Higher-Degree Generalizations of the Method of Moving Points

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Abstract

1 Introduction

This report will be presenting a generalization to a previous method, known as the "Method of Animation" [1] or the "Method of Moving Points", popularized in the United States math olympiad community by Zack Chroman, to solve Euclidean geometry problems by parameterizing all configurations of the problem in one variable, then writing the condition to be proven as proving a polynomial in this variable is always zero, then bounding the degree of this polynomial and checking the problem holds for a number of cases, one more than the degree of this polynomial.

This technique was later fully rigorized in 2025 by Molnar-Szabo Vilmos [2], giving a working definition of multiplicity, as well as fully working in homogeneous polynomials, and we will be extensively citing results from this publication.

In this report, we will generalize the "Method of Moving Points" to handle multiple intersections of higher-degree curves, such as conics and cubics, as well as rigorize it algebraically to provide a new framework for moving points on higher-genus curves.

Notate complex projective space as $\mathbb{CP}^n$. U_i will be used for an open set. We will often be working in the z-centered standard affine chart of $\mathbb{CP}^2$, $[x : y : z] \mapsto (x/z, y/z)$, defined on $U_z := \{[x : y : z], z \neq 0\}$, with the line at infinity denoted as $[x : y : 0]$.

Definition 1. Define the circular points at infinity as $[1 : i : 0], [1 : -i : 0]$. All circles are conics passing through these two points, and we will notate these two points as $\mathbf{I}$ and $\mathbf{J}$.

Definition 2. Let $\mathcal{K} \subset \mathbb{P}^2$ be a curve of degree d defined by the homogeneous polynomial $F(x : y : z) \in \mathcal{O}_{\mathbb{P}^{\not{2}}}(d)$. For a point $P = [p_0 : p_1 : p_2] \in \mathbb{P}^2$, define the n-th polar of P with respect to $\mathcal{K}$ (for $1 \leq n < d$) as the curve defined by the homogeneous polynomial of degree $d - n$:

$$\Delta_P^n(F) := \left(p_0 \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial y} + p_2 \frac{\partial}{\partial z}\right)^n F = 0.$$

Definition 3. Define a **moving point** of degree d as a regular map from $\mathbb{CP}^1 \to \mathbb{CP}^2$, $[s : t] \mapsto [F(s,t) : G(s,t); H(s,t)]$ where F, G, H are homogeneous polynomials, all of degree d. Define the **degree** of a moving point to be d.

We will often interpret a moving point as the parameterization of all possible locations of a point P in the plane.

Example 4. Let P be a point on line ℓ, and let Γ be a circle. Let X be a point on Γ, let line XP intersect Γ again at point Q. We can construct a degree-1 moving point parameterizing P. Let A, B be two points on the line with coordinates $[A_x : A_y : A_z]$ and $[B_x : B_y : B_z]$. Define the (bi)regular map

$$\begin{aligned} \mathcal{P} : \mathbb{CP}^1 &\to \mathbb{CP}^2 \\ [m : n] &\mapsto mA + nB. \end{aligned}$$

Let P have coordinates $[\mathbf{P_x} : \mathbf{P_y} : \mathbf{P_z}]$, where $\mathbf{P_i} := mA_x + nB_x$. The image of $\mathbb{CP}^1$ after this map would be line $\ell \in \mathbb{CP}^2$, would give all possible positions for P. Similarily, Q can be given as a degree-2 moving point, as line XP can be given as the line, with coefficients degree-1 homogeneous polynomials,

$$\begin{bmatrix} \mathbf{P_y}Q_z - \mathbf{P_z}Q_y \\ \mathbf{P_z}Q_x - \mathbf{P_x}Q_z \\ \mathbf{P_x}Q_y - \mathbf{P_y}Q_x \end{bmatrix} \cdot [x : y : z] = 0$$

and its intersection with the circle can be given as effectively the construction of the well-known homogeneous degree-2 polynomial parameterization of a circle, by its isomorphism to $\mathbb{CP}^1$.

We will cite Theorem 8.1 and Theorem 9.2 from Vilmos's paper, [2], reproduced below, with the proofs omitted.

Theorem 5 (Locus-degree Theorem)**.** H is a degree n polynomial moving point. Then the locus of H is a degree k algebraic curve, where $k \mid n$, and for every smooth point P on the curve (i.e. tangent space is nondegenerate), then P will have $\frac{n}{k}$ preimages in $\mathbb{CP}^1$. For every order d singularity B, it will have $\frac{n}{k} \cdot d$ preimages in $\mathbb{CP}^1$.

Define the degree of a moving line as the degree of its equation's coefficients as homogeneous polynomials.

Theorem 6 (Maszo-theorem)**.** For two moving points A, B moving on a common conic, then $\deg \overline{AB} = \frac{\deg A + \deg B}{2}$.

We will also cite its definitions of "multiplicity" of certain conditions.

1. The point-point multiplicity of the coincidence of two moving points A, B is the order of the zero of $A - B$ at the intersection point, calculated by taking a chart of $\mathbb{CP}^1$ containing the intersection point.

2. The point-line multiplicity of the coincidence of a moving point and a moving line is given by the order of the zero of their dot product at the coincidence, calculated by taking a chart of $\mathbb{CP}^1$ containing their intersection point.

We will cite a lemma from Vilmos which allows us to work in the affine charts of $\mathbb{CP}^1$ to count multiplicity much easier.

