum state $|1\rangle (= \psi)$. In the following discussions, we assume ψ is the specific quantum state $|1\rangle$. And the statistical and state-dependent Kochen-Specker theorem is considered.

First, we newly define the functional rule almost everywhere with respect to μ_ψ as:

$$f(g(O)) = g(f(O)), (\mu_\psi - a.e.) \quad (90)$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. We introduce the above assumption (the functional rule) for hidden variable theories. We want to investigate if the algebraic structure of quantum mechanical operators could be mirrored in the algebraic structure of the preexistence value of the operators. Second, the sum rule almost everywhere with respect to μ_ψ is newly defined as:

$$f(A + B) = f(A) + f(B), (\mu_\psi - a.e.) \quad (91)$$

where A, B are two Hermitian operators. Thirdly, the product rule almost everywhere with respect to μ_ψ is newly defined as:

$$f(A \cdot B) = f(A) \cdot f(B), (\mu_\psi - a.e.). \quad (92)$$

And an additional rule almost everywhere with respect to μ_ψ is proposed as:

$$f(A) \cdot f(B) = f(A) + f(B), (\mu_\psi - a.e.). \quad (93)$$

We have [4, 12, 14] the following implications between the rules:

The functional rule $\Rightarrow$ The sum rule

$$\text{The functional rule} \Rightarrow \text{The product rule} \quad (94)$$

and

$$\text{The functional rule} \wedge \text{Three definitions} \Rightarrow \text{An additional rule} \quad (95)$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. In [4, 12, 14], we discuss that we can define functions j and k and there exists a nondegenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned} f(A \cdot B) &= f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O)) \\ &= f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B), (\mu_\psi - a.e.) \end{aligned} \quad (96)$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned} f(A + B) &= f(l(O)) = l(f(O)) = j(f(O)) + k(f(O)) \\ &= f(j(O)) + f(k(O)) = f(A) + f(B), (\mu_\psi - a.e.) \end{aligned} \quad (97)$$

where we use the functional rule. Additionally, we may introduce a function m , if the following definitions are satisfied, such that $m(l(O)) := h(O)$, $m(j(O)) := j(O)$, and $m(k(O)) := k(O)$. These three definitions imply the following three definitions:

1. $m(l(O)) := h(O)$, that is, $m(A + B) := A \cdot B$.
2. $m(j(O)) := j(O)$, that is, $m(A) := A$.
3. $m(k(O)) := k(O)$, that is, $m(B) := B$.

For the first example, we may have

$$A = B = H_3(0) = |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (98)$$

The definition $m(A + B) := A \cdot B$ is satisfied when $m(1 + 1) = 1 \cdot 1$ and $m(0 + 0) = 0 \cdot 0$, that is, $m(2) = 1$ and $m(0) = 0$. The definition $m(A) := A$ is satisfied when $m(1) = 1$ and $m(0) = 0$. The definition $m(B) := B$ is satisfied when $m(1) = 1$ and $m(0) = 0$. Hence, we take the function m as $m(2) = 1$, $m(0) = 0$, and $m(1) = 1$. Thus we can introduce the function m that satisfies the three requirement definitions. Then we have an additional rule as:

$$\begin{aligned} f(A) \cdot f(B) &= f(A \cdot B) = f(h(O)) = f(m(l(O))) \\ &= l(f(m(O))) \\ &= j(f(m(O))) + k(f(m(O))) \\ &= f(m(j(O))) + f(m(k(O))) \\ &= f(A) + f(B), (\mu_\psi - a.e.) \end{aligned} \quad (99)$$

where we use the functional rule and the product rule. In fact, the sum rule is equivalent to the product rule for this specific case. Thus we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication when $A = B = |1\rangle\langle 1|$ and the quantum state is $|1\rangle$.

For the next example, we may have

$$A = B = H_3(1) = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (100)$$

The definition $m(A + B) := A \cdot B$ is satisfied when $m(1 + 1) = 1 \cdot 1$, that is, $m(2) = 1$. The definition $m(A) := A$ is satisfied when $m(1) = 1$. The definition $m(B) := B$ is satisfied when $m(1) = 1$. Hence, we take the function m as $m(2) = 1$ and $m(1) = 1$. Thus we can introduce the function m that satisfies the three requirement definitions. Then we have the additional rule (93). In fact, the sum rule is equivalent to the product rule for two identity operators also. Thus we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication when $A = B = H_3(1)$, where $H_3(1)$ is the three-dimensional identity operator and the quantum state is $|1\rangle$.

