On Properties of a Smooth Income Tax Function with Application to the UK

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Abstract

This paper examines smooth, progressive income tax functions as an alternative to the current piecewise UK tax function. The piecewise function applies constant rates within income brackets leading to a discontinuous first derivative or marginal tax rate which can create incentives for individuals to limit their earnings. Building on Odland's work on the Norwegian income tax function, we derive and analyse a general class of smooth tax functions that maintain progressivity while ensuring continuity in marginal rates. We find that the function Odland proposed as a replacement for the Norwegian piecewise tax function also provides a good fit for the UK piecewise tax function. Despite theoretical advantages of a smooth tax function, the implementation of such a system in the UK and elsewhere is likely to be constrained for several reasons including administrative complexity and political considerations.

1 Introduction

As in many countries [3], the current UK tax system is *piecewise*: a flat tax rate is applied within each tax bracket. Consequently, the *marginal tax rate*—the proportion of tax paid on the next unit of income—is *discontinuous* at the bracket boundaries. This discontinuity can create *perverse incentives*, whereby individuals deliberately limit their earnings to avoid entering a higher bracket and thereby paying a larger proportion of their income in tax.

A *smooth tax function*, defined mathematically as a function with continuous derivatives, eliminates these discontinuities and thus avoids such distortions in taxpayer behaviour.

The motivation for this paper arises from Odland's discussion of a smooth, progressive tax function in the context of the Norwegian tax system [7]. Following a similar approach, we analyse the properties of a smooth, progressive tax function and derive a more general formulation of such a function.

Throughout this paper, we denote the tax and after-tax income functions by T and A, respectively, defined as

$$T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad A : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad A(x) := x - T(x).$$

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The function $T(x)$ describes how the income tax burden evolves with income, while the function $A(x) = x - T(x)$ captures the corresponding after-tax income and thus the economic incentives individuals face. Analysing both is essential: T reveals the structure and progressivity of the tax system, whereas A shows how that structure translates into disposable income, affecting behaviour, labour supply, and overall welfare.

As found in Odland [7], we find that a suitable candidate for a smooth, progressive tax function capable of approximating the current UK piecewise tax function is

$$T(x) = q\left(x - \frac{1}{\gamma}(1 - e^{-\gamma x})\right), \quad \gamma > 0,\ q \in (0, 1),$$

where q represents the maximum proportion of income payable as tax, and γ is a free parameter calibrated to approximate the existing piecewise tax function. The parameter γ is responsible for the rate in which the tax function reaches the asymptotic limit of qx.

2 Current UK Piecewise Income Tax Function

In the United Kingdom (excluding Scotland, whose system is similar), income tax for the 2025 tax year is determined by a band tax after a personal allowance [5]. We will also include national insurance contributions [6] in the income tax. Let x denote an individual's *gross income* (income before tax).

Personal Allowance

Each individual receives a *personal allowance* — a portion of income that is tax-free. For 2025, the allowance is £12,570 for incomes up to £100,000. Above £100,000, it is reduced by £1 for every £2 earned over £100,000, until it is reduced to 0 at an income of £125,140.

Hence, the personal allowance $p(x)$ is defined as

$$p(x) = \begin{cases} 12570, & x \leq 100{,}000, \\ 12570 - \dfrac{x - 100{,}000}{2}, & 100{,}000 < x < 125{,}140, \\ 0, & x \geq 125{,}140. \end{cases}$$

Income Tax Bands

Tax is charged only on income *above* the personal allowance, $x - p(x)$. Thus the tax is described by the following bands:

- **Personal allowance (0%)**: less than or equal to the personal allowance;
- **Basic rate (20%)**: up to £37,700 after the personal allowance;
- **Higher rate (40%)**: between £37,701 and £125,140 after the personal allowance;
- **Additional rate (45%)**: above £125,140 total income.

