Criticizing Feynman's Path Integrals

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Abstract

I present an argument that shows that in general Feynman's path integrals are not equivalent with Schrödinger equations, and conclude that Feynman's path integrals are not a serious formulation of Quantum Mechanics. The path integral formulation appears to be equivalent with Schrödinger equation only in the special case where the Lagrangian depends on the velocity via a quadratic term.

According to mainstream physics Feynman's path integral formulation is an equivalent alternative to Schrödinger equation [1] [2]. However, the proof of this claim uses an assumption that we would only be interested in systems where the Lagrangian depends on the velocity via a quadratic term. People often speak about this equivalence result as if it could be extrapolated to other kind of systems too, but there is ambiguity about how justified that belief is.

Suppose we are interested in a system defined by a Lagrangian $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, \dot{x}) \mapsto L(x, \dot{x})$

$$L(x,\dot{x}) \;=\; \varepsilon \dot{x}^4 \;+\; \frac{1}{2} m \dot{x}^2 \;-\; U(x),$$

where $x \mapsto U(x)$ is some potential function, $m > 0$ is some mass constant, and $\varepsilon \geq 0$ is some constant. If $\varepsilon = 0$, this is the well known non-relativistic Lagrangian, and if $\varepsilon > 0$, then this is something new that's not so well known. Here the relation between canonical momentum p and velocity $\dot{x}$ is

$$p \;=\; 4\varepsilon \dot{x}^3 \;+\; m\dot{x}.$$

Is it possible to solve $\dot{x}$ out of this? If we try an ansatz

$$\dot{x} \;=\; \frac{1}{m} p \;+\; a_2 p^2 \;+\; a_3 p^3 \;+\; a_4 p^4 \;+\; a_5 p^5 \;+\; \cdots,$$

with some constants $a_2, a_3, a_4, \ldots$, then

$$\dot{x}^3 \;=\; \frac{1}{m^3} p^3 \;+\; \frac{3a_2}{m^2} p^4 \;+\; \Big(\frac{3a_2^2}{m} + \frac{3a_3}{m^2}\Big) p^5 \;+\; \Big(a_2^3 + \frac{6a_2a_3}{m} + \frac{3a_4}{m^2}\Big) p^6$$
$$+\; \Big(3a_2^2 a_3 + \frac{3a_3^2}{m} + \frac{6a_2a_4}{m} + \frac{3a_5}{m^2}\Big) p^7 \;+\; \cdots,$$

and this seems to work. The solution looks like

$$\dot{x} \;=\; \frac{1}{m}p \;-\; \frac{4\varepsilon}{m^4}p^3 \;+\; \frac{48\varepsilon^2}{m^7}p^5 \;-\; \frac{768\varepsilon^3}{m^{10}}p^7 \;+\; \cdots,$$

although the convergence is uncertain. Alternatively, we can apply Cardano's formula [3], which in this case gives

$$\dot{x} \;=\; \sqrt[3]{\frac{p}{8\varepsilon} \;+\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}} \;+\; \sqrt[3]{\frac{p}{8\varepsilon} \;-\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}}.$$

There is a problem that the cube roots can be interpreted in several different ways, and most of the interpretations are not what we want. The series $\dot{x} = \frac{1}{m}p-\cdots$ looked good, so if we state that the cube roots should be interpreted in a such way that the expression coincides with the series $\dot{x} = \frac{1}{m}p - \cdots$, that should specify the interpretation right.

We know from the basics of complex analysis that the root expressions are non-analytic at points where their input equals zero, and this will bound the domain of convergence of the series $\dot{x} = \frac{1}{m}p - \cdots$. This series does not work for all $p \in \mathbb{R}$, but it has some non-trivial domain of convergence instead.

