

ON THE PROJECTION ANALOGUE OF THE SLICING PROBLEM

JOHAN ASPEGREN

ABSTRACT. We give a proof for a sharp projection analogue of the slicing problem. Moreover, we show a geometric proof of the slicing problem.

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1. INTRODUCTION

The isotropic conjecture or Bourgain's slicing problem asks for the existence of a universal constant c such that

Theorem 1.1. *There exists an affine hyperplane H and a universal constant c such that*

$$m_{n-1}(E \cap K) > c,$$

for convex bodies K of unit volume.

A classic reference for these questions is [6]. The isotropic constant conjecture has already been proven by Klartag and Lehec [5]. The entries of the covariance matrix of a convex body K are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|} - \frac{\int_K x_i}{|K|} \frac{\int_K x_j}{|K|}.$$

We define the isotropic constant of any convex body K in a scaling-invariant way using

$$L_K^{2n} := \frac{\text{Det}(\text{Cov}K)}{|K|^2}.$$

The isotropic position is a position when the covariance matrix is diagonal and all the diagonal entries are the same. This kind of position exists [6].

Theorem 1.2. *The isotropic constant is universally bounded for symmetric convex bodies.*

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The above is equivalent to the slicing problem [6]. So is the following formulation.

Theorem 1.3. *For any symmetric convex body K of unit volume there exists a position $T(K)$ such that*

$$c \leq |B(0, R) \cap T(K)|,$$

where

$$|B(0, R)|^{1/n} \leq C,$$

and C and c are universal constants.

We use the next theorem to study a non-symmetric "almost counterexample" to the isotropic constant conjecture in the above formulation with fixed, natural setup $c = 1$ and $C = \frac{1}{2}$. We prove that

Theorem 1.4 (Angular Thickness Condition). *Let $K \subset \mathbb{R}^n$ be a bounded measurable set of measure $|K| = 1$ that is star-shaped with respect to the origin. Then there exists a measurable radial function $R(\theta) \in [0, \infty]$ on S^{n-1} such that*

$$K = \{r\theta : \theta \in S^{n-1}, 0 \leq r \leq R(\theta)\}.$$

Let $R > 0$ be such that $|B(0, R)| = c$. If the set of directions where K reaches at least radius R satisfies

$$(1.1) \quad \omega(\{\theta \in S^{n-1} : R(\theta) \geq R\}) \geq C \omega(S^{n-1}),$$

then

$$|K \cap B(0, R)| \geq Cc.$$

Moreover, constant $\frac{1}{2}$ with set up $c = 1$ is sharp: if the left-hand side of (1.1) is strictly less than $\frac{1}{2} \omega(S^{n-1})$, the conclusion may fail.

Our theorem for the projection analog is the following:

Theorem 1.5. *For every convex body $K \subset \mathbb{R}^n$ there exists a universal constant $c > 0$ such that*

$$\mathbb{E}_{\varphi \in S^{n-1}} [m_{n-1}(\text{Proj}_{\varphi} K)] \geq c |K|^{\frac{n-1}{n}},$$

where the expectation is taken with respect to surface measure on S^{n-1} . Moreover, the inequality is sharp, with equality for Euclidean balls.

Now, if the convex body is central symmetric, then the maximal slice is always attained in the subspace, say, H_{ϕ} . If we project K to that subspace in general the projection is bigger since $H_{\phi} \cap K \subset \text{Proj}_{\phi}(K)$. If K is symmetric with respect to H_{ϕ} , then the projection and central slice are the same with respect to H_{ϕ} . Moreover, a fundamental relationship between projections and central slices uses the polar body

$$K^{\circ} = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.$$

The projections and slices are related by polarity as follows:

$$(\text{Proj}_{\phi}(K))^{\circ} = K^{\circ} \cap H_{\phi}, \quad (K \cap H_{\phi})^{\circ} = \text{Proj}_{\phi}(K^{\circ}).$$

2. AN ELEMENTARY BOUND FOR THE RANDOM PROJECTIONS

We use Cauchy's surface area formula:

$$|\partial K| = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} m_{n-1}(\text{Proj}_\varphi(K)) d\omega(\varphi),$$

where $d\omega$ is the standard surface measure on S^{n-1} and $\omega_{n-1} = |B^{n-1}|$.

