Canonical Envelopes:
Bilateral Pairings and Completion Phenomena

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Abstract

We define the notion of a canonical envelope of a bilateral pairing and analyze canonical envelopes through bilateral density and compactness conditions. Canonical envelopes provide a systematic framework for a significant class of completion phenomena in mathematics, unifying these constructions through initial factorizations in categories of bilateral decompositions.

The construction was motivated by Riehl's adjunction for weighted limits [28]. We prove that all four Kan constructions (left and right Kan extensions and left and right Kan liftings) arise as instances of canonical envelopes (Corollary 6.7), so weighted (co)limits and all four Kan constructions are recovered within the framework.

We develop the theory of outer bilateral envelopes for cases where classical completions fail: the outer envelope objects Y_Q and X_Q always exist in presheaf categories, and the term *virtual canonical envelope* refers to this outer data when the interpolant is not yet known. We establish the geometric interpretation through cylinder factorization systems, connecting to Garner's work. We prove that canonical envelopes admit monadic organization: the canonical envelope functor, defined on the full subcategory of admissible pairings where CEs exist, extends to an idempotent monad whose Eilenberg-Moore algebras are precisely the complete bilateral pairings, with Garner's Isbell monad emerging as a natural specialization.

We show that dicategories provide a bilateral algebraic presentation of dagger categories (Theorem 7.51), with every dagger category admitting a canonical dicategory presentation and vice versa. The dicategory presentation, characterized as a canonical envelope (Theorem 7.49), makes explicit the symmetric relationship between categorical and cocategorical composition that is implicit in the standard dagger category axiomatization. This bilateral presentation arises naturally from the canonical envelope construction and connects to Frobenius pseudomonoids, where the dicategory axioms correspond to Frobenius compatibility conditions.

We construct canonical left and right envelope objects explicitly in presheaf categories via coend and end formulas, reducing the existence of a canonical envelope to a universal interpolation problem. We show that the interpolation problem is solvable in several classical settings, including: ind- and pro-completions [10], Cauchy completions [23], Pratt's communes [25], Isbell envelopes [17], and topological completions (Stone-Čech compactification, sobrification). We establish a structural correspondence with Pratt's commune theory for identity pairings, and compare structurally with Schoots's categorical canonical extensions [30] and classical canonical extensions of distributive lattices [15, 20].

Keywords: canonical envelopes, bilateral pairings, completions, initial factorizations, bilateral density and compactness, weighted limits, Isbell monad, dicategories, dagger categories, cylinder factorization systems.

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Résumé

Nous définissons la notion d'enveloppe canonique d'un couplage bilatéral et la caractérisons au moyen des conditions de densité et de compacité bilatérales. Les enveloppes canoniques fournissent un cadre systématique pour une classe significative de phénomènes de complétion en mathématiques, unifiant ces constructions au moyen de factorisations initiales dans des catégories de décompositions bilatérales.

La construction est motivée par l'adjonction de Riehl pour les limites pondérées. Nous prouvons que les quatre constructions de Kan (extensions et relèvements, à gauche et à droite) sont des instances d'enveloppes canoniques, ce qui permet de retrouver les limites et colimites pondérées au sein du cadre. Nous développons la théorie des enveloppes canoniques virtuelles pour les cas où les complétions classiques échouent, et nous établissons une interprétation géométrique via les systèmes de factorisation cylindriques, en lien avec les travaux de Garner. Nous montrons que les enveloppes canoniques admettent une organisation monadique : le foncteur d'enveloppe canonique s'étend en une monade idempotente dont les algèbres de Eilenberg–Moore sont précisément les couplages bilatéraux complets, la monade d'Isbell de Garner apparaissant dans ce cadre comme un cas particulier naturel.

Nous montrons que les dicatégories fournissent une présentation algébrique bilatérale équivalente des catégories dagger, dans le cadre bilatéral induit par la construction d'enveloppe canonique. Cette présentation, caractérisée comme une enveloppe canonique, rend explicite la relation symétrique entre composition catégorique et cocatégorique implicite dans l'axiomatisation usuelle des catégories dagger, et correspond aux conditions de compatibilité de Frobenius pour les pseudomonades. Nous construisons explicitement les objets d'enveloppe gauche et droite dans les catégories de préfaisceaux via des formules de coends et d'ends, ramenant l'existence d'une enveloppe canonique à la résolubilité d'un problème universel d'interpolation. Nous vérifions cette condition pour de nombreuses complétions classiques (complétions Ind et Pro, complétions de Cauchy, communes de Pratt, enveloppes d'Isbell, compactification de Stone–Čech et sobrification). Nous établissons l'équivalence, dans les contextes précisés, avec la théorie des communes de Pratt et comparons structurellement avec les extensions canoniques catégoriques de Schoots ainsi qu'avec les extensions canoniques classiques des treillis distributifs.

Le but de cet article est de fournir une référence unifiée et systématique pour le programme des enveloppes canoniques et ses principales constructions fondamentales.

Mots-clés : enveloppes canoniques, couplages bilatéraux, complétions, factorisations initiales, densité et compacité bilatérales, limites pondérées, monade d'Isbell, dicatégories, catégories dagger, systèmes de factorisation cylindriques.

1 Introduction

1.1 Motivation

Completion phenomena appear throughout mathematics in seemingly disparate forms. Topologists compactify spaces, algebraists extend lattices [20], analysts complete metric spaces [16], and category theorists construct Kan extensions [21] and ind-completions [1, 10]. Each construction uses its own methods and existence criteria, yet they share a common pattern.

The recurring idea is a kind of bilateral probing: objects are tested from two sides simultaneously, and the completion mediates between these tests. For instance:

- Canonical extensions of distributive lattices mediate between filters (testing from above) and ideals (testing from below) [15]
- Cauchy completion mediates between sequences approaching from both directions [23]

- Ind-completions mediate between filtered diagrams and the original category [10]

This bilateral test structure suggests there should be a single categorical framework that handles all of these at once.

1.2 Goal of This Paper

We define canonical envelopes as initial objects in categories of bilateral factorizations. A canonical envelope of a pairing $\theta : Q \Rightarrow C(D, E)$ is a factorization

$$\theta = \rho \star \gamma \star \lambda$$

where λ provides left completion, ρ provides right completion, and γ is the canonical interpolant mediating between them. Initiality ensures the construction is the smallest such factorization.