For the thirdly, we may have

$$A = B = \begin{pmatrix} 1 - \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (101)$$

The value of ϵ , ($0 \leq \epsilon < 1$) is interpreted as the reduction factor of the contrast observed in a single-particle experiment. Hence we have

$$A + B = \begin{pmatrix} 2(1 - \epsilon) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A \cdot B = \begin{pmatrix} (1 - \epsilon)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (102)$$

The definition $m(A + B) := A \cdot B$ is satisfied when $m((1 - \epsilon) + (1 - \epsilon)) = (1 - \epsilon) \cdot (1 - \epsilon)$ and $m(0 + 0) = 0 \cdot 0$, that is, $m(2(1 - \epsilon)) = (1 - \epsilon)^2$ and $m(0) = 0$. The definition $m(A) := A$ is satisfied when $m(1 - \epsilon) = 1 - \epsilon$ and $m(0) = 0$. The definition $m(B) := B$ is satisfied when $m(1 - \epsilon) = 1 - \epsilon$ and $m(0) = 0$. Hence, we take the function m as $m(2(1 - \epsilon)) = (1 - \epsilon)^2$, $m(1 - \epsilon) = 1 - \epsilon$, and $m(0) = 0$. Thus we can introduce the function m that satisfies the three requirement definitions. Then we have the additional rule (93).

As the final example, we may have

$$A = B = \begin{pmatrix} 1 - \epsilon & 0 & 0 \\ 0 & 1 - \epsilon & 0 \\ 0 & 0 & 1 - \epsilon \end{pmatrix}. \quad (103)$$

Hence we have

$$A + B = \begin{pmatrix} 2(1-\epsilon) & 0 & 0 \\ 0 & 2(1-\epsilon) & 0 \\ 0 & 0 & 2(1-\epsilon) \end{pmatrix} \text{ and } A \cdot B = \begin{pmatrix} (1-\epsilon)^2 & 0 & 0 \\ 0 & (1-\epsilon)^2 & 0 \\ 0 & 0 & (1-\epsilon)^2 \end{pmatrix}. \quad (104)$$

The definition $m(A+B) := A \cdot B$ is satisfied when $m((1-\epsilon)+(1-\epsilon)) = (1-\epsilon) \cdot (1-\epsilon)$, that is, $m(2(1-\epsilon)) = (1-\epsilon)^2$. The definition $m(A) := A$ is satisfied when $m(1-\epsilon) = 1-\epsilon$. The definition $m(B) := B$ is satisfied when $m(1-\epsilon) = 1-\epsilon$. Hence, we take the function m as $m(2(1-\epsilon)) = (1-\epsilon)^2$ and $m(1-\epsilon) = 1-\epsilon$. Thus we can introduce the function m that satisfies the three requirement definitions. Then we have the additional rule (93).

Hence, we understand, for the four specific examples, we may have the additional rule (93) as follows:

$$f(A) \cdot f(B) = f(A) + f(B), (\mu_\psi - a.e.). \quad (105)$$

In what follows, we consider only the following cases including the case of $\epsilon = 0$:

Theorem.

The operation Addition ($f(A) + f(B)$) is equal to the operation Multiplication ($f(A) \cdot f(B)$) when $A = B = \begin{pmatrix} 1-\epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1-\epsilon & 0 & 0 \\ 0 & 1-\epsilon & 0 \\ 0 & 0 & 1-\epsilon \end{pmatrix}$ and the quantum state is $|1\rangle$ that is a simultaneous eigenstate of A, B .

The set for an additional rule is different from our common set in which Newton's mechanics is created. In the set holding the additional rule, the operation Addition and the operation Multiplication are the same as each other. In the normal set where Newton's mechanics is held, the operation Addition and the operation Multiplication are different from each other.

VII. THE INCONSISTENCY WITH HIDDEN VARIABLE THEORIES

Let $H_3(x) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$ be a quantum operator described by a fixed constant $x \in \{0, 1\}$. We consider two patterns of the operator as:

1. $H_3(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (The one-dimensional projection operator in a three dimensional space).
2. $H_3(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (The three-dimensional identity operator).

Let $|1\rangle$ be an eigenstate of $H_3(x)$ such that $H_3(x)|1\rangle = 1|1\rangle$. The preexistence value of quantum measurement outcome is 1 in the ideal case.

We are in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The preexistence value is 1 by measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$. The two measurements of $H_3(x)$ are corresponding to the same x . It turns out that we cannot assign the preexistence value for quantum measurement outcome as 1 when measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$.

We consider a value V which is the sum of two preexistence values in a thought experiment. The preexistence value of outcome is 1. If the number of outcomes is two, then we have

$$V = 1 + 1 = 2. \quad (106)$$

We derive a value $V = 2$.