Lemma 7 (point-point multiplicity is preserved, lemma 4.1 in Vilmos)**.** Let $p, q : \mathbb{CP}^1 \to M$ be two holomorphic maps and $\phi : M \to N$ a holomorphic map, where M and N are m and n dimensional complex manifolds. Then for every $t_0 \in U \in \mathbb{CP}$ (where U is an affine chart) where $p(t_0) = q(t_0)$, the point-point multiplicity of $\phi \circ p$ and $\phi \circ q$'s concurrence at t_0 is at least the point-point multiplicity of p and q's concurrence at t_0. Furthermore if ϕ has a Jacobian matrix of rank m at $p(t_0)$, then the two multiplicities are equal.

2 Moving Curves

2.1 Resultants and the Veronese embedding

Definition 8. Given two polynomials in x with coefficients $a_m \dots a_0$ and $b_m \dots b_0$, define the **resultant** of this system as this determinant

$$\begin{bmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m & a_{m-1} & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n & b_{n-1} & \cdots & b_0 \end{bmatrix}$$

Here is a well-known property of the resultant $\mathrm{Res}_t(-,-)$ of two polynomials $\mathcal{P}, \mathcal{Q}$ in $K[x,y,z,\dots][t]$:

Theorem 9. This satisfies $\sqrt{\langle \mathcal{P}, \mathcal{Q}\rangle \cap K[x,y,z,\dots]} = \sqrt{\mathrm{Res}_t(\mathcal{P},\mathcal{Q})}$ (where $\sqrt{I}$ denotes radical ideal).

First, let's generalize the method of moving points to "moving curves". Let our parameterization variable be $[s:t] \in \mathbb{CP}^1$. We will usually work in the chart $t \neq 0$.

Definition 10 (Degree of a moving curve)**.** The **moving degree** d of a moving curvem is the degree of its coefficients, as a polynomial in $[s:t]$.

As a convention to avoid confusion, in this section, we will use capital letters to denote the algebraic degree of a curve, and we will use lowercase letters to denote the moving degree of a curve.

Definition 11 (Veronese map)**.** Define the degree-2 Veronese map as such [5]:

$$\mathbb{CP}^2 \to \mathbb{CP}^5$$

$$[x:y:z] \mapsto \left[x^2 : xy : xz : y^2 : yz : z^2\right].$$

Analogously, define the degree-N Veronese map for higher degree monomials, sending

$$\mathbb{CP}^2 \to \mathbb{CP}^{\binom{N+2}{2}-1}$$

$$[x:y:z] \mapsto \left[x^N : x^{N-1}y : x^{N-1}z : x^{N-2}y^2 : x^{N-2}yz : \cdots : y^N : y^{N-1}z : \cdots : yz^{N-1} : z^N\right].$$

The image of the projective plane after this map is the Veronese surface, lying in a subvariety of $\mathbb{CP}^{\binom{N+2}{2}-1}$.

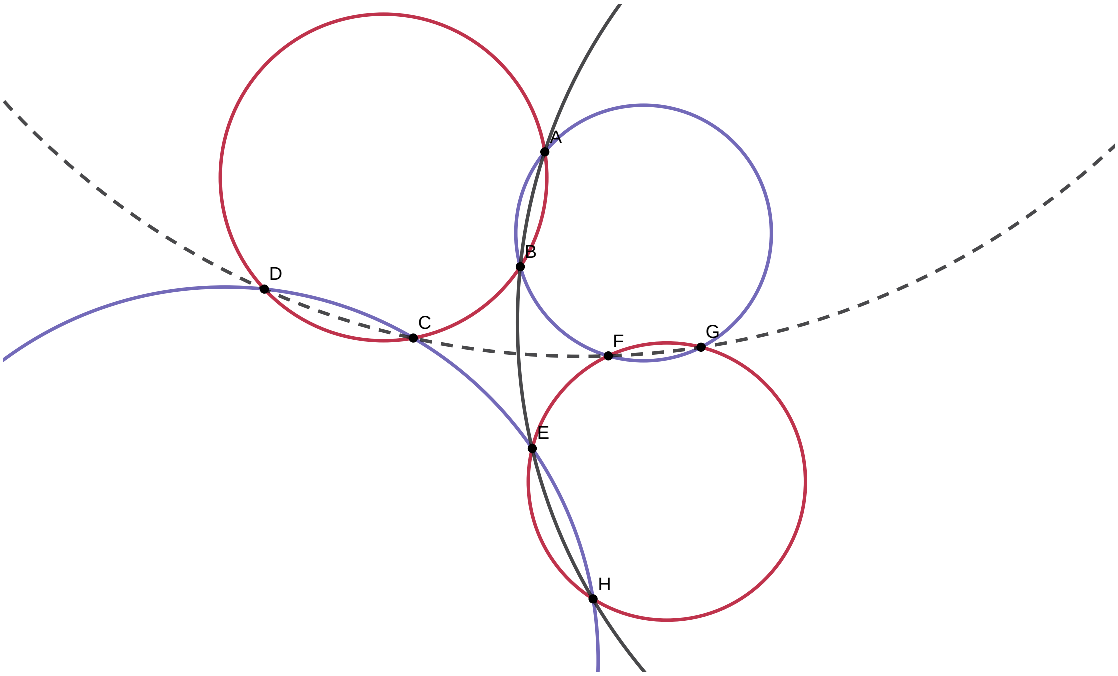
The image of any degree-N plane curve, after the degree N Veronese map, will lie in the intersection of a hyperplane with dimension $\binom{N+2}{2} - 2$ with the Veronese surface.

Further, after the Veronese map, a degree-d moving curve corresponds to a hyperplane in $\mathbb{CP}^{\binom{N+2}{2}-2}$ moving with degree d, given by the matrix transpose of its coefficients.

It is often useful to dualize everything in the $\mathbb{CP}^{\binom{n+2}{2}-1}$, swapping the hyperplanes (degree-N curves) with the points.

As an example of this technique, we give a clean solution to a classical theorem.