Thus, the band tax function is $B(x)$ is given by

$$B(x) = \begin{cases} 0, & x \le 12{,}570, \\ 0.2(x - p(x)), & 0 < x - p(x) \le 37{,}700, \\ 0.4\big(x - p(x) - 37{,}700\big) + 0.2 \times 37{,}700, & 37{,}700 < x - p(x) \le 125{,}140, \\ 0.45\big(x - 125{,}140\big) + 0.4(125{,}140 - 37{,}700) \\ \qquad + 0.2 \times 37{,}700, & x > 125{,}140. \end{cases}$$

National Insurance

In addition to income tax, employees pay *National Insurance contributions* on earnings after the personal allowance:

- 8% on income up to £37,700 (after the allowance);
- 2% on income above £37,700 (after the allowance).

Tax Functions

Including National Insurance (NI), the total effective tax liability represented by the tax function $T(x)$ becomes

$$T(x) = \begin{cases} 0, & x \le 12{,}570, \\ 0.28(x - p(x)), & 0 < x - p(x) \le 37{,}700, \\ 0.42\big(x - p(x) - 37{,}700\big) + 0.28 \times 37{,}700, & 37{,}700 < x - p(x) \le 125{,}140, \\ 0.47\big(x - 125{,}140\big) + 0.42(125{,}140 - 37{,}700) \\ \qquad + 0.28 \times 37{,}700, & x > 125{,}140. \end{cases} \tag{1}$$

The after-tax income function $A(x) = x - T(x)$ is then:

$$A(x) = \begin{cases} x, & x \le 12{,}570, \\ (1 - 0.28)(x - p(x)) + p(x), & 0 < x - p(x) \le 37{,}700, \\ (1 - 0.42)\big(x - p(x) - 37{,}700\big) + (1 - 0.28) \times 37{,}700 \\ \qquad + p(x), & 37{,}700 < x - p(x) \le 125{,}140, \\ (1 - 0.47)\big(x - 125{,}140\big) + (1 - 0.42)(125{,}140 - 37{,}700) \\ \qquad + (1 - 0.28) \times 37{,}700, & x > 125{,}140. \end{cases} \tag{2}$$

Remarks

These formulations approximate the main UK income tax and National Insurance structure for 2025 and is a representative income tax function for most taxpayers. Additional complexities, such as different NI classes, dividends, pension contributions, or marriage allowances, are not included here for simplicity.

3 Smooth Tax Functions

We state the mathematical definition of a smooth function:

Definition 1 (Smooth Function)**.** *A* smooth function f *is a function that has continuous derivatives up to a certain order.*

In this paper we shall assume that the order is at least **two** i.e that the first and second derivatives are continuous. We want the second derivative to be continuous for the concave/convex properties on the tax functions below.

3.1 Arguments For and Against a Smooth Tax Function

If the tax function T (equivalently, the after-tax income function A) is smooth, then its marginal tax rate $T'(x)$ (equivalently $A'(x)$) is continuous. In other words, the proportion of a person's tax increase relative to their income increase varies continuously with income. By contrast, a piecewise function such as the UK tax function (1) is not smooth and exhibits discontinuities at the bracket boundaries.

3.1.1 Arguments For

A smooth tax function eliminates the perverse incentives found in traditional bracket-based systems, where individuals may alter their behaviour to avoid moving into a higher tax bracket. Such distortions can reduce productivity and discourage innovation within the economy. How much distortion the bracket based system causes is up for debate, however there has been clear evidence of significant distortion for lower brackets in the US, see [9]. From a mathematical perspective, a smooth tax function is also much cleaner and more elegant than the piecewise form described in Section 2.

3.1.2 Arguments Against

Implementing a smooth tax function is generally more mathematically complex than using a piecewise function. This added complexity may reduce public understanding and transparency, as taxpayers are accustomed to clear bracket structures. Moreover, implementing a smooth tax function could involve higher administrative costs and practical complications, such as the need to round fractional values of tax liabilities [10]. Finally, there would be debate over how to select and justify the parameters of the chosen smooth functional form [3].