Combining the Cauchy surface area formula with the isoperimetric inequality

$$|\partial K| \geq n \omega_n^{1/n} |K|^{(n-1)/n},$$

we obtain

$$|K|^{n/(n-1)} \leq \omega_n^{-1/n} n^{-1} \left(\frac{1}{\omega_{n-1}} \int_{S^{n-1}} m_{n-1}(\text{Proj}_\varphi(K)) d\omega(\varphi) \right).$$

Writing the integral in terms of expectation with respect to the normalized probability measure

$$d\sigma(\varphi) = \frac{1}{|S^{n-1}|} d\omega(\varphi), \quad |S^{n-1}| = n\omega_n,$$

we obtain

$$\int_{S^{n-1}} m_{n-1}(\text{Proj}_\varphi(K)) d\omega(\varphi) = n\omega_n \mathbb{E}[m_{n-1}(\text{Proj}_\varphi(K))].$$

Substituting this into the previous inequality yields

$$|K|^{n/(n-1)} \leq \omega_n^{-1/n} n^{-1} \cdot \frac{n\omega_n}{\omega_{n-1}} \mathbb{E}[m_{n-1}(\text{Proj}_\varphi(K))].$$

Hence,

$$\mathbb{E}[m_{n-1}(\text{Proj}_\varphi(K))] \geq \omega_{n-1} \omega_n^{(1-n)/n} |K|^{(n-1)/n}.$$

Since equality in the isoperimetric inequality holds only for Euclidean balls, these are the extremizers for the above inequality. This ends the proof of Theorem 1.5.

3. ON ANGULAR THICK STAR-CONVEX SETS

In this section we prove the theorem 1.4.

Proof. Since K is star-shaped, polar coordinates yield

$$|K \cap B(0, R)| = \int_{S^{n-1}} \int_0^{\min(R, R(\theta))} r^{n-1} dr d\omega(\theta) = \frac{1}{n} \int_{S^{n-1}} \min(R^n, R(\theta)^n) d\omega(\theta).$$

The volume of the ball is

$$|B(0, R)| = \frac{\omega(S^{n-1})}{n} R^n = c \quad (\text{by assumption}).$$

Define the density

$$f(R) := \frac{|K \cap B(0, R)|}{|B(0, R)|} = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \min\left(1, \left(\frac{R(\theta)}{R}\right)^n\right) d\omega(\theta).$$

Let

$$A := \{\theta \in S^{n-1} : R(\theta) \geq R\}.$$

On A we have $(R(\theta)/R)^n \geq 1$, so the integrand equals 1. Therefore

$$f(R) \geq \frac{1}{\omega(S^{n-1})} \int_A 1 d\omega(\theta) = \frac{\omega(A)}{\omega(S^{n-1})} \geq C \quad \text{by assumption (1.1).}$$

Thus

$$|K \cap B(0, R)| = f(R) \cdot |B(0, R)| \geq Cc,$$

which proves the main statement.

Sharpness. Let $0 < \varepsilon < \frac{1}{2}$. Choose a measurable set $A \subset S^{n-1}$ with

$$\omega(A) = \left(\frac{1}{2} - \varepsilon\right)\omega(S^{n-1}).$$

Define a star-shaped set

$$K = \{r\theta : \theta \in A, 0 \leq r \leq L\},$$

where $L > 0$ is chosen so that $|K| = 1$. Then

$$|K| = \int_A \int_0^L r^{n-1} dr d\omega(\theta) = \frac{\omega(A)}{n} L^n = 1 \implies L^n = \frac{n}{\omega(A)}.$$