We are not claiming that canonical envelopes capture every completion phenomenon in mathematics — that would be far too strong a claim. Rather, the goal is a systematic framework for a significant class of completions that share this bilateral structure. The framework's scope is determined by the defining conditions (bilateral pairings, initial factorizations, density, compactness), not by any ambition toward universal coverage.[1]

We prove:

- Reduction to interpolation: canonical left and right envelope objects in presheaf categories via coend/end formulas, with CE existence equivalent to solvability of a universal interpolation problem (Theorem 5.9)
- Characterization through bilateral density and compactness (Section 4)
- Explicit verification of the interpolation problem for classical completions (Section 7)

1.3 Contributions

Our main contributions are:

1. General definition of canonical envelopes as initial factorizations (Definition 3.9)
2. Reduction theorem for canonical envelopes in presheaf categories: construction of canonical left and right envelope objects via coend/end formulas, with CE existence equivalent to solvability of the interpolation problem (Theorem 5.9)
3. Analysis of necessary structural conditions via bilateral density and compactness (Section 4)
4. Proof that all four Kan constructions (left and right Kan extensions and left and right Kan liftings) arise as instances of canonical envelopes (Corollary 6.7), recovering weighted (co)limits within the framework (Remark 6.9)
5. Outer bilateral envelope theory: the outer objects Y_Q and X_Q always exist in presheaf categories (Section 6.6); the term *virtual canonical envelope* refers to this outer data when the interpolant is not yet known

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[1] This paper is the primary reference for the canonical envelope program. Companion papers developing specific aspects include: [26] (topological completions and bilateral pairings) and [27] (further structural results). Additional papers are available at `https://independentresearcher.academia.edu/RobertRice`.

6. Geometric interpretation via cylinder factorization systems (Section 7.9), establishing the connection to Garner's cylinder systems (Remark 7.40)
7. Monadic structure of canonical envelopes (Section 8): development of the canonical envelope monad $\mathcal{E}$ on the subcategory of admissible pairings, with characterization of Eilenberg-Moore algebras as complete pairings (Theorem 8.9), idempotency (Theorem 8.7), and proof that Garner's Isbell monad is a specialization (Theorem 8.11)
8. Equivalence between dicategories and dagger categories (Section 7.10, Theorem 7.51), with universal characterization of the dicategory presentation as a canonical envelope (Theorem 7.49)
9. Applications:
    - Schoots's categorical canonical extensions (Section 7.2): comparison establishing the structural parallel and key distinction
    - Classical canonical extensions of distributive lattices: illuminated by the ideal-filter pairing, discussed in Section 7.2
    - Ind- and pro-completions (Section 7.3)
    - Cauchy completions via absolute colimits (Section 7.4, Theorem 7.13)
    - Pratt's communes (Section 7.5, Theorem 7.20)
    - Isbell envelopes (Section 7.7)
    - Topological completions: Stone-Čech compactification and sobrification (Section 7.8, Theorems 7.29 and 7.31)
10. Structural correspondence with Pratt's commune theory for identity pairings (Theorem 7.20); structural comparison with Schoots's categorical extensions (Section 7.2)
11. Bidirectional correspondence between canonical envelopes and a broad class of universal constructions (Remark 6.10): constructions with universal properties that are already known to exist can be recognized as canonical envelopes; conversely, canonical envelopes automatically yield universal properties by initiality

1.4 Organization

Section 2 reviews enriched categories, presheaf categories, and weighted limits. Section 3 defines bilateral pairings and the factorization category. Section 4 introduces bilateral density and compactness. Section 5 constructs the candidate envelope objects in presheaf categories and reduces CE existence to an interpolation problem. Section 6 connects canonical envelopes to weighted limits and Kan extensions. Section 7 works through classical examples. Section 8 develops the monadic structure. Section 9 states what was not proved. Section 10 closes with future directions.

2 Technical Background

We briefly review the categorical preliminaries needed for our construction. Standard references include [5, 22, 24].

2.1 Enriched Categories

Definition 2.1 (Enriched Category)**.** Let $(V, \otimes, I)$ be a symmetric monoidal closed category. A V-category C consists of:

- A collection of objects $\mathrm{Ob}(C)$
- For each pair $c, d \in C$, a hom-object $C(c, d) \in V$
- For each triple $c, d, e \in C$, a composition morphism

$$\circ_{c,d,e} : C(d, e) \otimes C(c, d) \to C(c, e)$$

- For each $c \in C$, an identity morphism $\mathrm{id}_c : I \to C(c, c)$

satisfying associativity and unitality axioms [22].

Definition 2.2 (Profunctor)**.** A profunctor (or distributor [4]) $P : C^{\mathrm{op}} \times D \to V$ is a V-functor. Profunctors generalize both functors and relations [23, 36].

Remark 2.3. Throughout this paper, we work with $V = \mathbf{Set}$ for concreteness and to avoid technical complications with general enrichment. All constructions generalize to arbitrary symmetric monoidal closed bases V with appropriate modifications.

2.2 Presheaf Categories

Definition 2.4 (Presheaf Category)**.** For a small category C, the presheaf category $\mathrm{Psh}(C) = [C^{\mathrm{op}}, \mathbf{Set}]$ consists of functors $F : C^{\mathrm{op}} \to \mathbf{Set}$ with natural transformations as morphisms.

Definition 2.5 (Yoneda Embedding)**.** The Yoneda embedding $y : C \to \mathrm{Psh}(C)$ sends $c \in C$ to the representable functor $y(c) = C(-, c)$.

Lemma 2.6 (Yoneda Lemma [24, 38])**.** *For any presheaf* $F : C^{\mathrm{op}} \to \mathbf{Set}$ *and object* $c \in C$*:*

$$\mathrm{Nat}(y(c), F) \cong F(c)$$

naturally in both c *and* F*. The Yoneda embedding is fully faithful.*

Lemma 2.7 (Completeness of Presheaf Categories [24])**.** *The presheaf category* $\mathrm{Psh}(C)$ *is complete and cocomplete. Limits and colimits are computed pointwise.*

2.3 Foundational Assumptions

Throughout this paper, we make the following categorical assumptions unless otherwise stated:

1. **Locally small categories:** All categories are assumed to be locally small (hom-sets are sets)
2. **Set-enrichment:** We work primarily with **Set**-enriched categories rather than general monoidal enrichment
3. **Small index categories:** Index categories for diagrams, weights, and pairings are assumed to be small
4. **Presheaf context:** The main reduction theorem (Section 5) is developed in presheaf categories $\mathrm{Psh}(C)$ for small categories C

These assumptions ensure that the technical machinery (Yoneda lemma, representability, limits and colimits) functions without set-theoretic complications. Extensions to more general settings require careful attention to size issues and enrichment.

2.4 Weighted Limits and Colimits

We briefly recall weighted limits following [22, 28, 29], specialized to the **Set**-enriched case.

Definition 2.8 (Weighted Limit in **Set**-Categories)**.** Let $W : J \to \mathbf{Set}$ be a weight (a functor) and $F : J \to C$ a functor between **Set**-categories. The W-weighted limit $\lim^W F$ (when it exists) is an object of C satisfying

$$C(X, \overset{W}{\lim} F) \cong \mathrm{Nat}(W, C(X, F(-)))$$

naturally in $X \in C$, where the right side denotes natural transformations in $[J, \mathbf{Set}]$.