3.2 Properties of Income After Tax and Tax Functions

We start with the after-tax income function $A(x)$ and set the following four properties:

a.1 $A(x)$ is a smooth function.

a.2 $A(x)$ is a strictly increasing function.

a.3 The following asymptotic relations[1] hold

$$A(x) \underset{x\to 0^+}{\sim} x\,, \quad A(x) \underset{x\to\infty}{\sim} px\,, \quad p \in (0,1)\,. \tag{3}$$

a.4 $A(x)$ is a strictly concave function.

Now we explain the rational for choosing the above properties.

a.1 assumes that the after-tax income function is smooth. The advantages of imposing smoothness were discussed above.

a.2 requires that individuals with higher pre-tax income also have higher after-tax income - a natural monotonicity property.

a.3 ensures that individuals with the lowest income retain almost all of it, while those with the highest income take home a proportion $p \in (0,1)$ of their income. Since $A(x)$ is smooth and hence continuous, **a.3** implies in particular that $A(0) = 0$.

a.4 guarantees that the tax function is progressive; that is, individuals with higher income pay a larger share of their income in taxes. This can be formalised in the following proposition, which follows from standard properties of the derivative of a strictly concave function (see, for example, [8, part 1, §4]). Proofs of all statements in this section are provided in the appendix.

Proposition 1.

$$\frac{A(x)}{x} \textit{ is strictly decreasing for all } x > 0\,.$$

A simple corollary of Proposition 1 is

Corollary 1.

$$\frac{T(x)}{x} \textit{ is strictly increasing for all } x > 0\,.$$

The four conditions for $A(x)$ give rise to the following equivalent four conditions for $T(x)$.

Proposition 2. *With the above conditions for $A(x)$, **a.1** - **a.4**, we have the following equivalent conditions for $T(x)$.*

t.1 *$T(x)$ is a smooth function.*

t.2 *$T(x)$ is a strictly increasing function.*

t.3 *The following asymptotic relations[2] hold*

$$T(x) \underset{x\to 0^+}{=} o(x)\,, \quad T(x) \underset{x\to\infty}{\sim} (1-p)x\,, \quad p \in (0,1)\,. \tag{4}$$

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[1] Where we define $f(x) \underset{x\to a}{\sim} g(x) := \lim_{x\to a} \frac{f(x)}{g(x)} = 1\,.$

[2] Where we define (the little-o notation) $f(x) \underset{x\to a}{=} o(g(x)) := \lim_{x\to a} \frac{f(x)}{g(x)} = 0\,.$

t.4 *$T(x)$ is a strictly convex function.*

By a simple application of L'Hôpital's rule, see e.g. §7.12-7.15 of [1], one can show the following proposition.

Proposition 3. *If $f(x)$ is strictly increasing and smooth on $\mathbb{R}_{\geq 0}$ with $f(0) = 0$ and $\beta \in \mathbb{R} \setminus \{0\}$ then for $k = 0^+$ or $k = \infty$*

$$f(x) \underset{x\to k}{\sim} \beta x \Leftrightarrow f'(x) \underset{x\to k}{\sim} \beta\,,$$

or

$$f(x) \underset{x\to k}{=} o(x) \Leftrightarrow f'(x) \underset{x\to k}{=} o(1)\,.$$

Therefore by Proposition 3 we have the following asymptotic properties on the marginal rates equivalent to properties **a.3** and **t.3**.