Example 12 (Bundle theorem)**.** With points defined in the diagram, prove if the quadrilateral $ABEH$ is cyclic, then quadrilateral $DCFG$ is cyclic.



Proof. After applying the degree-2 Veronese map, circles become dim-4 hyperplanes passing through the line containing the images of $\mathbf{I}, \mathbf{J}$. Choose a fixed dim-3 linear subspace $\mathscr{P}$ in $\mathbb{CP}^5$ not including the line containing the images of $\mathbf{I}, \mathbf{J}$ as a subvariety, and map each circle uniquely to the intersection of its corresponding dim-4 hyperplane with $\mathscr{P}$. Thus, by projection down from $\mathbb{CP}^5$ to $\mathbb{CP}^3$, the set of circles in $\mathbb{CP}^2$ are all planes in $\mathbb{CP}^3$. Take the projective dual of all planes in this $\mathbb{CP}^3$ to get a bijection of circles in $\mathbb{CP}^2$ to the set of points in $\mathbb{CP}^{3*} \cong \mathbb{CP}^3$. Since all operations we have taken are given by linear homogeneous polynomials, a pencil of coaxal circles, all linear combinations of each other, will map to the line connecting two points in $\mathbb{CP}^3$. Intersection of two lines corresponds to two pencils of circles having a common circle, implying the two pairs of real fixed points of these pencils form a cyclic quadrilateral.

Then, our problem is translated to constructing four points $c_{12}, c_{23}, c_{34}, c_{14}$ in $\mathbb{CP}^3$ such that the line connecting c_{23}, c_{12} and the line connecting c_{34}, c_{14} intersect, then the line connecting c_{23}, c_{34} and the line connecting c_{12}, c_{14} also intersect. However, this is true since the four points must all be coplanar. $\square$

Theorem 13 (Interpolation formula)**.** For $\binom{N+2}{2} - 1$ points P_i in the plane, the points on a degree-N curve passing through all of the points will be given by the preimages of all $Q \in$ Veronese surface $\in \mathbb{CP}^{\binom{N+2}{2}-1}$ satisfying

$$\det \begin{bmatrix} \text{coordinates of } P_1 \text{ after the Veronese} \\ \text{coordinates of } P_2 \text{ after the Veronese} \\ \vdots \\ \text{coordinates of } P_{\binom{N+2}{2}-1} \text{ after the Veronese} \\ Q_1, Q_2, Q_3, \dots, Q_{\binom{N+2}{2}-2}, Q_{\binom{N+2}{2}-1} \end{bmatrix} = 0.$$

Currently this is only well-defined if P_i all lie on a codimension-1 hyperplane, and thus not everywhere-regular - we will address this shortly.

Proof. All points $Q = [Q_1 : Q_2 : \cdots : Q_{\binom{N+2}{2}-1}]$ satisfying this determinant condition will be coplanar on a codimension-1 hyperplane with all of P_i, and the intersection of this hyperplane with the Veronese surface gives the image of our desired curve after the Veronese map. □

The components of this determinant give a rational map from $\mathbb{CP}^1 \dashrightarrow \mathbb{CP}^{\binom{N+2}{2}-1}$, with each of the $\left(\binom{N+2}{2} - 2\right) \times \left(\binom{N+2}{2} - 2\right)$ minors' determinants corresponding to one coordinate in $\mathbb{CP}^{\binom{N+2}{2}-1}$.

It is important to recognize that this map is defined on the open subset of $\mathbb{CP}^1$ of when the previous determinant is not equal to the zero polynomial, as then all components of the determinant will be identically zero and our map will not be well-defined there.

Theorem 14. If $P_1, P_2, P_3, \dots$ are moving points, with there existing no value $(s : t) \in \mathbb{CP}^1$ such that the previous determinant is the zero polynomial, then the degree-N moving curve through all of them has moving degree

$$N \cdot (\text{sum of degrees of the moving points}).$$

Proof. After the degree-N Veronese map is applied, the coordinates of a moving point P_i moving with degrees $\deg P_i$ will be polynomials of degrees $\deg P_i \cdot N$. The row in the above matrix corresponding to P_i will thus have homogeneous terms of degree $\deg P_i \cdot N$. Then, each component of our determinant will have degree $\sum(\deg P_i \cdot N) = N \cdot \sum(\deg P_i)$. Since this determinant is never the zero polynomial, the components in each coordinate will never be zero at the same $(s : t) \in \mathbb{CP}^1$ so our rational map is everywhere-defined and thus regular. □

In fact, for any projective smooth curve, all rational functions on it extend to locally regular functions in each affine open on it. [8]

Suppose there is a point $(r_s : r_t) \in \mathbb{CP}^1$ such that this determinant is the zero polynomial. Let U_i be a chart of $\mathbb{CP}^1$ containing a neighborhood of $(r_s : r_t)$, sending $(r_s : r_t) \mapsto r \in \mathbb{C}$. Restricted to U_i, we get a well-defined multiplicity for each coefficient of the determinant vanishing, by the number of iterated derivatives of it which vanish at r. Define the **multiplicity of vanishing** at r analogously.

Note when the determinant vanishes at r with multiplicity n, then in U_i, the determinant can be given by the polynomial $F_1(z)(z-r)^n \cdot Q_1 + F_2(z)(z-r)^n \cdot Q_2 + \cdots = 0$ for some polynomials $F_i(z)$, with at least one F_i being nonzero at r.

If the determinant is the zero polynomial ever, then we get degree-reduction lemmas, via the procedure to extend a rational map from $\mathbb{CP}^1 \dashrightarrow \mathbb{CP}^n$ to a regular map.