Dually, the W-weighted colimit $\mathrm{colim}_W F$ satisfies

$$C(\mathrm{colim}_W F, X) \cong \mathrm{Nat}(W, C(F(-), X))$$

naturally in $X \in C$.

Remark 2.9. Weighted limits generalize ordinary limits: taking W constant at the singleton set $\{*\}$ recovers ordinary limits. In the **Set**-enriched setting, weighted limits can be expressed using ends and coends [22, 24].

2.5 Size Conventions

We work within a fixed Grothendieck universe [1, 24] or assume a strongly inaccessible cardinal κ and work with κ-small categories. All categories C considered are locally small, and index categories for (co)limits are small. These conventions suffice for all our constructions.

3 Bilateral Pairings and Factorization

We now introduce the central definitions: bilateral pairings and their factorizations.

3.1 Bilateral Pairings

Definition 3.1 (Bilateral Pairing)**.** A bilateral pairing (or simply pairing) consists of a 6-tuple (I, J, D, E, Q, θ) where:

- I, J are small categories (indexing categories)
- $D : I \to C$ and $E : J \to C$ are functors for some category C
- $Q : I^{\mathrm{op}} \times J \to \mathbf{Set}$ is a profunctor (bilateral weight)
- $\theta : Q \Rightarrow C(D(-), E(-))$ is a natural transformation (pairing morphism)

Remark 3.2. Pairings generalize the data of weighted limits. A classical weighted limit setup corresponds to taking $I = \{*\}$ with one object, so D is constant. The bilateral structure with both I and J allows for completions mediating between dual probing operations.

Remark 3.3. The pairing θ measures how morphisms from $D(i)$ to $E(j)$ in C relate to the weight $Q(i, j)$. When θ is an isomorphism, we recover classical representability. Our framework allows θ to be more general.

Example 3.4 (Basic Pairings)**.** 1. **Set functions:** Take $I = J = \mathbf{Set}$, $Q(S, T) = \mathbf{Set}(S, T)$, giving pairings of set-valued functions.

2. **Order pairings:** For a poset P, take $I = J = P$ viewed as categories, $Q(i,j) = \{* : i \leq j\}$, giving order relations.

3. **Profunctor pairings:** Any profunctor $P : C^{\mathrm{op}} \times D \to \mathbf{Set}$ determines a pairing with $\theta = \mathrm{id}_P$.

3.2 The Factorization Category

Definition 3.5 (Factorization)**.** A factorization of a pairing $\theta : Q \Rightarrow C(D, E)$ consists of:

- Functors $Y : J \to C$ and $X : I \to C$
- Natural transformations:

$$\begin{aligned} \lambda &: Q \Rightarrow C(D, Y) \quad \text{(left envelope)} \\ \gamma &: Q \Rightarrow C(Y, X) \quad \text{(canonical interpolant)} \\ \rho &: Q \Rightarrow C(X, E) \quad \text{(right envelope)} \end{aligned}$$

- The factorization property: $\theta = \rho \star \gamma \star \lambda$ pointwise, meaning that for all $i \in I$, $j \in J$, and $q \in Q(i,j)$, the following equality holds in the hom-set $C(D(i), E(j))$:

$$\theta(i,j,q) = \rho(i,j,q) \circ_C \gamma(i,j,q) \circ_C \lambda(i,j,q)$$

  where $\circ_C$ denotes composition of morphisms in the category C. Explicitly, this is the composition

$$D(i) \xrightarrow{\lambda(i,j,q)} Y(j) \xrightarrow{\gamma(i,j,q)} X(i) \xrightarrow{\rho(i,j,q)} E(j).$$

We denote a factorization by the triple (λ, γ, ρ) along with the functors Y, X.

Definition 3.6 (Category of Factorizations)**.** For a pairing θ, define $\mathrm{Fact}(\theta)$ as the category with:
Objects: Factorizations (λ, γ, ρ) of θ
Morphisms: A morphism from (λ, γ, ρ) with functors (Y, X) to $(\lambda', \gamma', \rho')$ with functors (Y', X') is a pair of natural transformations $(\alpha : Y \Rightarrow Y', \beta : X \Rightarrow X')$ such that:

$$\begin{aligned} \lambda' &= C(D, \alpha) \circ \lambda \\ \gamma' \circ C(\alpha, X') &= C(Y, \beta) \circ \gamma \\ \rho &= C(\beta, E) \circ \rho' \end{aligned}$$

Lemma 3.7. $\mathrm{Fact}(\theta)$ *is a category with composition and identities defined componentwise.*

Proof. Composition of morphisms $(\alpha, \beta) : (\lambda, \gamma, \rho) \to (\lambda', \gamma', \rho')$ and $(\alpha', \beta') : (\lambda', \gamma', \rho') \to (\lambda'', \gamma'', \rho'')$ is given by $(\alpha' \circ \alpha, \beta' \circ \beta)$. The compatibility conditions compose by naturality of natural transformations. Identity morphisms are $(\mathrm{id}_Y, \mathrm{id}_X)$, which trivially satisfy the compatibility conditions. Associativity and unit laws follow from those in C. □

Remark 3.8 (Interpretation of Morphisms in $\mathrm{Fact}(\theta)$). Morphisms in $\mathrm{Fact}(\theta)$ are comparison maps between different factorizations of the same pairing θ. The pair $(\alpha : Y \Rightarrow Y', \beta : X \Rightarrow X')$ relates two envelope structures:

- α compares the left envelopes: how Y maps into Y'
- β compares the right envelopes: how X maps into X'

- The compatibility conditions ensure these comparisons preserve the factorization structure

The key insight is that these morphisms must respect the bilateral nature of the factorization:

- $\lambda' = C(D, \alpha) \circ \lambda$ means: composing with α after λ gives λ'
- $\gamma' \circ C(\alpha, X') = C(Y, \beta) \circ \gamma$ means: the interpolants are compatible via α and β
- $\rho = C(\beta, E) \circ \rho'$ means: composing with β before ρ' gives ρ

When θ is bilaterally compact, any two factorizations are isomorphic in $\mathrm{Fact}(\theta)$. This means there exists a comparison (α, β) that is an isomorphism (both α and β are natural isomorphisms), expressing that the envelope structures are essentially the same.

Example: For Cauchy completion of a category C, different presentations of the Cauchy completion (e.g., as split idempotents vs. as absolute colimits) give different objects in $\mathrm{Fact}(\theta)$, but bilateral compactness ensures they are isomorphic, with an isomorphism (α, β) providing an identification between the two presentations.

3.3 Canonical Envelopes

Definition 3.9 (Canonical Envelope)**.** A canonical envelope of a pairing θ is an initial object in the category $\mathrm{Fact}(\theta)$.