Using $|B(0, R)| = 1$ gives $\frac{\omega(S^{n-1})}{n} R^n = 1$, so $R^n = \frac{n}{\omega(S^{n-1})}$. Therefore

$$\left(\frac{L}{R}\right)^n = \frac{n/\omega(A)}{n/\omega(S^{n-1})} = \frac{\omega(S^{n-1})}{\omega(A)} = \frac{1}{\frac{1}{2} - \varepsilon} > 1,$$

and hence $L \geq R$, so $R(\theta) \geq R$ for all $\theta \in A$ and $R(\theta) = 0$ otherwise. Thus

$$f(R) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \min\left(1, \left(\frac{R(\theta)}{R}\right)^n\right) d\omega(\theta) = \frac{\omega(A)}{\omega(S^{n-1})} = \frac{1}{2} - \varepsilon < \frac{1}{2}.$$

Consequently

$$|K \cap B(0, R)| = f(R) \cdot |B(0, R)| = \frac{1}{2} - \varepsilon,$$

which violates the desired inequality. Since ε is arbitrary, the constant $\frac{1}{2}$ in (1.1) is a lower bound for $c = 1$. \square

4. A GEOMETRIC PROOF OF THE SLICING PROBLEM

We start with a known theorem [4]

Theorem 4.1. *Let K be a convex body. Denote*

$$L_{K,\theta} := \left(\int_K \langle \theta, x \rangle^2 dx \right)^{1/2}.$$

Then there exists universal constants c_1 and c_2 such that

$$\frac{c_1}{L_{K,\theta}} \leq |H_\theta \cap K| \leq \frac{c_2}{L_{K,\theta}},$$

for all $\theta \in S^{n-1}$.

We use the following symmetrization method.

Definition 4.2 (Cylinder Symmetrization). Let Ω be a bounded, measurable set in \mathbb{R}^n . The cylinder symmetrization of Ω , with respect to direction θ , denoted by Ω_θ^* , is the closed spherical cylinder with the center of mass at the origin. Moreover

$$(4.1) \quad \mathcal{L}^n(\Omega^*) = \mathcal{L}^n(\Omega),$$

and

$$(4.2) \quad \mathcal{L}^{n-1}(H_\theta \cap K_\theta^*) = \mathcal{L}^{n-1}(H_\theta \cap K)$$

where \mathcal{L}^n denotes the n -dimensional Lebesgue measure (volume).

We use the following lemma.

Lemma 4.3. *Let*

$$C = B_2^{n-1}(r) \times [-h, h] \subset \mathbb{R}^n$$

be a spherical cylinder with $|C| = 1$. Assume that

$$r \leq B\sqrt{n} \quad \text{and} \quad r \geq A\omega_{n-1}^{-1/(n-1)},$$

for some universal constants $B > 0$ and $A > 1$, where

$$\omega_{n-1} = |B_2^{n-1}(1)|.$$

Then for every $\theta \in S^{n-1}$,

$$\int_C \langle x, \theta \rangle^2 dx \leq c,$$

where $c > 0$ depends only on A and B . In particular, c is universal whenever A and B are universal.

Proof. Write points $x \in \mathbb{R}^n$ as

$$x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

so that

$$C = \{(y, t) : |y| \leq r, |t| \leq h\}.$$

Since $|C| = 1$, we have

$$(1) \quad 2h\omega_{n-1}r^{n-1} = 1.$$

Fix $\theta \in S^{n-1}$, and decompose it as

$$\theta = (u, s) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad |u|^2 + s^2 = 1.$$

Then

$$\langle x, \theta \rangle = \langle y, u \rangle + st.$$

Therefore,

$$\int_C \langle x, \theta \rangle^2 dx = \int_C (\langle y, u \rangle + st)^2 dx.$$

Expanding,

$$\int_C \langle x, \theta \rangle^2 dx = \int_C \langle y, u \rangle^2 dx + 2s \int_C t \langle y, u \rangle dx + s^2 \int_C t^2 dx.$$

The mixed term vanishes by symmetry, since C is symmetric under $t \mapsto -t$. Hence

$$(2) \quad \int_C \langle x, \theta \rangle^2 dx = \int_C \langle y, u \rangle^2 dx + s^2 \int_C t^2 dx.$$

For the first term, Fubini gives

$$\int_C \langle y, u \rangle^2 dx = 2h \int_{B_2^{n-1}(r)} \langle y, u \rangle^2 dy.$$