ã.3 The following asymptotic relations on the after-tax marginal rate hold

$$A'(x) \underset{x\to 0^+}{\sim} 1\,, \quad A'(x) \underset{x\to\infty}{\sim} p\,.$$

t̃.3 The following asymptotic relations on the tax marginal rate hold

$$T'(x) \underset{x\to 0^+}{=} o(1)\,, \quad T'(x) \underset{x\to\infty}{\sim} 1-p\,.$$

Proposition 4. *A function $A(x)$ satisfying the properties **a.1-a.4** can be written as*

$$A'(x) = p + (1-p)f(x)\,, \quad A(x) = px + (1-p)\int_0^x f(t)\mathrm{d}t\,, \quad p \in (0,1)$$

such that the function f has the following five properties

f.1 *f is smooth,*

f.2 $f(x) > 0\,, \quad x > 0\,,$

f.3 $f'(x) < 0\,, \quad x > 0\,,$

f.4 $f(0) = 1\,,$

f.5 $f(x) \underset{x\to\infty}{=} o(1)\,.$

We note from Proposition 4 we also have an alternative form for the tax function and marginal tax rate:

$$T'(x) = (1-p)(1-f(x))\,, \quad T(x) = (1-p)\left(x - \int_0^x f(t)\mathrm{d}t\right) \tag{5}$$

such that f follows **f.1-f.5**.

3.3 Smooth Function Satisfying the Properties and Fitting the Piecewise Function

We identify an after-tax income function $A(x)$ that satisfies properties **a.1–a.4** by first finding a simple function f that meet the same conditions in Proposition 4. Once $A(x)$ is obtained, the corresponding function $T(x)$, satisfying properties **t.1–t.4**, can be readily derived from (5).

We take the exponential function

$$f(x) = e^{-\gamma x}\,, \quad \gamma > 0 \tag{6}$$

which satisfies **f.1-f.5**.

Set $p \in (0,1)$. The marginal after-tax income function and after-tax income function for (6) are then

$$A'(x) = p + (1-p)e^{-\gamma x}\,, \quad A(x) = px + \frac{1-p}{\gamma}(1 - e^{-\gamma x})\,. \tag{7}$$

The corresponding marginal tax function and tax function are

$$T'(x) = q(1 - e^{-\gamma x})\,, \quad T(x) = q\Big(x - \frac{1}{\gamma}(1 - e^{-\gamma x})\Big)\,, \quad q = 1 - p\,. \tag{8}$$

Now we set an additional property for the after-tax income function (equivalently tax function):

a.5/t.5 The income after-tax/tax functions fits reasonably well the existing piecewise income after-tax/tax functions, see (2), (1) in Section 2.

Corresponding to (2) we choose $p = 1 - 0.47 = 0.53$ and we find γ so that the smooth function $A(x)$ in (7) fits the piecewise function (2) with non-linear least squares via the `curve_fit` function from the SciPy library [11], see the author's GitHub for details [4]. The fit for γ is

$$\hat{\gamma} \approx 3.588 \cdot 10^{-5}\,.$$

We see that this fit for γ is reasonable for the piecewise function (2) in Figure 1. The corresponding fits for $T(x)$ and the marginal rates $A'(x)$ and $T'(x)$ are also shown in Figure 1.