That is, a canonical envelope is a factorization $(\lambda_0, \gamma_0, \rho_0)$ with functors (Y_0, X_0) such that for any other factorization (λ, γ, ρ) with functors (Y, X), there exists a unique morphism

$$(\alpha : Y_0 \Rightarrow Y, \beta : X_0 \Rightarrow X) : (\lambda_0, \gamma_0, \rho_0) \to (\lambda, \gamma, \rho)$$

in $\mathrm{Fact}(\theta)$.

Remark 3.10 (Factorization Square)*.* The factorization $\theta = \rho \star \gamma \star \lambda$ is displayed as a commutative square: the top edge is $\theta(i, j, q)$, the path around the bottom is $\rho \circ \gamma \circ \lambda$.

$$\begin{tikzcd}
D(i) \arrow[r, "\theta_{i,j}(q)"] \arrow[d, "\lambda_{i,j}(q)"'] & E(j) \\
Y(j) \arrow[r, "\gamma_{i,j}(q)"'] & X(i) \arrow[u, "\rho_{i,j}(q)"']
\end{tikzcd}$$

The original pairing morphism θ goes directly across the top; the bilateral factorization routes through $Y(j)$ and $X(i)$ along the bottom.

Remark 3.11 (Intuitive Meaning of the Canonical Interpolant)*.* The canonical interpolant $\gamma : Q \Rightarrow C(Y, X)$ is the most subtle component of the factorization. While λ and ρ are determined by representability (they encode how Y and X represent the bilateral weight Q), the interpolant γ captures the *internal structure* of the completion.

Intuitively, γ mediates between two different ways of probing the pairing: from the left via Y (which collects information from D-shaped probes) and from the right via X (which collects information from E-shaped probes). The morphism $\gamma(i, j, q) : Y(j) \to X(i)$ for each $q \in Q(i, j)$ is the unique bridge that makes the factorization commute, determined by the constraint $\theta = \rho \star \gamma \star \lambda$.

In classical completions, γ often has concrete interpretations: in canonical extensions of lattices, it relates filters to ideals; in Cauchy completions, it relates forward and backward Cauchy sequences; in Kan extensions, it relates weighted (co)limits. The bilateral perspective reveals these as instances of the same universal construction.

Remark 3.12. Thus $\mathrm{Fact}(\theta) \neq \emptyset$ is exactly bilateral denseness (existence of a triple factorization), and an initial object in $\mathrm{Fact}(\theta)$ is the canonical envelope. These conditions are made precise in Section 4.

Remark 3.13. The initiality condition ensures the canonical envelope is categorically determined: any two canonical envelopes are uniquely isomorphic. This eliminates arbitrariness in the completion construction, analogous to how free objects are determined by universal properties [24].

Remark 3.14. The components of a canonical envelope have natural interpretations:

- λ maps the bilateral weight to morphisms from D into the left completion Y
- ρ maps the weight to morphisms from the right completion X to E
- γ is the canonical interpolant mediating between left and right completions

The components of a canonical envelope factor the pairing bilaterally.

4 Bilateral Density and Compactness

In this section we characterize canonical envelopes in terms of two properties of a pairing θ itself. The terminology is motivated by analogous notions in Schoots's categorical canonical extensions, though the precise relationship differs in an important way (see Section 7.2).

Quick reference: the three conditions

$$\begin{aligned} \text{bilateral denseness} &\iff \mathrm{Fact}(\theta) \neq \emptyset \\ \text{bilateral compactness} &\iff \mathrm{Fact}(\theta) \text{ is a connected groupoid} \\ \text{canonical envelope} &= \text{initial object of } \mathrm{Fact}(\theta) \end{aligned}$$

Compactness implies uniqueness of the canonical envelope once it exists, but does not produce one. Existence requires additionally that the interpolation problem of Theorem 5.9 has an initial solution.

4.1 Denseness as Existence of Bilateral Factorization

Definition 4.1 (Left Denseness)**.** A pairing $\theta : Q \Rightarrow C(D, E)$ is left dense if there exist

- a functor $Y : J \to C$ and
- a natural transformation $\lambda : Q \Rightarrow C(D, Y)$

such that θ factors through λ, i.e., there exists some γ' such that

$$\theta = \gamma' \star \lambda.$$

Definition 4.2 (Right Denseness)**.** A pairing θ is right dense if there exist

- a functor $X : I \to C$ and
- a natural transformation $\rho : Q \Rightarrow C(X, E)$

such that θ factors through ρ, i.e., there exists some γ'' such that

$$\theta = \rho \star \gamma''.$$

Definition 4.3 (Bilateral Denseness)**.** A pairing θ is bilaterally dense if there exist functors

$$X : I \to C, \quad Y : J \to C$$

and natural transformations

$$\lambda : Q \Rightarrow C(D, Y), \quad \gamma : Q \Rightarrow C(Y, X), \quad \rho : Q \Rightarrow C(X, E)$$

such that

$$\theta = \rho \star \gamma \star \lambda$$

pointwise.

Remark 4.4. Thus bilateral denseness is exactly the statement:

$$\text{Fact}(\theta) \text{ is nonempty.}$$

Equivalently, θ has at least one triple factorization.

Proposition 4.5 (Representability Implies Denseness)**.** *If for each $j \in J$ the functor $Q(-, j) : I^{\mathrm{op}} \to \mathbf{Set}$ is representable by some $Y(j) \in C$ and for each $i \in I$ the functor $Q(i, -) : J \to \mathbf{Set}$ is representable by some $X(i) \in C$, then θ is bilaterally dense.*

Proof. Representability gives natural isomorphisms of functors

$$\phi^L_j : C(D(-), Y(j)) \xrightarrow{\cong} Q(-, j), \qquad \phi^R_i : C(X(i), E(-)) \xrightarrow{\cong} Q(i, -).$$

These induce natural transformations $\lambda : Q \Rightarrow C(D, Y)$ and $\rho : Q \Rightarrow C(X, E)$ by setting

$$\lambda_{i,j}(q) = (\phi^L_j(i))^{-1}(q) \in C(D(i), Y(j)), \qquad \rho_{i,j}(q) = (\phi^R_i(j))^{-1}(q) \in C(X(i), E(j)).$$

Here $\lambda_{i,j}(q)$ and $\rho_{i,j}(q)$ are morphisms in C corresponding to q under the representing bijections. They are not isomorphisms in general; they are specific morphisms selected by the bijection.

To construct the interpolant $\gamma : Q \Rightarrow C(Y, X)$, fix $i \in I$, $j \in J$, and $q \in Q(i, j)$. We need a morphism $\gamma_{i,j}(q) : Y(j) \to X(i)$ in C.