By rotational symmetry of the Euclidean ball,

$$\int_{B_2^{n-1}(r)} \langle y, u \rangle^2 dy = |u|^2 \int_{B_2^{n-1}(r)} y_1^2 dy = \frac{|u|^2}{n-1} \int_{B_2^{n-1}(r)} |y|^2 dy.$$

Also,

$$\int_{B_2^{n-1}(r)} |y|^2 dy = \frac{n-1}{n+1} \omega_{n-1} r^{n+1}.$$

Therefore,

$$\int_{B_2^{n-1}(r)} \langle y, u \rangle^2 dy = \frac{|u|^2}{n+1} \omega_{n-1} r^{n+1},$$

and thus, using (1),

$$(3) \quad \int_C \langle y, u \rangle^2 dx = 2h \cdot \frac{|u|^2}{n+1} \omega_{n-1} r^{n+1} = \frac{r^2}{n+1} |u|^2.$$

For the second term in (2),

$$\int_C t^2 dx = |B_2^{n-1}(r)| \int_{-h}^h t^2 dt = \omega_{n-1} r^{n-1} \cdot \frac{2h^3}{3}.$$

Using again $2h \omega_{n-1} r^{n-1} = 1$, we get

$$(4) \quad \int_C t^2 dx = \frac{h^2}{3}.$$

Combining (2), (3), and (4), we obtain

$$(5) \quad \int_C \langle x, \theta \rangle^2 dx = \frac{r^2}{n+1} |u|^2 + \frac{h^2}{3} s^2.$$

Now we bound the two terms separately. Since $r \leq B\sqrt{n}$,

$$(6) \quad \frac{r^2}{n+1} |u|^2 \leq \frac{B^2 n}{n+1} |u|^2 \leq B^2.$$

On the other hand, from

$$r \geq A \omega_{n-1}^{-1/(n-1)},$$

we obtain

$$\omega_{n-1} r^{n-1} \geq A^{n-1}.$$

Hence, by (1),

$$h = \frac{1}{2\omega_{n-1} r^{n-1}} \leq \frac{1}{2A^{n-1}} \leq \frac{1}{2},$$

since $A > 1$. Therefore,

$$(7) \quad \frac{h^2}{3} s^2 \leq \frac{1}{12}.$$

Finally, by (5), (6), and (7),

$$\int_C \langle x, \theta \rangle^2 dx \leq B^2 + \frac{1}{12}.$$

This proves the claim, with

$$c = B^2 + \frac{1}{12}.$$

□

One of our main results is the following.

Theorem 4.4 (A proof on the slicing problem). *Let K be an isotropic symmetric convex body. Then for any direction $\theta \in S^{n-1}$*

$$|H_\theta \cap K| \geq c,$$

where c is a universal constant and H_θ is the hyperplane through the origin perpendicular to θ .

Proof. We proceed by steps.

Step 1: Cylinder Symmetrization.

Assume that a slice volume is attained in a slice orthogonal to θ -axis. Apply cylinder symmetrization to K with respect to the θ -axis. This produces a body K_θ^* with the following properties:

- $|K_\theta^*| = |K| = 1$ (volume preservation),
- K_θ^* is a body of revolution (trivial),
- $|H_\theta \cap K_\theta^*| = |H_\theta \cap K|$. (central slice preservation).

Step 2: Isotropic position.

Transform K_θ^* to its isotropic position $T(K_\theta^*)$. Let $\theta \in S^{n-1}$ be the fixed direction. Choose an orthonormal basis for \mathbb{R}^n with $e_n \parallel \theta$. In this coordinate system, the isotropic transformation T can be represented as a diagonal matrix:

$$T = \text{diag}(a_1, \dots, a_n), \quad \prod_{i=1}^n a_i = 1,$$

since the isotropic position is rotation invariant. On hyperplane $\{x_n = 0\}$, points $(x_1, \dots, x_{n-1}, 0)$ map to $(a_1 x_1, \dots, a_{n-1} x_{n-1}, 0)$. The $(n-1)$ -dimensional volume scales by

$$\prod_{i=1}^{n-1} a_i.$$

Now, K_θ^* is a spherical cylinder, so its isotropic position is a spherical cylinder. So

$$a_i = a$$

for all $1 \leq i \leq n-1$ and

$$a_n = a^{-n+1}.$$

Step 3: Bounding the slice size We split by cases.