Theorem 15 (Degree-bound on interpolated curve)**.** For $\binom{N+2}{2} - 1$ moving points on the plane, then the moving degree of a degree-N curve passing through all of them will be maximally bounded by

$$N \cdot (\text{sum of degrees of the moving points})$$

- (number of times the previous determinant is zero, including multiplicity).

Proof. We defined a rational map from $\mathbb{CP}^1 \mapsto \mathbb{CP}^{\binom{N+2}{2}-1}$. Since our rational map is given in each coordinate by polynomials homogeneous in $[s : t]$ (call them $F_1(s : t)$), the only time this map is not defined is for $[r_s : r_t]$ when all of the homogeneous polynomials F_i are zero, with determinant multiplicity m as defined before. When this happens, we can write our map (globally) as $[s : t] \mapsto [F_1(s,t)(r_s t - r_t s)^m + F_2(s,t)(r_s t - r_t s)^m + \dots] = 0$ such that there exists $F_i \neq 0$ at $[r_s : r_t]$. However, this rational map is identical to the regular map $[s : t] \mapsto [F_1(s,t) + F_2(s,t) + \dots] = 0$, dropping the expected degree by m. Repeating this procedure over every time the rational map is undefined, we get our desired result. □

When n moving points P_i concur, we get more reduction. The amount of reduction will be much easier to calculate if we work in an affine chart, as we will look at derivatives near the concurrence. Let the points be parameterized by t near the concurrence.

Then we get the multiplicity of vanishing of the determinant is given by its corank, plus additional terms from first and higher order derivatives vanishing simultaneously as well. However, the corank of the higher terms is generally hard to fully count nicely, and usually in geometry problems this exact reduction is not needed to solve them.

2.2 Geometric degree-reduction lemmas

For this section, we will give a few lemmas for commonly-seen situations when degree of an interpolated moving curve drops, and maximally bound it in each case, helping to connect the geometric picture with the algebraic picture. Assume all moving points defined here are non-cuspidal; that is, they always have a well-defined first derivative in an appropriate affine chart.

Lemma 16 (Concurrence degree-reduction Lemma)**.** When n_0 moving points P_i concur, the degree of the interpolated curve will drop by at least $n_0 - 1$ (summed over all concurrences).

Proof. If n_0 moving points concur, then by row-reduction, after the Veronese map, we can construct a matrix with $n_0 - 1$ rows which all vanish simultaneously at the time of concurrence. This gives the determinant a multiplicity of vanishing with at least $n_0 - 1$ at this root.

Remark 17. Note how this immediately gives the formula for the degree of a moving line through moving points A and B as $\deg A + \deg B - \{\#(A = B)\}$. By duality in $\mathbb{CP}^2$, the degree of the point of intersection of two moving lines ℓ_A and ℓ_B is given by $\deg \ell_A + \deg \ell_B - \{\#(\ell_A = \ell_B)\}$.

□

Lemma 18 (Collinearity degree-reduction Lemma)**.** For a conic through five points, whenever four points are collinear, the moving degree of the conic drops by at least 1, summed over all occurrences of collinearity.

Proof. The image of a line after the Veronese map is a conic in $\mathbb{CP}^5$ lying on some dim-2 plane. Thus, when four points are collinear, after the Veronese map, they will go to four points lying on a common dimension-2 plane. Thus the interpolating conic's matrix will have corank at least 1, so the determinant will vanish with multiplicity at least 1. □

For conics, we have a special case.

Lemma 19 (5-collinear degree-reduction Lemma)**.** For a conic through five points, whenever all five points are collinear, the degree drops by at least 2.

Proof. Similar to the previous proof, this gives five points in $\mathbb{CP}^5$ lying on a dim-2 plane, and thus a corank of at least 2, since for five points lying on a common dim-2 plane, at least two must be linear combinations of the other three, as three non-collinear points determine a plane. □

Theorem 20 (Intersection locus degree formula)**.** Two curves of degrees A and B in $\mathbb{CP}^2$ moving with degrees m and n, never sharing a common irreducible component, will intersect in a locus of maximally degree $An + Bm$.

Proof. In a standard affine chart of $\mathbb{CP}^1$, the locus of intersections will be given by the zero set of a resultant in t of the two curves. By the determinant definition of resultant, we have m terms of degree A multiplied by n terms of degree B, so the total degree of the resultant is $An + Bm$ in t. In the other chart of $\mathbb{CP}^1$, since we have homogenity of the polynomials defining our moving curves, we also have degree $An + Bm$. □

If they do share a common irreducible component at some point, then the resultant must contain the whole irreducible component, and thus has to have the irreducible component as a factor. This allows for further degree-reduction sometimes, and will be used in the examples.

Remark. Note how this is compatible with the degree-formula for the intersection of two moving lines ℓ_A and ℓ_B as $\deg \ell_A + \deg \ell_b - \{\#(\ell_A = \ell_B)\}$, as when $\ell_A = \ell_B$, we have a line in the resultant, dropping its degree by 1. Thus the degree of the resultant here exactly corresponds to the degree of the moving point formed by intersecting the two lines.