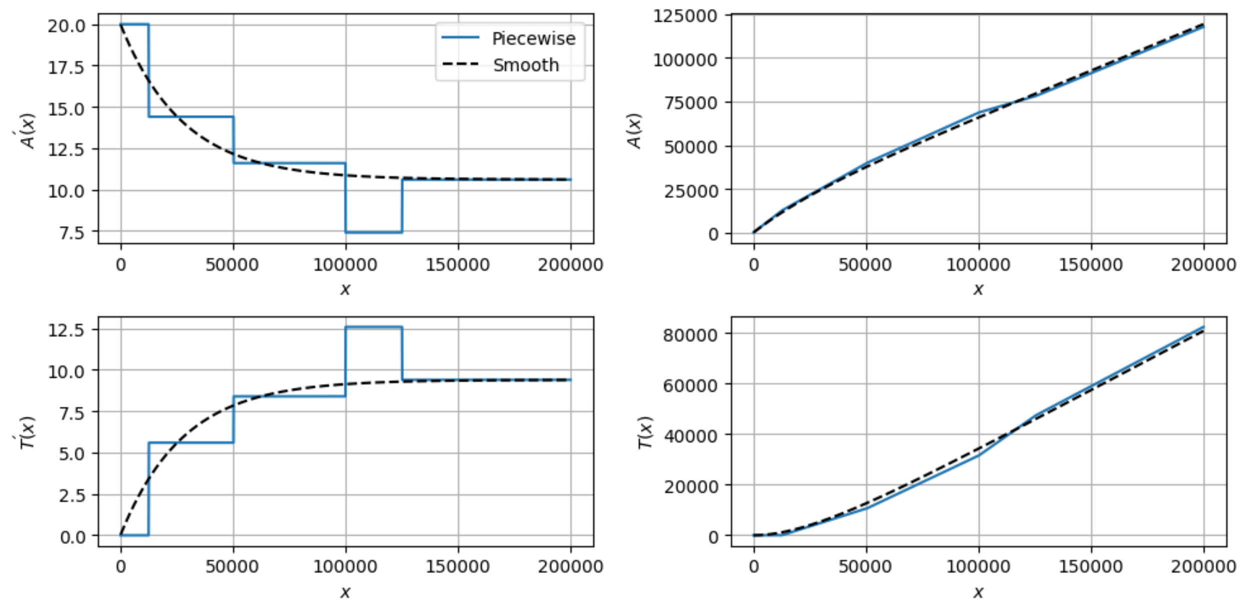

Figure 1: The free parameter γ in the smooth after-tax income function $A(x)$ (7) was calibrated to the piecewise after-tax income specification (2) as described above. The corresponding tax function $T(x)$ and marginal rates $A'(x)$, $T'(x)$ of the piecewise functions and smooth fits are also plotted.

4 Conclusion

We conducted an in-depth analysis of the properties of a progressive, smooth tax function, from which we derived a general form (5). We have shown that the smooth tax function (8), originally proposed by Odland [7], provides a reasonable alternative to the current UK piecewise tax function. Such a function would remove the perverse economic incentives associated with crossing tax brackets. However, its practical implementation in the near future appears unlikely due to perceived functional complexity, administrative costs, and political constraints. Future work could relate the properties of a smooth, progressive tax function to the extensive literature on optimal tax theory, see for example [2].

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Appendix

Proofs of Statements in Section 3.2

Proposition 1.

$$\frac{A(x)}{x} \text{ is strictly decreasing for all } x > 0\,.$$

Proof.

$$\begin{aligned}
A(x) \text{ strictly concave } \Rightarrow A(y) &< A(x) + A'(x)(y - x) \\
\text{Set } y = 0, \quad 0 = A(0) &< A(x) + A'(x)(-x) \\
A'(x)x - A(x) &< 0 \\
\frac{A'(x)x - A(x)}{x^2} &< 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{A(x)}{x}\right) &< 0 \Rightarrow \frac{A(x)}{x} \text{ strictly decreasing.}
\end{aligned}$$

■

Proposition 2. *With the above conditions for $A(x)$,* ***a.1*** *-* ***a.4****, we have the following equivalent conditions for $T(x)$.*

t.1 *$T(x)$ is a smooth function.*

t.2 *$T(x)$ is a strictly increasing function.*

t.3 *The following asymptotic relations[3] hold*

$$T(x) \underset{x\to 0^+}{=} o(x)\,, \quad T(x) \underset{x\to\infty}{\sim} (1-p)x\,, \quad p \in (0,1)\,. \tag{4}$$

t.4 *$T(x)$ is a strictly convex function.*

Proof. **t.1** $T(x) = x - A(x)$ is smooth as both x and $A(x)$ are smooth and a linear transformation of two smooth functions is also smooth.

t.3 We have

$$\begin{aligned}
\lim_{x\to 0^+} \frac{T(x)}{x} &= \lim_{x\to 0^+}\left(1 - \frac{A(x)}{x}\right) = 0 \\
\lim_{x\to\infty} \frac{T(x)}{x} &= \lim_{x\to\infty}\left(1 - \frac{A(x)}{x}\right) = 1 - p\,.
\end{aligned}$$

t.4 $T''(x) = -A''(x) > 0$ so $T(x)$ is strictly convex.