On the other hand, we can depict the preexistence values using v_1, v_2 as: $v_1 = 1$ and $v_2 = 1$. Let us write V as:

$$V = v_1 + v_2. \quad (107)$$

In the following, we evaluate another value for V . Then, we have, using an additional rule (93),

$$V = (v_1 + v_2) = (v_1 \times v_2) = 1, \quad (108)$$

where $f(A) + f(B) = f(A) \times f(B)$ is used. Here $f(\cdot) = \text{tr}[\rho(\cdot)]$ and the quantum state ρ is $|1\rangle\langle 1|$.

We cannot assign simultaneously the same two truth values (“1” and “1”) or (“0” and “0”) for the two propositions (106) and (108). We derive the inconsistency with hidden variable theories for quantum measurement outcome on $H_3(x)$. In the case that $x = 0$, the one-dimensional projection operator in a three dimensional space does not have a classical counterpart. In the case that $x = 1$, the three-dimensional identity operator does not have a classical counterpart. Our simplified version says that we cannot assign a preexistence value as 1 to both quantum operators (the projection operator and the identity operator) even though the assignment is independent of each other. Both of them do not have a classical counterpart.

Theorem: (*The statistical Kochen–Specker theorem based on the identity operator or the projection operator in a three-dimensional space*).

We are in the inconsistency with hidden variable theories when the following quantum measurement process happens twice: The preexistence value is 1 by measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$. The two measurements of $H_3(x)$ are corresponding to the same x . It turns out that we cannot assign the preexistence value for quantum measurement outcome as 1 when measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$.

VIII. THE STATISTICAL KOCHEN–SPECKER THEOREM BASED ON THE IDENTITY OPERATOR

We restrict ourselves to the identity operator. We want to modify the KS arguments provided in Sec. VII so as to relax the assumptions. In fact, VR and FUNC (8) are relaxed into D (35) and FUNC A.E. (44), respectively. In what follows, we assume HV and JD (41) hold. This implies that we can use D (35), FUNC A.E. (44), and PROD A.E. (54).

We shall modify the KS theorem presented in Sec. VII for the identity operator in a three-dimensional space (We assume $I = H_3(1)$). Let us try an attempt to assign a function $f : \Omega \mapsto \mathbf{R}$ to the identity operator in a three-dimensional space I in the quantum state $\psi (= |1\rangle)$. D (35) says $f_I(\omega) = 1, (\mu_\psi - a.e.)$. Then, one can see that

$$\sum_{\omega \in \{\omega_1, \omega_2\}} f_I(\omega) = f_I(\omega_1) + f_I(\omega_2) = 2, (\mu_\psi - a.e.). \quad (109)$$

By the way ($\epsilon = 0$) the operation Addition ($f(A) + f(B)$) is equal to the operation Multiplication ($f(A) \cdot f(B)$) almost everywhere with respect to μ_ψ when $A = B = I$ and the quantum state is ψ that is a simultaneous eigenstate of A, B . Hence we have the additional rule $f(A) \cdot f(B) = f(A) + f(B), (\epsilon = 0)$ almost everywhere with respect to μ_ψ . Hence we get $f_I(\omega_1) + f_I(\omega_2) = f_I(\omega_1) \times f_I(\omega_2)$ almost everywhere with respect to μ_ψ and

$$\sum_{\omega \in \{\omega_1, \omega_2\}} f_I(\omega) = f_I(\omega_1) + f_I(\omega_2) = f_I(\omega_1) \times f_I(\omega_2) = 1, (\mu_\psi - a.e.). \quad (110)$$

Noting the relation (109), we see that an attempt to assign function to the identity operator in a three-dimensional space in the quantum state $|1\rangle$ should fail.

IX. THE INCOMPLETENESS IN A REAL EXPERIMENT

In a real experiment, there are not perfect detectors, but the good ones with some errors. Also we consider an imperfect source. In this case, we increase measurement outcomes to an even number $2N (\gg 1)$ and then we change such errors into trivial things. In fact, such an error becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We suppose that the identity operator in a three-dimensional space can have a preexistence value for quantum measurement outcome as 1. We define the identity operator in a three-dimensional space $I (= H_3(1))$ as:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (111)$$

Let $|1\rangle$ be an eigenstate of I such that $I|1\rangle = 1|1\rangle$. The preexistence value for quantum measurement outcome is 1 in the ideal case.