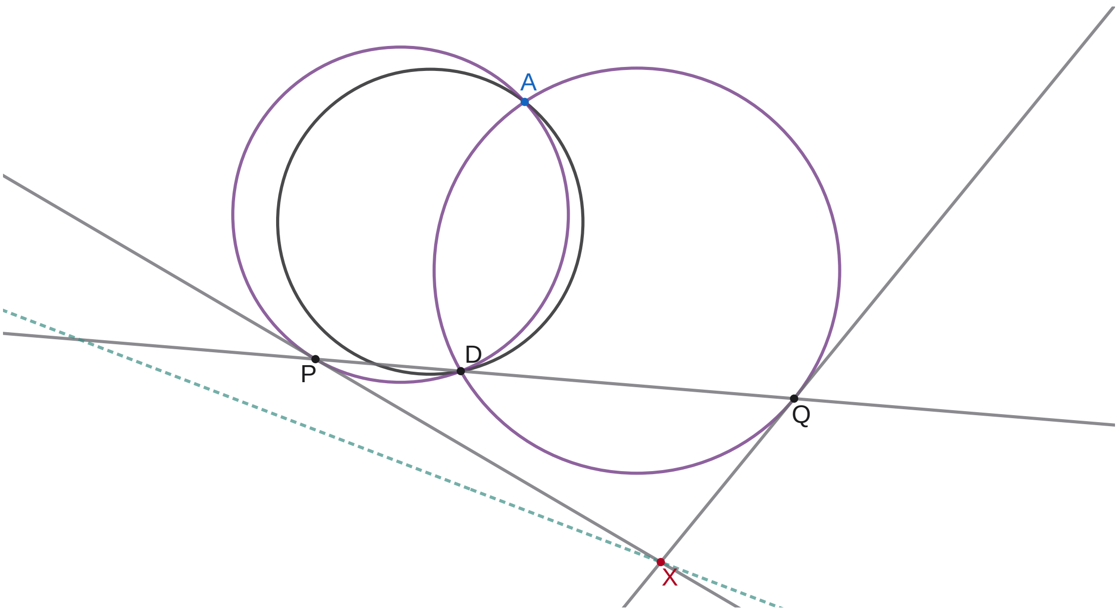
Lemma 21 (Condition for passing through a point)**.** A degree N curve moving with degree d will always pass through a point P if it passes through P for $d + 1$ cases.

Proof. The condition for a moving hyperplane to pass through a fixed point is a linear equation in the coordinates of the hyperplane, so there can be at most d solutions unless the hyperplane always contains the fixed point. □

Now we will solve some olympiad geometry problems with this technique, to demonstrate its utility and clarify the theory.

3 Applications and examples

Example 22 (2023 ELMO Shortlist G4)**.** Let D be a point on segment PQ. Let ω be a fixed circle passing through D, and let A be a variable point on ω. Let X be the intersection of the tangent to the circumcircle of $\triangle ADP$ at P and the tangent to the circumcircle of $\triangle ADQ$ at Q. Show that as A varies, X lies on a fixed line.



Solution. Let $\mathbf{I}, \mathbf{J}$ be the circular points at infinity. Parameterize $A \in \omega \in \mathbb{CP}^2$ as a degree-2 moving point on the circle ω. By the Locus-degree Theorem, A will cover the circle once.

Claim. Circles (ADP) and (ADQ) move with degree 1.

Note there are three positions for A where $A = \mathbf{I}, \mathbf{J}, D$, since A covers the circle once. By the generalized degree formula, we have the circle (ADP) maximally moves with degree

$$2(\deg \mathbf{I} + \deg \mathbf{J} + \deg D + \deg P + \deg A) - [\ \#\text{ of A=}\mathbf{I}\text{,}\mathbf{J}\text{,D}] = 2(0+0+0+0+2) - 3 = 1,$$

and similarily for (ADQ).

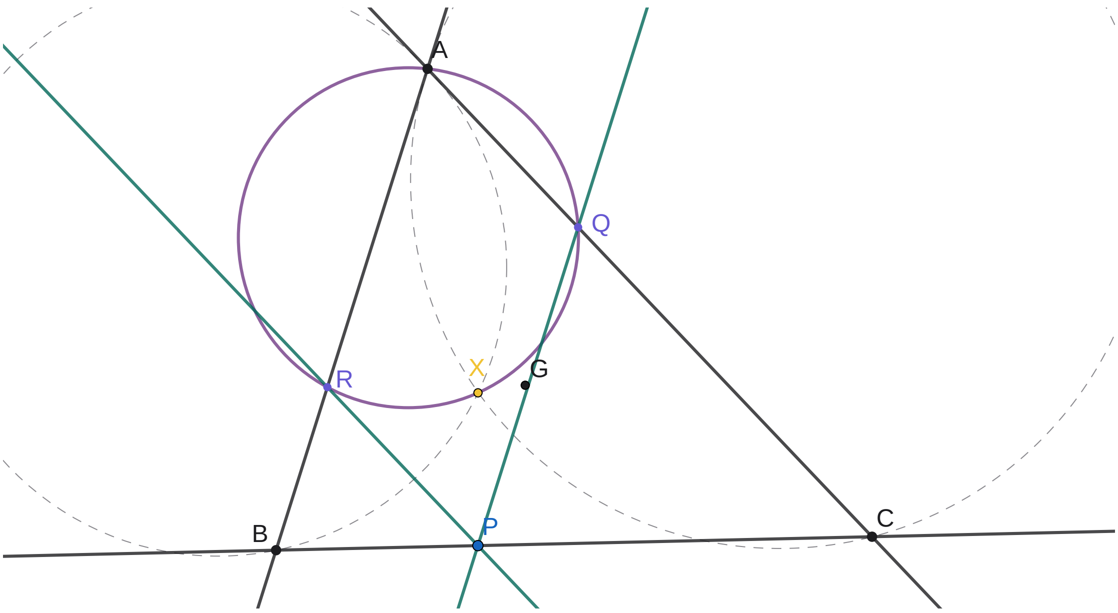
Let $\mathcal{C}_1(x:y:z) := (ADP)$. The tangent at P is given by the line

$$x\frac{\partial \mathcal{C}_1}{\partial x}(P) + y\frac{\partial \mathcal{C}_1}{\partial y}(P) + z\frac{\partial \mathcal{C}_1}{\partial z}(P) = 0.$$

Since P is a fixed point, and $\frac{\partial \mathcal{C}_1}{\partial x}$ varies linearly as the coefficients of $\mathcal{C}_1$ vary linearly, the tangent at P also moves with degree-1. Similarly, the tangent at Q moves with degree-1.