Hamiltonian function of this system can be written as

$$\begin{aligned} H(x,p) \;=\; & p\Big(\sqrt[3]{\frac{p}{8\varepsilon} \;+\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}} \;+\; \sqrt[3]{\frac{p}{8\varepsilon} \;-\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}}\Big) \\ & -\varepsilon\Big(\sqrt[3]{\frac{p}{8\varepsilon} \;+\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}} \;+\; \sqrt[3]{\frac{p}{8\varepsilon} \;-\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}}\Big)^4 \\ & -\frac{1}{2}m\Big(\sqrt[3]{\frac{p}{8\varepsilon} \;+\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}} \;+\; \sqrt[3]{\frac{p}{8\varepsilon} \;-\; \sqrt{\frac{p^2}{64\varepsilon^2}+\frac{m^3}{1728\varepsilon^3}}}\Big)^2 \\ & +\; U(x). \end{aligned}$$

We probably don't want to do much with this Hamiltonian, but it is noteworthy that some precise formula for it exists. The Hamiltonian can also be expressed as

$$H(x,p) \;=\; U(x) \;+\; \frac{p^2}{2m} \;-\; \frac{\varepsilon p^4}{m^4} \;+\; \frac{8\varepsilon^2 p^6}{m^7} \;-\; \frac{96\varepsilon^3 p^8}{m^{10}} \;+\; \cdots.$$

People, who study Quantum Mechanics, usually at some point do an exercise, where they show that with the model $L = \frac{1}{2}m\dot{x}^2 - U$ Feynman's path integral formulation is equivalent with Schrödinger equation. The exercise is not very difficult: All you need is the skill of using the Gaussian integral formulas and Taylor series. Next, suppose we are interested in a quantum mechanical description of the system with the parameter $\varepsilon > 0$. Here too we have two options. One option is that we use Schrödinger equation

$$i\hbar\partial_t\psi(t,x) \;=\; H(M_x, -i\hbar\partial_x)\psi(t,x),$$

where

$$(M_x\psi)(x) \;=\; x\psi(x),$$

and where the operator

$$\sqrt[3]{\frac{-i\hbar\partial_x}{8\varepsilon} + \sqrt{\frac{-\hbar^2\partial_x^2}{64\varepsilon^2} + \frac{m^3}{1728\varepsilon^3}}} + \sqrt[3]{\frac{-i\hbar\partial_x}{8\varepsilon} - \sqrt{\frac{-\hbar^2\partial_x^2}{64\varepsilon^2} + \frac{m^3}{1728\varepsilon^3}}}$$

is a pseudo-differential operator defined by using Fourier transforms [4]. Another option is that we use Feynman's path integral formulation

$$\begin{aligned}\psi(t+\Delta t, x) \;&\propto\; \int_{-\infty}^{\infty} e^{\frac{i\Delta t}{\hbar}L(x,\frac{x-x'}{\Delta t})}\psi(t,x')dx' \\ &=\; \int_{-\infty}^{\infty} e^{\frac{i\Delta t}{\hbar}\left(\varepsilon(\frac{x-x'}{\Delta t})^4+\frac{1}{2}m(\frac{x-x'}{\Delta t})^2-U(x)\right)}\psi(t,x')dx'.\end{aligned}$$

Now the big question is that are these two time evolutions equivalent or not? Both of these time evolution formulations are of such kind that it's difficult to see what they imply, so the question is somewhat difficult. One way of finding the answer is to examine the locality properties of these time evolutions.

When we define Hamiltonian operator $H(M_x, -i\hbar\partial_x)$ by putting differential operators inside root expressions, we obtain a non-local operator. This means that if we want to know the value $H(M_x, -i\hbar\partial_x)\psi(t,x)$ with some x, it is not sufficient to know the values of $\psi(t,\bullet)$ in some infinitesimal environment of the point x, but instead $H(M_x, -i\hbar\partial_x)\psi(t,x)$ depends on $\psi(t,x')$, where x' is far away from x. The reason for this is that if this Hamiltonian operator was local, then with $\psi(t,\bullet)$ with bounded support, it should be possible to write the Taylor series of the Fourier transform $\widehat{H\psi}(t,k)$ with respect to the momentum parameter k for all $k \in \mathbb{R}$. However, the non-analyticity of the root expressions makes the Taylor series diverge for some k, which proves that the Hamiltonian operator cannot be local. We went through how this argument works in more detail in an earlier article [5].