Let us introduce the random noise admixture $\rho_{\text{noise}} (= \frac{1}{3}I)$ into the quantum state $|1\rangle$, where the operator I is the three-dimensional identity operator. We consider the noisy quantum state emerged from an imperfect source as:

$$\rho = (1 - \epsilon)|1\rangle\langle 1| + \epsilon \times \rho_{\text{noise}}. \quad (112)$$

The value of $\epsilon (< 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\text{tr}[\rho I] = 1 - \epsilon$. The preexistence value for quantum measurement outcome is $1 - \epsilon$.

However, the quantum state ρ is not an eigenstate of I because it is a mixed state. Thus we change the definition of the measured operator and of the quantum state as:

$$I' = \begin{pmatrix} 1 - \epsilon & 0 & 0 \\ 0 & 1 - \epsilon & 0 \\ 0 & 0 & 1 - \epsilon \end{pmatrix} \quad (113)$$

and

$$\rho' = |1\rangle\langle 1|. \quad (114)$$

Then the quantum state ρ' is an eigenstate of I' . Further we have $\text{tr}[\rho' I'] = 1 - \epsilon$. Also the preexistence value for quantum measurement outcome is $1 - \epsilon$.

We assume the number of outcomes is an even number $2N (\gg 1)$. The odd number preexistence values are $1 - \epsilon$ by measuring the operator I' in the quantum state ρ' , the even number preexistence values are the same $1 - \epsilon$ as by measuring the same operator I' in the same quantum state ρ' .

We consider the following value $V_i (i = 1, 2, \dots, N)$ which is the sum of two preexistence values in the thought experiment:

$$V_i = (1 - \epsilon) + (1 - \epsilon) = 2(1 - \epsilon). \quad (115)$$

We introduce the following function $S(N)$ which is the sum over i of V_i in the thought experiment:

$$S(N) = V_1 + V_2 + \dots + V_N. \quad (116)$$

If the number of outcomes is an even number $2N (\gg 1)$, then we have

$$\begin{aligned} S(N) &= V_1 + V_2 + \dots + V_N = N \left((1 - \epsilon) + (1 - \epsilon) \right) \\ &= N(2(1 - \epsilon)) = 2N(1 - \epsilon), \end{aligned} \quad (117)$$

where we use $V_i = (1 - \epsilon) + (1 - \epsilon) = 2(1 - \epsilon)$. We derive a function value $S(N)$.

On the other hand, we can depict the preexistence values using $v_1, v_2, v_3, \dots, v_{2N}$ as: $v_1 = 1 - \epsilon$, $v_2 = 1 - \epsilon$, $v_3 = 1 - \epsilon, \dots$, $v_{2N} = 1 - \epsilon$. We can write V_i as:

$$V_i = v_{2i-1} + v_{2i}, \quad (118)$$

where

$$v_{2i-1} = 1 - \epsilon, \quad v_{2i} = 1 - \epsilon. \quad (119)$$

Thus, we have

$$V_i = (1 - \epsilon) + (1 - \epsilon). \quad (120)$$

In the following we evaluate another value of V_i and derive another function value of $S(N)$. Then, we have, using the additional rule (93),

$$V_i = (v_{2i-1} + v_{2i}) = (v_{2i-1} \times v_{2i}) = (1 - \epsilon)^2, \quad (121)$$

where $f(A) \cdot f(B) = f(A) + f(B)$ is used. Here $f(\cdot) = \text{tr}[\rho'(\cdot)]$ and the quantum state ρ' is $|1\rangle\langle 1|$. Thus, we have

$$S(N) = V_1 + V_2 + \dots + V_N = N(1 - \epsilon)^2. \quad (122)$$

We cannot assign simultaneously the same two truth values (“1” and “1”) or (“0” and “0”) for the two propositions (117) and (122). We derive the experimentally appearing inconsistency with hidden variable theories.

Theorem: (*An experimentally appearing inconsistency with hidden variable theories in quantum phenomena based on the identity operator in a three-dimensional space*).

We are in the experimentally appearing inconsistency with hidden variable theories in quantum phenomena based on the identity operator in a three-dimensional space when quantum measurement outcomes are an even number $2N (\gg 1)$, the odd number preexistence values are $1 - \epsilon$ by measuring the operator I' in the quantum state ρ' , and the even number preexistence values are the same $1 - \epsilon$ as by measuring the same operator I' in the same quantum state ρ' which is the quantum state $|1\rangle\langle 1|$.

X. THE STATISTICAL AND EXPERIMENTALLY APPEARING KOCHEN-SPECKER THEOREM BASED ON A QUANTUM OPERATOR

We want to modify the KS argument provided in Sec. IX so as to relax the assumptions. In fact, VR and FUNC (8) are relaxed into D (35) and FUNC A.E. (44), respectively. In what follows, we assume HV and JD (41) hold. This implies that we can use D (35), FUNC A.E. (44), and PROD A.E. (54).