The two tangents are identical when $A = PQ \cap \omega$, implying their intersection X moves with degree $1 + 1 - 1 = 1$, so X moves on a fixed line by the Locus-degree Theorem. $\square$

Example 23 (2008 USA IMO Team Selection Test P7)**.** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $\overline{PQ} \parallel \overline{AB}$ and $\overline{PR} \parallel \overline{AC}$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.



Solution. First, we want to prove X exists.

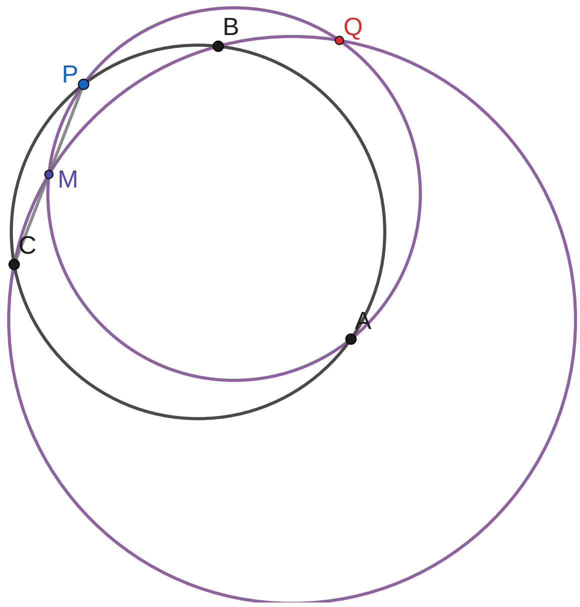
Move P, then points Q, R will move with degree 1 each, by projecting through the point at infinity along AB and AC respectively, so circle (AQR) will move maximally with degree $2(\deg Q + \deg R) = 4$. When $P = B$ and $P = C$, we have $Q = A$ and $R = A$ respectively, which decrease our degree bound to maximally 2 by the Concurrence degree-reduction Lemma applied twice.

When P is at the point at infinity along BC, we have Q, R as the points at infinity along AC and AB respectively. Since $\mathbf{I}, \mathbf{J}$ lie on the line at infinity, we have $Q, R, \mathbf{I}, \mathbf{J}$ collinear here, so we can apply the Collinearity degree-reduction Lemma to decrease the degree of the circle to degree 1, a tight bound. Since two general values of P give circles (AQR) intersecting at four distinct complex points, (AQR) will move in a pencil of conics, and thus pass through four fixed points. We know it passes through the three fixed points $A, \mathbf{I}, \mathbf{J}$, let the fourth fixed point be X'.

By setting $P = B, P = C$, we know that X' is the intersection of the circle through B tangent to AC at A, and the circle at C tangent to AB at A. In other words, X' is the Miquel point of degenerate quadrilateral $ABAC$, also known as the Dumpty point, known to lie on the A-symmedian, and thus $\angle BAG = \angle CAX$. $\square$

Next we give an example of the resultant bound for the degree of the intersection of two moving curves.

Example 24 (2014 IMO Shortlist G4)**.** Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