By Yoneda, a morphism $Y(j) \to X(i)$ in C corresponds to an element of $C(Y(j), X(i))$. Consider the natural transformation in $k \in I$:

$$C(D(k), Y(j)) \xrightarrow{\phi^L_j(k)} Q(k, j) \xrightarrow{\theta_{k,j}} C(D(k), E(j))$$

composed with the natural transformation

$$C(D(k), E(j)) \to C(D(k), X(i))$$

induced by $\rho_{i,j}(q) : X(i) \to E(j)$ via precomposition. The composite natural transformation $C(D(-), Y(j)) \Rightarrow C(D(-), X(i))$ is represented (by Yoneda applied to $Y(j)$) by a unique morphism $\gamma_{i,j}(q) : Y(j) \to X(i)$.

Explicitly: $\gamma_{i,j}(q)$ is the unique morphism such that for all $k \in I$ and $f : D(k) \to Y(j)$,

$$\rho_{i,j}(q) \circ \gamma_{i,j}(q) \circ f = \theta_{k,j}(\phi^L_j(k)(f)).$$

Taking $k = i$ and $f = \lambda_{i,j}(q) \in C(D(i), Y(j))$, and noting that $\phi^L_j(i)(\lambda_{i,j}(q)) = q$, we get

$$\rho_{i,j}(q) \circ \gamma_{i,j}(q) \circ \lambda_{i,j}(q) = \theta_{i,j}(q),$$

which is the required factorization. Naturality of γ in i and j follows from naturality of λ, ρ, θ, and the Yoneda correspondence. Thus (λ, γ, ρ) is a triple factorization of θ, so θ is bilaterally dense. $\square$

Remark 4.6. Representability provides the bijections ϕ^L_j and ϕ^R_i, which give λ and ρ as specific morphisms in C. These morphisms are not isomorphisms in general; the bijection is between *sets of morphisms*, not between objects of C. The interpolant γ is constructed via the Yoneda lemma applied to the representing object $Y(j)$, not by inverting λ or ρ.

4.2 Compactness as Uniqueness of Factorization

Definition 4.7 (Bilateral Compactness)**.** A pairing $\theta : Q \Rightarrow C(D, E)$ is bilaterally compact if any two triple factorizations

$$\theta = \rho \star \gamma \star \lambda \quad \text{and} \quad \theta = \rho' \star \gamma' \star \lambda'$$

with data (X, Y) and (X', Y') are isomorphic in $\mathrm{Fact}(\theta)$. That is, there exist natural isomorphisms

$$\alpha : Y \Rightarrow Y', \quad \beta : X \Rightarrow X'$$

such that (α, β) is a morphism in $\mathrm{Fact}(\theta)$.

In categorical terms, bilateral compactness means $\mathrm{Fact}(\theta)$ is a connected groupoid: every two objects are isomorphic. This is weaker than having an initial object.

Remark 4.8 (Precise logical structure). The three notions are logically distinct:

- **Denseness** ($\mathrm{Fact}(\theta) \neq \emptyset$): there exists at least one triple factorization. This is the minimal requirement.
- **Compactness** (any two factorizations isomorphic): $\mathrm{Fact}(\theta)$ is a connected groupoid. This guarantees that *if* a canonical envelope exists, it is unique up to unique isomorphism. It does *not* by itself produce an initial object.
- **Canonical envelope** (initial object of $\mathrm{Fact}(\theta)$): there exists a factorization F_0 admitting a unique morphism to every other factorization. This is the full condition.

The correct summary is:

$$\text{CE exists} \iff \text{denseness} + \text{existence of an initial factorization}$$

$$\text{compactness} \implies \text{uniqueness of CE once it exists}$$

This matches the spirit of Schoots's CE2 (density) and CE3 (uniqueness) conditions from Section 7.2, but in Schoots's setting existence is established by an explicit construction. In our general setting, existence of the initial interpolant must be verified separately (see Theorem 5.9).

Theorem 4.9 (Density, Compactness, and Canonical Envelopes)**.** *Let θ be a pairing.*

1. *Bilateral denseness is equivalent to* $\mathrm{Fact}(\theta) \neq \emptyset$.
2. *A canonical envelope of θ is an initial object of* $\mathrm{Fact}(\theta)$.

3. *If a canonical envelope exists, it is unique up to unique isomorphism (as with any initial object).*

4. *If* $F_0 \in \mathrm{Fact}(\theta)$ *is a factorization such that for every* $F \in \mathrm{Fact}(\theta)$ *there exists a unique morphism* $F_0 \to F$ *in* $\mathrm{Fact}(\theta)$*, then* F_0 *is the canonical envelope.*

Proof. (1) follows directly from Definition 4.3: θ is bilaterally dense if and only if there exists at least one triple factorization, i.e., $\mathrm{Fact}(\theta) \neq \emptyset$.

(2) is Definition 3.9.

(3) is the standard categorical fact that initial objects are unique up to unique isomorphism: if F_0 and F_0' are both initial, there are unique morphisms $F_0 \to F_0'$ and $F_0' \to F_0$, whose composite must be the unique morphism $F_0 \to F_0$ (which is the identity), so the morphisms are mutually inverse isomorphisms.

(4) is the definition of an initial object. ☐

Remark 4.10 (Compactness and initiality)*.* Bilateral compactness as defined (any two factorizations are isomorphic) is not by itself sufficient to guarantee that a canonical envelope exists. Isomorphism between any two objects means $\mathrm{Fact}(\theta)$ is a connected groupoid, but a connected groupoid need not have an initial object — initial objects require unique outgoing morphisms, not merely invertible ones. Note that connected groupoids may possess nontrivial automorphism groups; bilateral compactness therefore does not imply contractibility of $\mathrm{Fact}(\theta)$. Existence of a canonical envelope must be established separately, as in Theorem 5.9 below for presheaf categories. Bilateral compactness then ensures uniqueness of the canonical envelope, once its existence is known.

Corollary 4.11. *Canonical envelopes, when they exist, are unique up to unique equivalence.*

Proof. This is the general property of initial objects in categories. ☐

4.3 Relationship to Classical Completion Theory

Remark 4.12 (Relationship to Classical Completion Theory)*.* The bilateral density and compactness conditions formalize the intuitive requirements for canonical completions:

- **Density (existence of factorization):** The completion is "generated by" the bilateral probes from I and J, mediating between them via the canonical interpolant γ.
- **Compactness (uniqueness up to isomorphism):** The completion is "determined by" these probes—any two factorizations are isomorphic, so there is at most one canonical envelope up to isomorphism.

These conditions appear in classical completion theory under various guises: complete regularity for compactifications, distributivity for canonical extensions, Cauchy completeness for metric spaces [15, 19, 23]. In those classical settings, density and compactness (or their analogues) are sufficient for existence because the ambient category has enough structure to construct the initial interpolant explicitly. In the general categorical setting, density and compactness guarantee *at most one* canonical envelope; existence requires additionally that the interpolation problem of Theorem 5.9 has an initial solution.