Case 1: $a \leq 1$. There is almost nothing to prove in this case, except the change of variables, since the map T shrinks the base slice. It is well known that in the isotropic position the base slice volume of the cylinder is bounded from below.

Lemma 4.5 (Lower bound for the base slice of an isotropic spherical cylinder).

Let

$$Z = aB_2^{n-1} \times [-b, b] \subset \mathbb{R}^n$$

be a spherical cylinder in isotropic position, normalized by $|Z| = 1$. Denote its base slice by

$$S_{\text{base}}^{\text{isotropic}} := |Z \cap \{x_n = 0\}|_{n-1} = \omega_{n-1} a^{n-1}.$$

Then there exists a universal constant $c > 0$ such that

$$S_{\text{base}}^{\text{isotropic}} \geq c.$$

More precisely,

$$S_{\text{base}}^{\text{isotropic}} = \omega_{n-1}^{1/n} 2^{-(n-1)/n} \left(\frac{n+1}{3} \right)^{\frac{n-1}{2n}}.$$

Proof. Since $|Z| = 1$, we have

$$|Z| = 2b \omega_{n-1} a^{n-1} = 1.$$

Thus

$$2b S_{\text{base}}^{\text{isotropic}} = 1.$$

Because Z is isotropic, its coordinate second moments are equal:

$$\int_Z x_n^2 dx = \int_Z x_1^2 dx.$$

We compute each side explicitly.

For the axial coordinate,

$$\int_Z x_n^2 dx = \left(\int_{aB_2^{n-1}} dy \right) \left(\int_{-b}^b t^2 dt \right) = \omega_{n-1} a^{n-1} \cdot \frac{2b^3}{3}.$$

Using $2b\omega_{n-1}a^{n-1} = 1$, this becomes

$$\int_Z x_n^2 dx = \frac{b^2}{3}.$$

For a transverse coordinate,

$$\int_Z x_1^2 dx = \left(\int_{-b}^b dt \right) \left(\int_{aB_2^{n-1}} x_1^2 dy \right).$$

Since

$$\frac{1}{|aB_2^{n-1}|} \int_{aB_2^{n-1}} x_1^2 dy = \frac{a^2}{n+1},$$

we get

$$\int_{aB_2^{n-1}} x_1^2 dy = \omega_{n-1} a^{n-1} \frac{a^2}{n+1}.$$

Hence

$$\int_Z x_1^2 dx = 2b\omega_{n-1}a^{n-1} \frac{a^2}{n+1} = \frac{a^2}{n+1}.$$

Equating the two moments gives

$$\frac{b^2}{3} = \frac{a^2}{n+1}, \quad \text{so} \quad b = a\sqrt{\frac{3}{n+1}}.$$

Substituting this into the volume identity yields

$$1 = 2b\omega_{n-1}a^{n-1} = 2\omega_{n-1}a^n \sqrt{\frac{3}{n+1}},$$

hence

$$a^n = \frac{1}{2\omega_{n-1}} \sqrt{\frac{n+1}{3}}.$$

Therefore

$$S_{\text{base}}^{\text{isotropic}} = \omega_{n-1}a^{n-1} = \omega_{n-1}^{1/n} 2^{-(n-1)/n} \left(\frac{n+1}{3} \right)^{\frac{n-1}{2n}}.$$

It remains to show this is bounded below by a universal constant. By the standard estimate

$$\omega_m^{1/m} \asymp \frac{1}{\sqrt{m}},$$

there exists a universal constant $c_0 > 0$ such that

$$\omega_m^{1/m} \geq \frac{c_0}{\sqrt{m}} \quad \text{for all } m \geq 1.$$

Applying this with $m = n - 1$, we obtain

$$S_{\text{base}}^{\text{isotropic}} \geq \left(\frac{c_0}{\sqrt{n-1}} \cdot \frac{\sqrt{n+1}}{2\sqrt{3}} \right)^{\frac{n-1}{n}} \geq c$$

for some universal constant $c > 0$. This proves the claim. \square

Case 2: $a > 1$. The geometric definitions for the relevant volumes are:

$$(4.3) \quad \text{Total Volume: } V_n = S_{\text{base}} \cdot H$$

$$(4.4) \quad \text{Base Slice } ((n-1)\text{-ball}): S_{\text{base}} = |B_{n-1}(0, R)|$$

$$(4.5) \quad \text{Axial Slice } ((n-2)\text{-ball} \times \text{Height}): S_{\text{axial}} = |B_{n-2}(0, R)| \cdot H$$