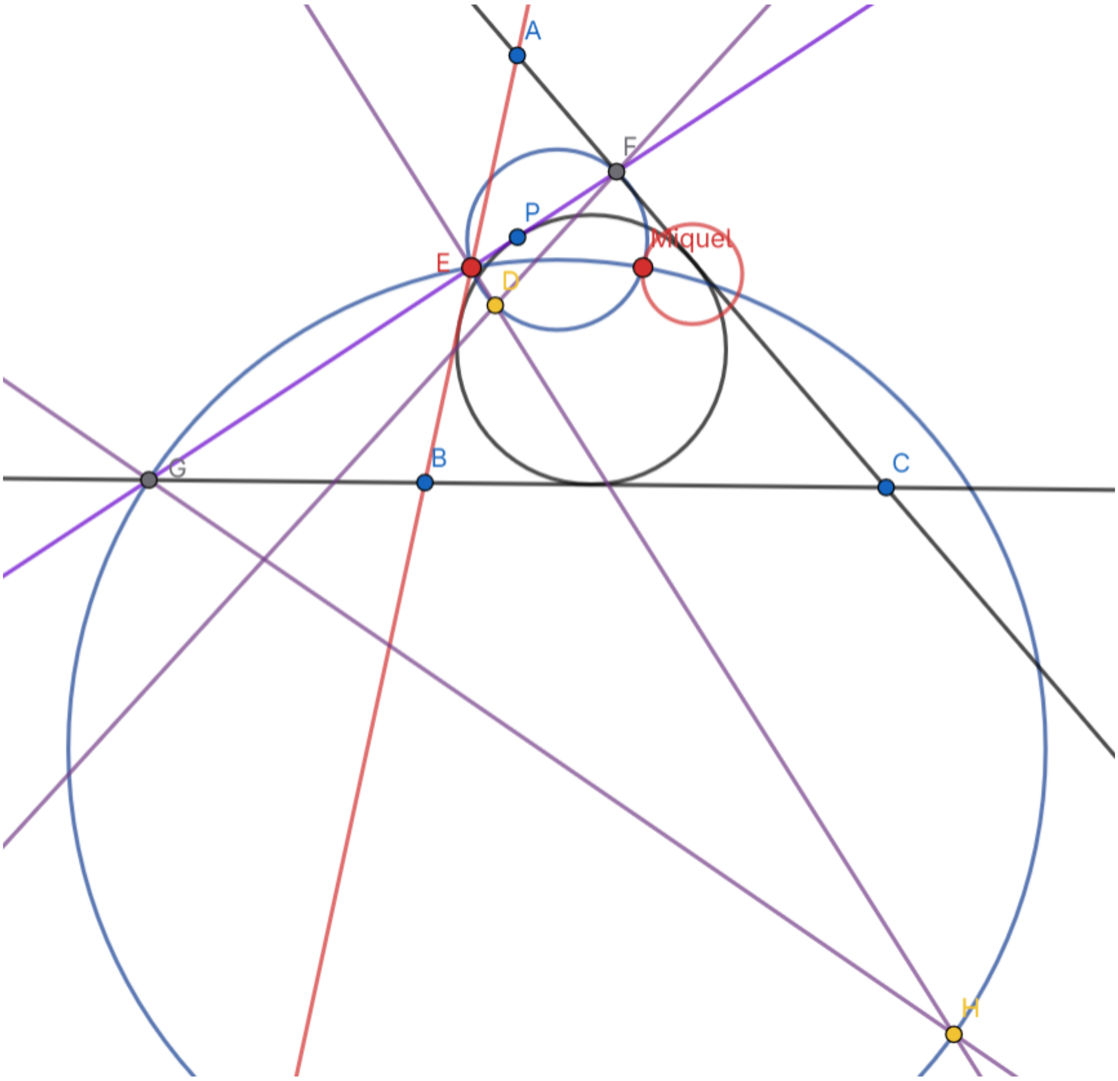


Move P with degree 2 on (ABC). Then M will also move with degree 2. (AMP) will then move with degree at most $2 \cdot (\deg P + \deg M) = 2 \cdot (4)$, with the degree reduced by $P = \mathbf{I}, P = \mathbf{J}, P = A, M = \mathbf{I}, M = \mathbf{J}, M = P$, so (AMP) will move with degree at most 2. (CMB) will then move with degree at most 4, dropping when $M = C, M = \mathbf{I}, M = \mathbf{J}$. Thus (CMB) moves with degree 1, a tight bound. The intersection of (CMB) with (AMP) will then be a degree 6 curve, singular at each of the two circle points as they are stationary. One of the points of intersection is M, which moves on a degree 2 circle.

When $P = C$, $(CMB) = (AMP)$, giving us a degree 2 irreducible component in the resultant. Since $\mathbf{I}$ and $\mathbf{J}$ are fixed, Q must move on a degree 2 curve, passing through the circle points when $P = \mathbf{I}$ and $P = \mathbf{J}$. Thus Q moves on a circle.

Example 25 (TelvCohl, AoPS)**.** Let I be the incenter of $\triangle ABC$. Let P be a point lie on the incircle (I) of $\triangle ABC$. Let ℓ be a line passing through P and tangent to (I) . Let ℓ_a, ℓ_b, ℓ_c be the reflection of ℓ in BC, CA, AB, respectively . Let M be the Miquel point of complete quadrilateral $\{\ell, \ell_a, \ell_b, \ell_c\}$.

Find the locus of M when P varies on (I).



Solution. Let ℓ intersect AB, AC, BC at E, F, G respectively. Move P as a degree-2 moving point on the incircle.

Claim. E moves with degree-1. Similarily, F and G move with degree 1.

Proof. Consider the tangent to P as a moving line, which is given by the first polar of P in the incircle, so it also moves with degree 2. This line is equal to line AB once when P is at the C-intouch point, so we can apply Concurrence degree-reduction Lemma (in the dual space $\mathbb{P}^{2\vee}$) to get E moves with degree 1 on AB. $\square$

By [2], the reflection of a degree n line across a fixed line is degree $2n$, so the lines ℓ_a, ℓ_b, ℓ_c are degree 4, and ℓ is degree 2. $D = \ell_c \cap \ell_b$ is degree $4 = 6 - 2$ by Concurrence degree-reduction Lemma, since the lines ℓ_c and ℓ_b are equal when the tangent at P passes through A.

Now our setup is done.

Claim 26. Circle (DEF) moves with deg 2.

Proof. We expect (DEF) to have degree $2(\deg D + \deg E + \deg F) = 2(4 + 1 + 1) = 12$. However, we actually have 10 cases where the degree drops, by Concurrence degree-reduction Lemma. When P is at the B and C-intouch points, E and F both concur. There are two cases when $\ell \perp AB$ (implying $D = F$), and another two cases where $\ell \perp AC$ (implying $D = E$). Finally, note D moves on a rational quartic (rational, because we have an explicit birational map from it to the incircle, given by parameterization in P). By the Riemann-Hurwitz formula, a rational quartic has three singularities, in this case two at $\mathbf{I}$ and $\mathbf{J}$ respectively. Thus we can apply Concurrence degree-reduction Lemma another four times, since (DEF) is a circle. $\square$

By an analogous argument, (EHG) moves with degree 2 also. Therefore the resultant of (EHG) and (DEF) is a curve of degree $2 \cdot 2 + 2 \cdot 2 = 8$. Now we start factorizing the resultant, similar to the previous example. We want to find a (real) circle in it that passes through the Miquel point for some P.

We know it includes AB and the line at infinity when P is at the antipode of the C-intouch point.

Now let's consider what happens when P is at a circular point at infinity. Let O be the incenter. When $P = \mathbf{I}$, we have the tangent at P is line $O\mathbf{I}$, as the tangents at $\mathbf{I}$ and $\mathbf{J}$ to the incircle will intersect at O (by the projective definition of foci of a conic). Here, point E is given by $OI \cap AB$, so $E, D, \mathbf{J}$ are collinear on the reflection of line $O\mathbf{I}$ across AB (the reflection of $\mathbf{I}$ is $\mathbf{J}$). Similarly, $F, D, \mathbf{J}$ are collinear, so $D = \mathbf{J}$ when $P = \mathbf{I}$ (by Bezout's theorem). Analogously, $H = \mathbf{J}$ as well. Thus we have that E, D, F, G, H are concyclic, so $(EFD) = (EHG)$, giving us a circle in the resultant. Similarly for $P = \mathbf{J}$ we have another circle in the resultant. Thus we have found two lines and two circles in the resultant.