We shall modify the KS theorem presented in Sec. IX with the operator I' . Let us try an attempt to assign a function $f : \Omega \rightarrow \mathbf{R}$ to the operator I' in the quantum state $\psi(=|1\rangle)$. D (35) says $f_{I'}(\omega) = 1 - \epsilon, (\mu_\psi - a.e.)$. Then, one can see that

$$N \times \sum_{\omega \in \{\omega_1, \omega_2\}} f_{I'}(\omega) = N(f_{I'}(\omega_1) + f_{I'}(\omega_2)) = 2N(1 - \epsilon), (\mu_\psi - a.e.). \quad (123)$$

By the way the operation Addition ($f(A) + f(B)$) is equal to the operation Multiplication ($f(A) \cdot f(B)$) almost everywhere with respect to μ_ψ when $A = B = \begin{pmatrix} 1 - \epsilon & 0 & 0 \\ 0 & 1 - \epsilon & 0 \\ 0 & 0 & 1 - \epsilon \end{pmatrix}$ and the quantum state is $\psi(=|1\rangle)$ that it is a simultaneous eigenstate of A, B . Hence we have $f(A) \cdot f(B) = f(A) + f(B)$ almost everywhere with respect to μ_ψ . Hence we get $f_{I'}(\omega) + f_{I'}(\omega) = f_{I'}(\omega) \times f_{I'}(\omega)$ almost everywhere with respect to μ_ψ and

$$N \times \sum_{\omega \in \{\omega_1, \omega_2\}} f_{I'}(\omega) = N(f_{I'}(\omega_1) + f_{I'}(\omega_2)) = N(f_{I'}(\omega_1) \times f_{I'}(\omega_2)) = N(1 - \epsilon)^2, (\mu_\psi - a.e.). \quad (124)$$

Noting the relation (123), we see that an attempt to assign function to the operator I' in the quantum state $|1\rangle$ should fail.

XI. ANOTHER WAY OF DERIVING THE KOCHEN-SPECKER THEOREM BASED ON THE IDENTITY OPERATOR

Theorem: (*Another way of deriving the Kochen-Specker theorem based on the identity operator in a three-dimensional space*).

$$\text{HV} \wedge \text{JD (41)} \Rightarrow \text{HV} \wedge f_I(\omega) = 1, (\mu_\psi - a.e.) \wedge \text{FUNC A.E. (44)} \Rightarrow \perp, \quad (125)$$

where $\perp$ means Contradiction. That is, these assumptions do not hold at the same time. We say “from this definition D (35), it is required that the value of $f_I(\omega)$ is 1 almost everywhere, i.e., $f_I(\omega) = 1, (\mu_\psi - a.e.)$, where $\psi = |1\rangle$. Here, $I(= H_3(1))$ represents the identity operator in a three-dimensional space.” Thus, we can use the weak assumption $f_I(\omega) = 1, (\mu_\psi - a.e.)$ instead of the strong assumption D (35) in order to construct our theorem. We do not need the weak hidden variable theory.

XII. CONCLUSIONS AND DISCUSSIONS

In conclusions, we have investigated the aspect of the basic two quantum observables: (1) an observable that is obtained through a single projection operator and (2) an observable that is obtained through a single identity operator. Both of quantum operators have been well studied by Kochen and Specker in terms of the inconsistency with hidden variable theories. They have shown that some value assignment to elements of a set of projection operators and identity operators should fail.

Now, we have simplified the Kochen-Specker theorem in a three-dimensional case. We here have formulated the

basic two quantum operators as: $H_3(x) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$, where $x \in \{0, 1\}$. x is a fixed constant. We have been in the

inconsistency with hidden variable theories when the following quantum measurement process happens twice: The preexistence value has been 1 by measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$. The two measurements of $H_3(x)$ have been corresponding to the same x . It has turned out that we cannot assign the preexistence value for quantum measurement outcome as 1 when measuring the quantum operator $H_3(x)$ in the quantum state $|1\rangle$. In the case that $x = 0$, the one-dimensional projection operator in a three dimensional space has not had a classical counterpart. In the case that $x = 1$, the three-dimensional identity operator has not had a classical counterpart.

And finally, we have been able to use the weak assumption (the identity operator possesses a preexistence value as 1) instead of the strong assumption (the existence of the weak hidden variable theory) in order to verify the three-dimensional Kochen–Specker theorem.

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The authors state that there is no conflict of interest.

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