Step 4. Bounding the parameter a from above We continue with the case Case 2: $a > 1$. Now,

$$\begin{aligned} nL_{K_\theta^*}^2 &\leq \int_{K_\theta^*} \langle \theta, x \rangle^2 dx + \sum_{i=2}^n \int_{K_\theta^*} \langle \theta_i, x \rangle^2 dx \\ &\leq \int_{K_\theta^*} \langle \theta, x \rangle^2 dx + (n-1) \int_{K_\theta^*} \langle \theta_i, x \rangle^2 dx \\ &\leq c_1 |H_\theta \cap K_\theta^*|^{-2} + (n-1) \int_{K_\theta^*} \langle \theta_i, x \rangle^2 dx \\ &\leq CL_K^2 + (n-1) \int_{K_\theta^*} \langle \theta_i, x \rangle^2 dx, \end{aligned}$$

where we used GM-AM inequality in the first inequality, and the theorem 4.1 in the third and the last inequality. So we have

$$(4.6) \quad L_{K_\theta^*}^2 \leq \frac{C}{n} L_K^2 + \frac{(n-1)}{n} \int_{K_\theta^*} \langle \theta_i, x \rangle^2 dx \leq \frac{C(\ln n)^2}{\sqrt{n}} + \int_{K_\theta^*} \langle \theta_i, x \rangle^2 dx,$$

where we used a bound from Bourgain $L_K \leq c(\ln n)n^{1/4}$. Combining (4.6) with

$$\int_{T(K_\theta^*)} \langle y, \theta_i \rangle^2 dy = \int_{K_\theta^*} \langle Tx, \theta_i \rangle^2 |\det T| dx = a^2 \int_{K_\theta^*} \langle x, \theta_i \rangle^2 dx,$$

since $y = Tx$, $|\det T| = 1$, and $T\theta_i = a\theta_i$ for every transverse direction θ_i , we obtain

$$a^2 \leq \frac{C(\log n)^2}{\sqrt{n}L_{K_\theta^*, \theta_i}^2} + \frac{L_{K_\theta^*, \theta_i}^2}{L_{K_\theta^*, \theta_i}^2}.$$

Hence from above and 4.1

$$(4.7) \quad a^2 = 1 + O\left(S_{\text{axial}}^2 \frac{(\log n)^2}{\sqrt{n}}\right),$$

Now, we have

$$(4.8) \quad S_{\text{axial}} = a \cdot S_{\text{axial}}^{\text{isotropic}},$$

via change of variables. So we from (4.7) that

$$(4.9) \quad 1 = \frac{1}{a^2} + O\left(S_{\text{axial}}^{\text{isotropic}} \frac{(\log n)^2}{\sqrt{n}}\right).$$

We next prove a lemma.

Lemma 4.6 (Axial slice of the isotropic spherical cylinder). *Let*

$$Z = aB_2^{n-1} \times [-b, b] \subset \mathbb{R}^n, \quad n \geq 2,$$

be in isotropic position and normalized by $|Z| = 1$. Let H be any hyperplane containing the axis $\mathbb{R}e_n$, and set

$$S_{axial} := |Z \cap H|_{n-1}.$$

Then

$$S_{axial} = \frac{\omega_{n-2}}{\omega_{n-1}} \left(\frac{2\sqrt{3}\omega_{n-1}}{\sqrt{n+1}} \right)^{1/n},$$

where

$$\omega_m = |B_2^m| = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2} + 1)}.$$

Equivalently,

$$S_{axial}^n = \frac{2\sqrt{3}}{\sqrt{\pi(n+1)}} \frac{\Gamma(\frac{n+1}{2})^{n-1}}{\Gamma(\frac{n}{2})^n}.$$