Suppose we define $\psi(t+\Delta t, x)$ by using Feynman's path integral formulation, and then define $\partial_t\psi(t,x)$ by using $\psi(t+\Delta t, x)$. Is this time derivative $\partial_t\psi(t,x)$ determined by the values of $\psi(t,\bullet)$ in an infinitesimal environment of the point x, or could it be that $\psi(t,x')$, where x' is far away from x, has some effect on $\partial_t\psi(t,x)$? We already know from the theory of ordinary path integrals that $\partial_t\psi(t,x)$ is determined by the values of $\psi(t,\bullet)$ in an infinitesimal environment of the point x, if $\varepsilon = 0$. If we replace $\varepsilon = 0$ with $\varepsilon > 0$, this change makes the integration kernel only oscillate even faster, so it can be deduced that also in this $\varepsilon > 0$ case $\partial_t\psi(t,x)$ must be determined by the values of $\psi(t,\bullet)$ in an infinitesimal environment of the point x.

If we define the time evolution of the wave function using Schrödinger equation, we obtain a kind of time evolution, where $\partial_t\psi(t,x)$ depends on $\psi(t,x')$, where x' is far away from x, and the time evolution is non-local in this way. If we define the time evolution of the wave function using Feynman's path integral formulation, we obtain a kind of time evolution, where $\partial_t\psi(t,x)$ is fully determined by $\psi(t,\bullet)$ in an infinitesimal environment of the point x, and the time evolution is local in this way. These non-local and local time evolutions cannot be equivalent, so we see from here that in general Feynman's path integral formulation is not equivalent with Schrödinger equation.

We shouldn't yet conclude that in general Feynman's path integral formulation would be wrong, because in the light of what we just learned above, it could still be that Feynman's path integral formulation would be giving the true time evolution, and Schrödinger equation would be giving a false one. At this point I would like to show the reader one interesting result about Schrödinger equations. Suppose $N \in \{1,2,3\ldots\}$ is some number, and that some Hamiltonian function has been defined as $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$

$$H(x,p) \;=\; \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta} \prod_{n=1}^{N} \big(x_n^{\alpha(n)} p_n^{\beta(n)}\big).$$

Here α and β are multi-indices that essentially are $\alpha = (\alpha(1),\alpha(2),\ldots,\alpha(N))$ and $\beta = (\beta(1),\beta(2),\ldots,\beta(N))$, where $\alpha(n)\in\mathbb{N}$ and $\beta(n)\in\mathbb{N}$. The values of the coefficients $a_{\alpha,\beta}\in\mathbb{R}$ come from some mapping

$$a : \mathbb{N}^{\{1,2,\ldots,N\}} \times \mathbb{N}^{\{1,2,\ldots,N\}} \to \mathbb{R}, \qquad (\alpha,\beta)\mapsto a_{\alpha,\beta}.$$

If we want our subsequent calculations to be rigorous, we can assume that $a_{\alpha,\beta}$ assumes a non-zero value only for a finite amount of multi-index values (α,β). This way the Hamiltonian essentially is a polynomial, and there will be no convergence issues. Let's make another assumption about $a_{\alpha,\beta}$ too. We assume that

$$\big(\exists\, n\in\{1,2,\ldots,N\}\quad \text{s.t.}\quad \alpha(n)>0\quad\text{and}\quad \beta(n)>0\big)\qquad\Longrightarrow\qquad a_{\alpha,\beta}=0.$$

So for example with $N=2$

$$\begin{aligned} H(x,p) \;=\; & a_{(0,0),(4,0)}p_1^4 \;+\; a_{(0,0),(0,4)}p_2^4 \;+\; a_{(3,0),(0,1)}x_1^3 p_2 \;+\; a_{(0,3),(1,0)}x_2^3 p_1 \\ & +\; a_{(6,0),(0,0)}x_1^6 \;+\; a_{(0,6),(0,0)}x_2^6 \end{aligned}$$

is allowed, but

$$\begin{aligned} H(x,p) \;=\; & a_{(0,0),(4,0)}p_1^4 \;+\; a_{(0,0),(0,4)}p_2^4 \;+\; a_{(3,0),(1,0)}x_1^3 p_1 \;+\; a_{(0,3),(0,1)}x_2^3 p_2 \\ & +\; a_{(6,0),(0,0)}x_1^6 \;+\; a_{(0,6),(0,0)}x_2^6 \end{aligned}$$

is forbidden, if $a_{(3,0),(1,0)} \neq 0$ or $a_{(0,3),(0,1)} \neq 0$. The reason for this assumption is that for simplicity of our example we want our relevant operators M_{x_n} and $-i\hbar\partial_{x_{n'}}$ to commute, and they must not appear in any term with the same coordinate index (without a zero in exponent).