[3]Where we define (the little-o notation) $f(x) \underset{x\to a}{=} o(g(x)) := \lim_{x\to a} \frac{f(x)}{g(x)} = 0\,.$

t.2 We have by $\tilde{\mathbf{t}}$**.3** which is equivalent to **t.3**

$$T'(x) \to 0 \text{ as } x \to 0^+ , \quad T'(x) \to 1 - p \text{ as } x \to \infty .$$

Also by **t.4** we have $T'(x)$ is strictly increasing. Therefore $T'(x) > 0$, $x > 0$ and thus $T(x)$ is strictly increasing.

In an almost identical way we can start with **t.1-t.4** and prove **a.1-a.4** showing equivalence.

■

Corollary 1.

$$\frac{T(x)}{x} \textit{ is strictly increasing for all } x > 0 .$$

Proof. By definition of Propostion 1

$$\frac{A(x)}{x} > \frac{A(y)}{y} \Leftrightarrow \frac{x - A(x)}{x} < \frac{y - A(y)}{y} \Leftrightarrow \frac{T(x)}{x} < \frac{T(y)}{y} , \quad 0 < x < y .$$

■

Proposition 3. *If $f(x)$ is strictly increasing and smooth on $\mathbb{R}_{\geq 0}$ with $f(0) = 0$ and $\beta \in \mathbb{R} \setminus \{0\}$ then for $k = 0^+$ or $k = \infty$*

$$f(x) \underset{x \to k}{\sim} \beta x \Leftrightarrow f'(x) \underset{x \to k}{\sim} \beta ,$$

or

$$f(x) \underset{x \to k}{=} o(x) \Leftrightarrow f'(x) \underset{x \to k}{=} o(1) .$$

Proof. As $f(0) = 0$ and $f(x)$ is strictly increasing we have

$$\lim_{x \to 0^+} f(x) = 0 , \quad \lim_{x \to \infty} f(x) = \infty .$$

Assuming either side of the first equivalence we have by L'Hôpital's rule ($\beta \neq 0$)

$$\lim_{x \to k} \frac{f(x)}{\beta x} = \lim_{x \to k} \frac{f'(x)}{\beta} = 1 .$$

Assuming either side of the second equivalence we have by L'Hôpital's rule

$$\lim_{x \to k} \frac{f(x)}{x} = \lim_{x \to k} \frac{f'(x)}{1} = 0 .$$

■

Proposition 4. *A function $A(x)$ satisfying the properties **a.1-a.4** can be written as*

$$A'(x) = p + (1 - p) f(x) , \quad A(x) = px + (1 - p) \int_0^x f(t) \mathrm{d}t , \quad p \in (0, 1)$$

such that the function f has the following five properties

f.1 *f is smooth,*

f.2 $f(x) > 0\,, \quad x > 0\,,$

f.3 $f'(x) < 0\,, \quad x > 0\,,$

f.4 $f(0) = 1\,,$

f.5 $f(x) \underset{x\to\infty}{=} o(1)\,.$

Proof. We can find such an f by taking

$$f(x) = \frac{1}{1-p}(A'(x) - p)\,.$$

Now we want to prove the properties for f given A satisfies **a.1-a.4**.

f.1 Follows as a linear transform of the derivative of a smooth function is also smooth.

f.2 By **a.4** $A'(x)$ is strictly decreasing. As $A'(x)$ also follows the assymptotic relations **ã.3** we have $A'(x) > p, \quad \forall x > 0$ and thus $f(x) > 0\,.$

f.3 We have by **a.4**

$$f'(x) = \frac{1}{1-p}A''(x) < 0$$

f.4 By **ã.3** and the fact A is smooth we have $A'(0) = 1$ and thus $f(0) = 1\,.$

f.5 By **ã.3** we have $\lim_{x\to\infty} A'(x) = p$ and thus $\lim_{x\to\infty} f(x) = 0\,.$

■