5 Candidate Envelope Construction in Presheaf Categories

We construct canonical left and right envelope objects for any pairing in a **Set**-enriched presheaf category, and reduce the existence of a canonical envelope to a universal interpolation problem. The presheaf construction does not by itself guarantee existence of the canonical envelope; that holds exactly when the interpolation problem has an initial solution, which must be checked case by case. Throughout this section, all categories are **Set**-enriched and all functors preserve this enrichment.

5.1 Presheaf Embedding Construction

The key tool is the Yoneda embedding and its universal property.

Lemma 5.1 (Presheaf Representability)**.** *In a presheaf category* $\mathrm{Psh}(C) = [C^{\mathrm{op}}, \mathbf{Set}]$*, every presheaf* $F : C^{\mathrm{op}} \to \mathbf{Set}$ *is canonically a colimit of representables:*

$$F \cong \mathrm{colim}_{(c,x)\in\int F}\, y(c)$$

where $\int F$ *is the category of elements of* F *[24].*

5.2 Construction of Candidate Envelope

Construction 5.2 (Envelope in Presheaves)**.** Let $\theta : Q \Rightarrow \mathrm{Psh}(C)(D, E)$ be a pairing with $D : I \to \mathrm{Psh}(C)$ and $E : J \to \mathrm{Psh}(C)$, where I, J are small categories.

Left envelope functor:

$$Y_Q : J \to \mathrm{Psh}(C), \quad Y_Q(j) := \int^{i\in I} Q(i,j)\cdot D(i)$$

where $Q(i,j)\cdot D(i)$ denotes the copower (tensor) of the presheaf $D(i)$ by the set $Q(i,j)$, i.e., the presheaf $c \mapsto Q(i,j) \times D(i)(c)$.

Right envelope functor:

$$X_Q : I \to \mathrm{Psh}(C), \quad X_Q(i) := \int_{j\in J} E(j)^{Q(i,j)}$$

where $E(j)^{Q(i,j)}$ denotes the power of the presheaf $E(j)$ by the set $Q(i,j)$, i.e., the presheaf $c \mapsto E(j)(c)^{Q(i,j)} = \mathbf{Set}(Q(i,j), E(j)(c))$.

Remark 5.3. The coend $Y_Q(j) = \int^i Q(i,j)\cdot D(i)$ is a colimit weighted by $Q(-,j)$, while the end $X_Q(i) = \int_j E(j)^{Q(i,j)}$ is a limit weighted by $Q(i,-)$. Both exist in $\mathrm{Psh}(C)$ since presheaf categories are complete and cocomplete [24]. These are the canonical constructions for left and right Kan extensions, appearing here as the universal left and right envelope objects.

5.3 Universal Properties of the Envelope Objects

Lemma 5.4 (Universal property of the left envelope)**.** *For every* $Z \in \mathrm{Psh}(C)$ *and* $j \in J$*, there is a natural isomorphism*

$$\mathrm{Psh}(C)(Y_Q(j), Z) \cong \mathrm{Nat}\big(Q(-,j),\, \mathrm{Psh}(C)(D(-), Z)\big).$$

Proof. By the coend formula,

$$\mathrm{Psh}(C)(Y_Q(j), Z) = \mathrm{Psh}(C)\left(\int^{i} Q(i,j)\cdot D(i),\, Z\right).$$

Since hom out of a colimit converts it to a limit,

$$\cong \int_i \mathbf{Set}\big(Q(i,j),\, \mathrm{Psh}(C)(D(i),Z)\big) = \mathrm{Nat}\big(Q(-,j),\, \mathrm{Psh}(C)(D(-),Z)\big).$$

Naturality in Z and in j follows from the naturality of the coend construction. $\square$

Lemma 5.5 (Universal property of the right envelope)**.** *For every $Z \in \mathrm{Psh}(C)$ and $i \in I$, there is a natural isomorphism*

$$\mathrm{Psh}(C)(Z, X_Q(i)) \cong \mathrm{Nat}\big(Q(i,-),\, \mathrm{Psh}(C)(Z, E(-))\big).$$

Proof. By the end formula,

$$\mathrm{Psh}(C)(Z, X_Q(i)) = \mathrm{Psh}(C)\left(Z,\, \int_j E(j)^{Q(i,j)}\right).$$

Since hom into a limit is a limit,

$$\cong \int_j \mathrm{Psh}(C)\big(Z,\, E(j)^{Q(i,j)}\big).$$

By the power adjunction $\mathrm{Psh}(C)(Z, E(j)^{Q(i,j)}) \cong \mathbf{Set}(Q(i,j), \mathrm{Psh}(C)(Z, E(j)))$,

$$\cong \int_j \mathbf{Set}\big(Q(i,j),\, \mathrm{Psh}(C)(Z,E(j))\big) = \mathrm{Nat}\big(Q(i,-),\, \mathrm{Psh}(C)(Z,E(-))\big).$$

Naturality in Z and in i follows from the naturality of the end construction. $\square$

Lemma 5.6 (Canonical left and right maps)**.** *There are canonical natural transformations*

$$\lambda_Q : Q \Rightarrow \mathrm{Psh}(C)(D, Y_Q), \qquad \rho_Q : Q \Rightarrow \mathrm{Psh}(C)(X_Q, E).$$

Proof. For each $q \in Q(i,j)$, the map $\lambda_{Q,i,j}(q) : D(i) \to Y_Q(j)$ is the coprojection of the q-indexed summand of $D(i)$ into the coend $\int^i Q(i,j)\cdot D(i)$. This gives λ_Q by the universal property of the copower.

For each $q \in Q(i,j)$, the map $\rho_{Q,i,j}(q) : X_Q(i) \to E(j)$ is the q-indexed projection from the end $\int_j E(j)^{Q(i,j)}$, given by evaluation at q. This gives ρ_Q by the universal property of the power.

Naturality of λ_Q and ρ_Q in all variables follows from the universal properties of coends and ends. $\square$

Remark 5.7. Note that $\lambda_{Q,i,j}(q)$ and $\rho_{Q,i,j}(q)$ are coprojection and projection maps, respectively. They are *not* isomorphisms in general. The universal properties of Y_Q and X_Q are the key tool; the old argument using inverses of λ and ρ was incorrect and is not used here.

5.4 The Interpolation Problem and the Main Theorem

Proposition 5.8 (The interpolation problem)**.** *Let $\theta : Q \Rightarrow \mathrm{Psh}(C)(D,E)$ be a pairing with canonical maps λ_Q and ρ_Q from Lemma 5.6. A natural transformation*

$$\gamma : Q \Rightarrow \mathrm{Psh}(C)(Y_Q, X_Q)$$

satisfying $\theta = \rho_Q \star \gamma \star \lambda_Q$ is called an interpolant *for θ.*

Explicitly, γ assigns to each (i,j,q) a morphism $\gamma_{i,j}(q) : Y_Q(j) \to X_Q(i)$ such that for all i,j,q:

$$\theta_{i,j}(q) = \rho_{Q,i,j}(q) \circ \gamma_{i,j}(q) \circ \lambda_{Q,i,j}(q).$$

By the universal properties of Y_Q and X_Q (Lemmas 5.4 and 5.5), specifying $\gamma_{i,j}(q) : Y_Q(j) \to X_Q(i)$ is equivalent to specifying a natural family

$$\big\{Q(i,k) \to \mathrm{Psh}(C)(D(k'),\, E(j'))\big\}$$

compatible with the coend and end structures, with the constraint that composing with λ_Q on the left and ρ_Q on the right recovers θ.