Under these assumptions, the Hamiltonian operator of this system is

$$H(M_x, -i\hbar\nabla_x) = \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta} \prod_{n=1}^{N} \left(M_{x_n}^{\alpha(n)}(-i\hbar D_{x_n})^{\beta(n)}\right).$$

If

$$\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}, \qquad (t,x)\mapsto \psi(t,x)$$

is some wave function, according to the axioms of Quantum Mechancis, the corresponding expectation values of the position and momentum are

$$\langle x(t)\rangle = \int_{\mathbb{R}^N} x|\psi(t,x)|^2 d^N x$$

and

$$\langle p(t)\rangle = \int_{\mathbb{R}^N} (\psi(t,x))^*(-i\hbar\nabla_x)\psi(t,x) d^N x,$$

assuming that ψ has been normalized.

The partial derivatives of the Hamiltonian function are

$$\partial_{x_n} H(x,p) = \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\alpha(n) x_n^{\alpha(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \left(x_{n'}^{\alpha(n')} p_{n'}^{\beta(n')}\right)$$

and

$$\partial_{p_n} H(x,p) = \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\beta(n) p_n^{\beta(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \left(x_{n'}^{\alpha(n')} p_{n'}^{\beta(n')}\right).$$

These partial derivatives also have their corresponding operators

$$\partial_{x_n} H(M_x, -i\hbar\nabla_x)$$
$$= \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\alpha(n) M_{x_n}^{\alpha(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \left(M_{x_{n'}}^{\alpha(n')}(-i\hbar D_{x_{n'}})^{\beta(n')}\right)$$

and

$$\partial_{p_n} H(M_x, -i\hbar\nabla_x)$$
$$= \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\beta(n)(-i\hbar D_{x_n})^{\beta(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \left(M_{x_{n'}}^{\alpha(n')}(-i\hbar D_{x_{n'}})^{\beta(n')}\right).$$

It might seem that a nonsensical operator $D_{x_n}^{-1}$ is present in this expression, but this nonsensical operator always gets multiplied by zero, so we can interpret the expression as this nonsensical operator not being present. The expectation values of the partial derivatives of the Hamiltonian are

$$\begin{aligned}\langle \partial_{x_n} H(t) \rangle \;=\; & \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\alpha(n) \int\limits_{\mathbb{R}^N} \\ & (\psi(t,x))^* x_n^{\alpha(n)-1} \prod_{\substack{n'=1 \\ n'\neq n}}^{N} \big(x_{n'}^{\alpha(n')} (-i\hbar\partial_{x_{n'}})^{\beta(n')} \big) \psi(t,x) d^N x\end{aligned}$$

and

$$\begin{aligned}\langle \partial_{p_n} H(t) \rangle \;=\; & \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\beta(n) \int\limits_{\mathbb{R}^N} \\ & (\psi(t,x))^* (-i\hbar\partial_{x_n})^{\beta(n)-1} \prod_{\substack{n'=1 \\ n'\neq n}}^{N} \big(x_{n'}^{\alpha(n')} (-i\hbar\partial_{x_{n'}})^{\beta(n')} \big) \psi(t,x) d^N x.\end{aligned}$$

Let's take a closer look at the time derivatives of the expectation values of the position and momentum. The calculation of the time derivative of $\langle x_n(t) \rangle$ begins as

$$\begin{aligned}D_t \langle x_n(t) \rangle \;&=\; D_t \int\limits_{\mathbb{R}^N} x_n (\psi(t,x))^* \psi(t,x) d^N x \\ &=\; \int\limits_{\mathbb{R}^N} x_n \big((\partial_t \psi(t,x))^* \psi(t,x) \;+\; (\psi(t,x))^* \partial_t \psi(t,x) \big) d^N x \\ &=\; \frac{i}{\hbar} \int\limits_{\mathbb{R}^N} x_n \Big(\big(H(M_x, -i\hbar\nabla_x) \psi(t,x) \big)^* \psi(t,x) \\ &\qquad -\; (\psi(t,x))^* H(M_x, -i\hbar\nabla_x) \psi(t,x) \Big) d^N x\end{aligned}$$