Moreover, as $n \rightarrow \infty$,

$$S_{axial} = \left(1 - \frac{\log n}{2n} + O\left(\frac{1}{n}\right) \right) \sqrt{e}.$$

Proof. By rotational symmetry in the first $n-1$ coordinates, all hyperplane sections containing the axis have the same $(n-1)$ -dimensional volume. Thus it is enough to take

$$H = \{x_1 = 0\}.$$

Then

$$Z \cap H = aB_2^{n-2} \times [-b, b],$$

and hence

$$(1) \quad S_{axial} = |Z \cap H|_{n-1} = 2b\omega_{n-2}a^{n-2}.$$

Since Z is isotropic and $|Z| = 1$, we have

$$\int_Z x_n^2 dx = \int_Z x_1^2 dx.$$

Now

$$\int_Z x_n^2 dx = \left(\int_{aB_2^{n-1}} dy \right) \left(\int_{-b}^b t^2 dt \right) = \omega_{n-1}a^{n-1} \cdot \frac{2b^3}{3}.$$

Using the volume normalization

$$(2) \quad |Z| = 2b\omega_{n-1}a^{n-1} = 1,$$

it follows that

$$\int_Z x_n^2 dx = \frac{b^2}{3}.$$

On the other hand,

$$\int_Z x_1^2 dx = \left(\int_{-b}^b dt \right) \left(\int_{aB_2^{n-1}} x_1^2 dx \right).$$

Since for the Euclidean ball in \mathbb{R}^{n-1} ,

$$\frac{1}{|aB_2^{n-1}|} \int_{aB_2^{n-1}} x_1^2 dx = \frac{a^2}{n+1},$$

we obtain

$$\int_{aB_2^{n-1}} x_1^2 dx = |aB_2^{n-1}| \frac{a^2}{n+1} = \omega_{n-1} a^{n-1} \frac{a^2}{n+1}.$$

Hence, using again (2),

$$\int_Z x_1^2 dx = 2b \omega_{n-1} a^{n-1} \frac{a^2}{n+1} = \frac{a^2}{n+1}.$$

Therefore

$$(3) \quad \frac{a^2}{n+1} = \frac{b^2}{3}, \quad \text{i.e.} \quad b = a \sqrt{\frac{3}{n+1}}.$$

Substituting (3) into (2), we obtain

$$2\omega_{n-1} a^n \sqrt{\frac{3}{n+1}} = 1,$$

that is,

$$(4) \quad a^n = \frac{1}{2\omega_{n-1}} \sqrt{\frac{n+1}{3}}.$$

Now combining (1) and (3),

$$S_{\text{axial}} = 2\omega_{n-2} a^{n-1} \sqrt{\frac{3}{n+1}}.$$

Using (4), this becomes

$$S_{\text{axial}} = \frac{\omega_{n-2}}{\omega_{n-1}} \frac{1}{a} = \frac{\omega_{n-2}}{\omega_{n-1}} \left(\frac{2\sqrt{3}\omega_{n-1}}{\sqrt{n+1}} \right)^{1/n}.$$

This proves the first formula.

Using

$$\omega_{n-1} = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})}, \quad \frac{\omega_{n-2}}{\omega_{n-1}} = \pi^{-1/2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})},$$

we get

$$S_{\text{axial}}^n = \left(\frac{\omega_{n-2}}{\omega_{n-1}} \right)^n \frac{2\sqrt{3}\omega_{n-1}}{\sqrt{n+1}} = \frac{2\sqrt{3}}{\sqrt{\pi(n+1)}} \frac{\Gamma(\frac{n+1}{2})^{n-1}}{\Gamma(\frac{n}{2})^n}.$$

It remains to derive the asymptotic formula. By Stirling's formula,

$$\omega_m = \frac{1}{\sqrt{\pi m}} \left(\frac{2\pi e}{m} \right)^{m/2} \left(1 + O\left(\frac{1}{m}\right) \right) \quad (m \rightarrow \infty).$$