None of these circles or lines include the locus of the Miquel point, so therefore the Miquel point's locus must be a conic. However, the Miquel point passes through $\mathbf{I}$ and $\mathbf{J}$ when $P = \mathbf{I}$ and $P = \mathbf{J}$ as the two circles are the same, as shown above, so we are done. $\square$

4 Moving points through invertible sheaves

Here we reformulate the theory of moving points and curves through invertible sheaves. This viewpoint is not fully explored.

Our definition of geometric conditions imposed on a degree-d moving point defines three global sections (one for each coordinate of $\mathbb{P}^2$) of the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(d)$, where each section restricts to a rational function on the open sets given by $s \neq 0$ and $t \neq 0$ respectively [6]. Note that Theorem 14 tells us that any globally rational function on $\mathbb{P}^1$ is identical to a locally regular function in each open set, and also prevents these three sections from all being simultaneously zero.

Analogously through the Veronese map, we can actually define a moving curve of degree D moving with degree d in $\mathbb{P}^n$ as a choice of $\binom{D+n}{n}$ sections of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$, through the Veronese map.

Further, a degree-D condition on a moving point can be given by a section of $\mathcal{O}_{\mathbb{P}^2}(D)$, which pulls back through the morphism induced by the moving point from $\mathbb{P}^1 \to \mathbb{P}^n$ to define an effective Weil divisor on $\mathbb{P}^1$.

The main reason to use this definition is to move points on non-rational (smooth) curves. For a curve $\mathcal{K} \in \mathbb{P}^2$, define a **$\mathcal{K}-$parameterized moving point** in $\mathbb{P}^n$ through a choice of $n+1$ sections of the line bundle $\mathcal{O}_{\mathcal{K}}(d)$.

Here is a toy example of how we can use this to "move points on curves".

Example 27 (Class of a smooth degree-d curve)**.** For a degree-d smooth curve $\mathcal{K} \in \mathbb{P}^2$ and a generic point P, there are $d(d-1)$ lines through P tangent to $\mathcal{K}$.

Proof. For a point on $\mathcal{K}$, we can get its tangent in $\mathbb{P}^2$ by taking its $(d-1)$th polar. Algebraically, this is given by a morphism from $\mathbb{P}^2$ to $\mathbb{P}^{2\vee}$ induced by sections of degree $d-1$ (in each coordinate).

Therefore we have a chain of maps from $\mathcal{K} \xrightarrow{d} \mathbb{P}^2 \xrightarrow{d-1} \mathbb{P}^{2\vee}$. We have a section of $\mathcal{O}_{\mathbb{P}^{2\vee}}(1)$ testing whether a line in $\mathbb{P}^2$ passes through P. Pulling everything back, we get a section of $\mathcal{O}_{\mathcal{K}}(d-1)$, and we want to count the number of zeroes on this when $\mathcal{K}$ is embedded in $\mathbb{P}^2$ as a section of $\mathcal{O}_{\mathbb{P}^2}(d)$. This is well-known to be given by $d \cdot (d-1)$. [7] □

Considering linearily moving systems of degree-d divisors is also very useful.

Theorem 28 (Generalized Desargues' Involution Theorem, for lines)**.** For a line ℓ in $\mathbb{P}^2$, choose $N-2$ points X_i on it and define a degree-N moving curve $\mathcal{K}$, moving linearily with degree 1, passing through these points. This will intersect ℓ at two points P, Q. Then P, Q will swap under a fixed involution of ℓ.

Proof. Let M be the formal sum of all X_i as a divisor of degree $N-2$. The set of $\mathcal{K}$ restricted to ℓ defines a linear system in $\mathcal{O}(N)$, parameterized by a $\mathbb{P}^1$ (let's say with coordinates $[a : b]$). Since M is on a $\mathbb{P}^1$ with coordinates $[s : t]$, it is defined by the rational function $m = (t_1 s - s_1 t)(t_2 s - s_2 t)\ldots$. This must be a factor of every $\mathcal{K}|_\ell$, which is given by a function in the form $af + bg$ for cubic homogeneous polynomials f, g. So $\frac{af+bg}{s}$ is fully defined as a section of $\mathcal{O}(2)$ away from X_i, and since we are working over a $\mathbb{P}^1$, we can apply the degree-reduction lemma to extend this to a global section.

In other words, we have $\mathcal{O}(N) \otimes \mathcal{O}(-(N-2)) = \mathcal{O}(2)$ over $\mathbb{P}^1$. [7] Then we get the other intersection points define a linear system of a degree-2 divisor, and then our problem is just the standard Desargues' Involution Theorem. □

An analogous theorem holds for conics, except we consider $2n - 2$ points on the conic. Here, the degree-2 divisor on it actually extends to a degree-1 moving line in $\mathcal{O}_{\mathbb{P}^2}(1)$ since a conic is of degree 2.

Further work is likely needed to fully characterize how linear systems of divisors behave geometrically. Notably, a linear system of deg-3 divisors on a conic appears to define triangles in a Poncelet configuration (i.e. all sharing a common inconic), however this is still unproven. Additionally, for degree-reduction lemmas (extension of a rational function to a locally regular function via removal of base points) to work on an arbitrary algebraic curve, we need the divisor we are removing to be rationally equivalent to $n \cdot P$ where P is a point outside of U_i.

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