$$\begin{aligned}=\; \frac{i}{\hbar} \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta} \int\limits_{\mathbb{R}^N} x_n \Big(\Big(\prod_{n'=1}^{N} \big(x_{n'}^{\alpha(n')} (-i\hbar\partial_{x_{n'}})^{\beta(n')} \big) \psi(t,x) \Big)^* \psi(t,x) \\ -\; (\psi(t,x))^* \prod_{n'=1}^{N} \big(x_{n'}^{\alpha(n')} (-i\hbar\partial_{x_{n'}})^{\beta(n')} \big) \psi(t,x) \Big) d^N x\end{aligned}$$

$$= \frac{i}{\hbar} \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta} \int_{\mathbb{R}^N} x_n \Big(\Big(\prod_{n'=1}^{N} \big(x_{n'}^{\alpha(n')}(i\hbar\partial_{x_{n'}})^{\beta(n')}\big)(\psi(t,x))^* \Big) \psi(t,x)$$

$$- \; (\psi(t,x))^* \prod_{n'=1}^{N} \big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x) \Big) d^N x$$

$$= \cdots$$

We must spend some time thinking about how integration by parts is going to work in this situation. If $n' \neq n$, we can use the substitution

$$x_n\big(\partial_{x_{n'}}^{\beta(n')}(\psi(t,x))^*\big)\psi(t,x) \; \mapsto x_n(\psi(t,x))^*(-\partial_{x_{n'}})^{\beta(n')}\psi(t,x).$$

If $n' = n$, then we assume we know how to prove

$$D_x^k M_x \; - \; M_x D_x^k \; = \; k D_x^{k-1}$$

by induction for $k \in \{1,2,3,\ldots\}$, and then carry out integration by parts with the substitution

$$x_n\big(\partial_{x_n}^{\beta(n)}(\psi(t,x))^*\big)\psi(t,x)$$
$$\mapsto \; x_n(\psi(t,x))^*(-\partial_{x_n})^{\beta(n)}\psi(t,x) \; - \; \beta(n)(\psi(t,x))^*(-\partial_{x_n})^{\beta(n)-1}\psi(t,x).$$

The calculation of the time derivative of $\langle x_n(t)\rangle$ comes to an end as

$$\cdots \; = \; \frac{i}{\hbar} \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}(-\beta(n)) \int_{\mathbb{R}^N}$$

$$(\psi(t,x))^*(i\hbar)^{\beta(n)}(-\partial_{x_n})^{\beta(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x) d^N x$$

$$= \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\beta(n) \int_{\mathbb{R}^N}$$

$$(\psi(t,x))^*(-i\hbar\partial_{x_n})^{\beta(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x) d^N x$$

$$= \; D_t\langle\partial_{p_n} H(t)\rangle.$$

The calculation of the time derivative of $\langle p_n(t)\rangle$ begins as

$$\begin{aligned} D_t\langle p_n(t)\rangle \;&=\; D_t \int\limits_{\mathbb{R}^N} (\psi(t,x))^*(-i\hbar\partial_{x_n})\psi(t,x)d^Nx \\ &=\; \int\limits_{\mathbb{R}^N} \big((\partial_t\psi(t,x))^*(-i\hbar\partial_{x_n})\psi(t,x) \;+\; (\psi(t,x))^*(-i\hbar\partial_{x_n})\partial_t\psi(t,x)\big)d^Nx \\ &=\; \frac{i}{\hbar}\int\limits_{\mathbb{R}^N} \Big(\big(H(M_x,-i\hbar\nabla_x)\psi(t,x)\big)^*(-i\hbar\partial_{x_n})\psi(t,x) \\ &\qquad\qquad -\; (\psi(t,x))^*(-i\hbar D_{x_n})H(M_x,-i\hbar\nabla_x)\psi(t,x)\Big)d^Nx \end{aligned}$$

$$\begin{aligned} =\; \frac{i}{\hbar}\sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\int\limits_{\mathbb{R}^N} \Big(\Big(\prod_{n'=1}^{N}\big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x)\Big)^*(-i\hbar\partial_{x_n})\psi(t,x) \\ -\; (\psi(t,x))^*(-i\hbar D_{x_n})\prod_{n'=1}^{N}\big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x)\Big)d^Nx \\ =\; \frac{i}{\hbar}\sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\int\limits_{\mathbb{R}^N} \Big(\Big(\prod_{n'=1}^{N}\big(x_{n'}^{\alpha(n')}(i\hbar\partial_{x_{n'}})^{\beta(n')}\big)(\psi(t,x))^*\Big)(-i\hbar\partial_{x_n})\psi(t,x) \\ -\; (\psi(t,x))^*(-i\hbar D_{x_n})\prod_{n'=1}^{N}\big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x)\Big)d^Nx \\ =\; \cdots \end{aligned}$$