Hence

$$\frac{\omega_{n-2}}{\omega_{n-1}} = \sqrt{\frac{n}{2\pi}} \left(1 + O\left(\frac{1}{n}\right) \right),$$

and also

$$\left(\frac{2\sqrt{3}\omega_{n-1}}{\sqrt{n+1}} \right)^{1/n} = \sqrt{\frac{2\pi e}{n}} \left(1 - \frac{\log n}{2n} + O\left(\frac{1}{n}\right) \right).$$

Multiplying these estimates, we obtain

$$S_{\text{axial}} = \left(1 - \frac{\log n}{2n} + O\left(\frac{1}{n}\right)\right) \sqrt{e}.$$

This completes the proof. \square

Since, by the lemma isotropic axial slice satisfies

$$S_{\text{axial}}^{\text{isotropic}} = O(1),$$

it follows from (4.9) that

$$(4.10) \quad \left|1 - \frac{1}{a^2}\right| \leq C \frac{(\log n)^2}{\sqrt{n}}$$

for some universal constant $C > 0$. In Case $a > 1$, we have

$$0 < \frac{1}{a^2} < 1,$$

hence

$$1 - \frac{1}{a^2} \geq 0.$$

Therefore (4.10) gives

$$0 \leq 1 - \frac{1}{a^2} \leq C \frac{(\log n)^2}{\sqrt{n}}.$$

Set

$$\varepsilon_n := C \frac{(\log n)^2}{\sqrt{n}}.$$

Then

$$1 - \varepsilon_n \leq \frac{1}{a^2} \leq 1.$$

For all sufficiently large n , we have $\varepsilon_n \leq \frac{1}{2}$. Hence

$$a^2 \leq \frac{1}{1 - \varepsilon_n}.$$

Using the elementary estimate

$$\frac{1}{1 - t} \leq 1 + 2t \quad (0 \leq t \leq \tfrac{1}{2}),$$

we obtain

$$a^2 \leq 1 + 2\varepsilon_n = 1 + O\left(\frac{(\log n)^2}{\sqrt{n}}\right).$$

Taking square roots yields

$$a = 1 + O\left(\frac{(\log n)^2}{\sqrt{n}}\right).$$

In particular, $a \leq c$ for some universal constant $c > 0$.

For the finitely many remaining dimensions, the constant c can be enlarged if necessary. Thus, in all dimensions,

$$a = 1 + O\left(\frac{(\log n)^2}{\sqrt{n}}\right), \quad \text{and in particular} \quad a \leq c.$$

Step 5. Bounding the radius R from below

We continue with the case

Case 2: $a > 1$. Since $a \leq c$ via the previous step and the lemma we have

$$(4.11) \quad R^{-1} \frac{|B_{n-2}(0, 1)|}{|B_{n-1}(0, 1)|} = S_{\text{axial}} \leq c S_{\text{axial}}^{\text{isotropic}} = c \left(1 - \frac{\log n}{2n} + O\left(\frac{1}{n}\right) \right) \sqrt{e}.$$

Thus, we have a lower bound for R .

$$R \geq C' \sqrt{n},$$

where we used that

$$\frac{|B_{n-2}(0, 1)|}{|B_{n-1}(0, 1)|} \leq C \sqrt{n}.$$

Step 6. Using the two results. We continue with

Case 2: $a > 1$. By the previous step we have a lower bound $R \geq c\sqrt{n}$. We may also assume $R \leq C\sqrt{n}$, since otherwise

$$S_{\text{base}} = |B_{n-1}(0, R)|$$

is already bounded below by a universal constant.

Hence Lemma 4.3, applied to the cylinder K_θ^* , gives

$$L_{K_\theta^*, \theta} = \left(\int_{K_\theta^*} \langle x, \theta \rangle^2 dx \right)^{1/2} \leq c_1,$$

for a universal constant c_1 . Therefore, by Theorem 4.1,

$$|H_\theta \cap K_\theta^*| \geq c_2$$

for some universal constant $c_2 > 0$. Finally, by the slice-preserving property of the cylinder symmetrization,

$$|H_\theta \cap K| = |H_\theta \cap K_\theta^*| \geq c_2.$$

This completes the proof in Case $a > 1$. □

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Email address: jaspegren@outlook.com