Theorem 5.9 (Presheaf envelope theorem)**.** *Let C be a small* **Set***-enriched category, and let $\theta : Q \Rightarrow \mathrm{Psh}(C)(D,E)$ be a pairing in $\mathrm{Psh}(C) = [C^{\mathrm{op}}, \mathbf{Set}]$. The constructions*

$$Y_Q(j) = \int^{i\in I} Q(i,j)\cdot D(i), \qquad X_Q(i) = \int_{j\in J} E(j)^{Q(i,j)}$$

provide canonical left and right envelope objects, together with natural transformations

$$\lambda_Q : Q \Rightarrow \mathrm{Psh}(C)(D, Y_Q), \qquad \rho_Q : Q \Rightarrow \mathrm{Psh}(C)(X_Q, E).$$

A canonical envelope of θ in $\mathrm{Psh}(C)$ exists if and only if the interpolation problem

$$\gamma_Q : Q \Rightarrow \mathrm{Psh}(C)(Y_Q, X_Q), \quad \theta = \rho_Q \star \gamma_Q \star \lambda_Q$$

admits an initial solution. When such an initial interpolant γ_Q exists, the triple $(\lambda_Q, \gamma_Q, \rho_Q)$ is the canonical envelope of θ.

Proof. Y_Q **is the universal left envelope.** By Lemma 5.4, for any $Z \in \mathrm{Psh}(C)$:

$$\mathrm{Psh}(C)(Y_Q(j), Z) \cong \mathrm{Nat}(Q(-,j), \mathrm{Psh}(C)(D(-), Z)).$$

This means that any left envelope object Y' with natural transformation $\lambda' : Q \Rightarrow \mathrm{Psh}(C)(D, Y')$ receives a unique comparison map $\alpha : Y_Q \Rightarrow Y'$ with $\lambda' = \mathrm{Psh}(C)(D,\alpha) \circ \lambda_Q$. Thus Y_Q is the initial left envelope.

X_Q **is the universal right envelope.** Dually, by Lemma 5.5, X_Q is the terminal right envelope: any right envelope object X' with $\rho' : Q \Rightarrow \mathrm{Psh}(C)(X', E)$ receives a unique comparison map $\beta : X_Q \Rightarrow X'$ with $\rho' = \mathrm{Psh}(C)(\beta, E) \circ \rho_Q$.

Reduction to the interpolation problem. For any factorization $(\lambda, \gamma, \rho) \in \mathrm{Fact}(\theta)$ with objects (Y, X), the universal properties of Y_Q and X_Q give unique comparison maps $\alpha : Y_Q \Rightarrow Y$ and $\beta : X_Q \Rightarrow X$ compatible with λ and ρ. A morphism $(\lambda_Q, \gamma_Q, \rho_Q) \to (\lambda, \gamma, \rho)$ in $\mathrm{Fact}(\theta)$ requires additionally that the middle compatibility condition

$$\gamma \circ \mathrm{Psh}(C)(\alpha, X) = \mathrm{Psh}(C)(Y_Q, \beta) \circ \gamma_Q$$

holds. The existence and uniqueness of (α, β) from the outer universal properties reduces the initiality of $(\lambda_Q, \gamma_Q, \rho_Q)$ entirely to the question of whether γ_Q exists as an initial solution of the interpolation problem $\theta = \rho_Q \star \gamma_Q \star \lambda_Q$.

Sufficiency. If an initial interpolant γ_Q exists, the above argument shows $(\lambda_Q, \gamma_Q, \rho_Q)$ is initial in $\mathrm{Fact}(\theta)$: for any (λ, γ, ρ), the unique morphism is (α, β) where α, β come from the outer universal properties and compatibility of the middle follows from initiality of γ_Q.

Necessity. Conversely, if θ has a canonical envelope $(\lambda_0, \gamma_0, \rho_0)$, its outer components factor uniquely through Y_Q and X_Q by the universal properties, yielding an interpolant at (Y_Q, X_Q) that is initial by the initiality of $(\lambda_0, \gamma_0, \rho_0)$. ☐

Theorem 5.10 (Uniqueness of canonical envelopes)**.** *If the canonical envelope of θ exists, it is unique up to unique isomorphism in* $\mathrm{Fact}(\theta)$*.*

Proof. This is the standard uniqueness of initial objects: any two initial objects in a category are connected by a unique isomorphism. ☐

Corollary 5.11. *If θ is bilaterally dense in* $\mathrm{Psh}(C)$ *and the interpolation problem has an initial solution, then the canonical envelope exists and is unique up to unique isomorphism.*

Remark 5.12 (The real content of the theorem). The coend Y_Q and end X_Q are always constructible and always satisfy the outer universal properties. The genuine mathematical content is the existence of an initial interpolant γ_Q. For many classical completions (Kan extensions, canonical extensions of lattices, Cauchy completions), this interpolant exists and can be constructed explicitly—see Section 7. The theorem cleanly separates what is always true (the outer envelope objects and their universal properties) from what requires proof case by case (the middle interpolant).

Lemma 5.13 (Morphisms between factorizations)**.** *Let (λ, γ, ρ) with functors (Y, X) and $(\lambda', \gamma', \rho')$ with functors (Y', X') be two factorizations of θ. There are unique natural transformations $\alpha : Y_Q \Rightarrow Y$ and $\beta : X_Q \Rightarrow X'$ induced by the universal properties of Y_Q and X_Q. These satisfy the outer compatibility conditions*

$$\lambda' = \mathrm{Psh}(C)(D, \alpha) \circ \lambda_Q, \qquad \rho = \mathrm{Psh}(C)(\beta, E) \circ \rho_Q.$$

Proof. Immediate from Lemmas 5.4 and 5.5: morphisms out of $Y_Q(j)$ correspond to natural transformations $Q(-, j) \Rightarrow \mathrm{Psh}(C)(D(-), -)$, and λ' provides exactly such data, giving α uniquely. Dually for β. ☐

6 Weighted Limits and Kan Extensions

6.1 Virtual Weighted Limits

When classical weighted limits fail to exist, canonical envelopes provide a notion of virtual weighted limits. The key point is that the outer envelope objects Y_Q and X_Q always exist; what may fail is the interpolant.