Now we can relocate the $(i\hbar\partial_{x_{n'}})^{\beta(n')}$ operators with integration by parts easily. We want to relocate the $x_{n'}^{\alpha(n')}$ factors too so that some cancellation can happen, but we must spend some time thinking about how this works. If $n'\neq n$, we can use the equation

$$x_{n'}^{\alpha(n')}D_{x_n}\psi(t,x) \;=\; D_{x_n}\big(x_{n'}^{\alpha(n')}\psi(t,x)\big).$$

If $n'=n$, then we assume we know how to prove

$$M_x^kD_x \;-\; D_xM_x^k \;=\; -kM_x^{k-1}$$

by induction for $k\in\{1,2,3,\ldots\}$, and then use the equation

$$x_n^{\alpha(n)}D_{x_n}\psi(t,x) \;=\; D_{x_n}\big(x_n^{\alpha(n)}\psi(t,x)\big) \;-\; \alpha(n)x_n^{\alpha(n)-1}\psi(t,x).$$

The calculation of the time derivative of $\langle p_n(t)\rangle$ comes to an end as

$$\begin{aligned}
\cdots &= \frac{i}{\hbar} \sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}(-\alpha(n)) \int\limits_{\mathbb{R}^N} \\
&\quad (\psi(t,x))^*(-i\hbar)x_n^{\alpha(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x)d^Nx \\
&= -\sum_{\alpha,\beta\in\mathbb{N}^{\{1,2,\ldots,N\}}} a_{\alpha,\beta}\alpha(n) \int\limits_{\mathbb{R}^N} \\
&\quad (\psi(t,x))^* x_n^{\alpha(n)-1} \prod_{\substack{n'=1\\ n'\neq n}}^{N} \big(x_{n'}^{\alpha(n')}(-i\hbar\partial_{x_{n'}})^{\beta(n')}\big)\psi(t,x)d^Nx \\
&= -\langle \partial_{x_n} H(t)\rangle.
\end{aligned}$$

So there exists an implication

$$\begin{aligned}
&i\hbar\partial_t\psi(t,x) \;=\; H(M_x,-i\hbar\nabla_x)\psi(t,x) \qquad \Longrightarrow \\
&\Big(D_t\langle x(t)\rangle \;=\; \langle\nabla_p H(t)\rangle \qquad \text{and} \qquad D_t\langle p(t)\rangle \;=\; -\langle\nabla_x H(t)\rangle\Big).
\end{aligned}$$

In other words, Schrödinger equation implies Hamiltonian equations of motion. Notice, when we prove that Schrödinger equation implies Hamiltonian equations of motion, there is no need to ease the task by assuming that Hamiltonian would depend on the canonical momentum only via a quadratic term. Hamiltonian can have higher order terms such as p_1^4, $p_1^6p_2^2$, $p_1^8p_2^6p_3^4$ and so on, and even terms such as x_1p_2, $x_1^3p_2p_3^2$ and so on that combine position and momentum coordinates, and even then Schrödinger equation still implies Hamiltonian equations of motion. This is a very interesting result, and it supports the view that the generic Schrödinger equation $i\hbar\partial_t\psi(t,x) = H(M_x,-i\hbar\nabla_x)\psi(t,x)$ the most apparently is a very good equation, and there probably is not much wrong with it. Then, if an alternative Feynman's path integral formulation fails to be equivalent with this generic Schrödinger equation, it very likely is the Feynman's path integral formulation that is flawed. There is no similar rigorous proof for the hypothesis that Feynman's path integral formulation would always imply Euler-Lagrange equations. Only a heuristic argument exists, which isn't similarly convincing.

References

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