Definition 6.1 (Virtual Weighted Limit)**.** Let $W : J \to \mathbf{Set}$ be a weight and $F : J \to C$ a functor. Define a pairing:

- $I = \{*\}$ (terminal category)
- J (as given)

- $D : \{*\} \to C$ constant at some object $c \in C$
- $E = F$
- $Q(*, j) = W(j)$
- $\theta_{*,j} : W(j) \to C(c, F(j))$

The canonical envelope of this pairing (when it exists) defines the *virtual W-weighted limit of F at c.*

Dually, taking $J = \{*\}$ and I as the index category defines virtual W-weighted colimits.

Remark 6.2 (Relationship to Existing Notions)*.* The virtual weighted limit should be distinguished from:

- **Absolute Kan extensions:** While both concepts involve universality without requiring existence in the ambient category, virtual weighted limits arise from the envelope construction's bilateral factorization, whereas absolute Kan extensions are defined via preservation by all functors. When the weight W is absolute (preserved by all functors), these notions may coincide, but this requires verification.
- **Street's virtual equipment limits:** Street's notion applies in the context of proarrow equipments and double categories. Our virtual weighted limits are defined purely in terms of bilateral pairings in ordinary categories. The relationship, if any, would require embedding ordinary categories into appropriate double categorical structures.

Proposition 6.3 (Basic Properties of Virtual Weighted Limits)**.** *Virtual weighted limits satisfy the following structural properties:*

1. ***Uniqueness:*** *When the virtual W-weighted limit exists and the pairing is bilaterally compact, it is unique up to unique isomorphism by Theorem 5.10.*
2. ***Stability under isomorphism:*** *If $F \cong F'$ naturally and $W \cong W'$ naturally, then their virtual weighted limits (when they exist) are naturally isomorphic. This follows from functoriality of the envelope construction.*
3. ***Reduction to classical case:*** *When the classical W-weighted limit exists, the virtual weighted limit coincides with it (Proposition 6.4 below).*

Proof. (1) follows directly from Theorem 5.10.

(2) Natural isomorphisms $\alpha : F \Rightarrow F'$ and $\beta : W \Rightarrow W'$ induce a natural isomorphism of pairings $\theta \cong \theta'$. Since the envelope construction is functorial (it arises from the initial object in a category of factorizations), isomorphic pairings have isomorphic envelopes.

(3) is proved in Proposition 6.4. ☐

Proposition 6.4. *When the classical W-weighted limit $\lim^W F$ exists, it coincides with the virtual weighted limit defined via canonical envelopes.*

Proof. The classical weighted limit satisfies $C(c, \lim^W F) \cong [J, \mathbf{Set}](W, C(c, F(-)))$ naturally in c. This is precisely the universal property characterizing representability of $Q(*, -)$ in the pairing above. By definition, this representability is the left envelope in our factorization, which by initiality is the canonical envelope. ☐

6.2 General Weighted Limits as Canonical Envelopes

We now establish the general result that all weighted limits are canonical envelopes. The key observation is that the weighted limit universal property directly translates to bilateral representability.

Theorem 6.5 (Weighted Limits are Canonical Envelopes)**.** *Let* $W : J \to \mathbf{Set}$ *be a weight and* $F : J \to C$ *a functor. Suppose the* W*-weighted limit* $\lim^W F$ *exists in* C*. Then* $\lim^W F$ *arises as the canonical envelope of the pairing defined in Definition 6.1.*

Proof. The weighted limit $\lim^W F$ is characterized by the natural isomorphism:

$$C(c, \overset{W}{\lim} F) \cong [J, \mathbf{Set}](W, C(c, F(-)))$$

for all $c \in C$.

In the pairing from Definition 6.1, we have $Q(*, j) = W(j)$, and the natural isomorphism above is precisely the statement that $\lim^W F$ represents the functor $Q(*, -) : J \to \mathbf{Set}$.

By Proposition 4.5, this representability implies bilateral density. The representing object is exactly $\lim^W F$, and by Theorem 5.9, the canonical envelope exists and is unique up to unique isomorphism. Since $\lim^W F$ satisfies the universal property, it must be the canonical envelope. □

Corollary 6.6 (Weighted Colimits are Canonical Envelopes)**.** *Weighted colimits* $\operatorname{colim}^W F$ *are canonical envelopes by duality: reverse all arrows in Theorem 6.5.*

6.3 Kan Extensions and Liftings as Canonical Envelopes

Since Kan extensions and liftings are expressible as weighted (co)limits [22], they are immediately instances of canonical envelopes.

Corollary 6.7 (All Four Kan Constructions are Canonical Envelopes)**.** *Let* $K : A \to B$ *be a functor and* $F : A \to C$ *(for extensions) or* $F : B \to C$ *(for liftings). Then:*

1. $\mathrm{Lan}_K F$ *is a canonical envelope (as a weighted colimit)*
2. $\mathrm{Ran}_K F$ *is a canonical envelope (as a weighted limit)*
3. $\mathrm{Lift}_K F$ *is a canonical envelope (as a weighted colimit in slice category)*
4. $\mathrm{Rift}_K F$ *is a canonical envelope (as a weighted limit in slice category)*

Proof. It is well-known [22] that:

- Left Kan extensions are colimits weighted by the representable functor $B(K(-), b)$
- Right Kan extensions are limits weighted by the representable functor $B(b, K(-))$
- Kan liftings are Kan extensions in appropriate slice categories

By Theorem 6.5 and its corollary, all weighted (co)limits are canonical envelopes. Therefore all four Kan constructions are canonical envelopes. □

Remark 6.8. This derivation is much cleaner than proving each Kan direction separately. The logical flow is:

Weighted limits/colimits are envelopes ⇒ Kan constructions are envelopes

rather than proving left Kan extensions separately and appealing to duality for the others.

6.4 Conceptual Summary

The canonical envelope construction subsumes all four Kan constructions and all weighted (co)limits by the following clean logical flow:

1. Weighted limits/colimits are characterized by universal properties (representability)
2. Universal properties translate to bilateral density and compactness
3. Therefore weighted (co)limits are canonical envelopes (Theorem 6.5)
4. Kan extensions/liftings are weighted (co)limits [22]
5. Therefore Kan constructions are canonical envelopes (Corollary 6.7)

In each of these cases, the interpolation problem of Theorem 5.9 is solvable: the universal property of the weighted limit or Kan extension provides exactly the initial interpolant γ_Q. This is the concrete content behind the general reduction theorem—CE existence is not automatic from the presheaf construction, but for constructions already known to exist by universal properties, the interpolant is furnished by that very universal property.

Remark 6.9 (Scope). Within this section, we have shown that weighted (co)limits and all four Kan constructions are instances of canonical envelopes. The framework also covers limits and colimits (as weighted limits with trivial weights), adjunctions (via Kan extensions), and the completions in Section 7. We do not claim this covers all universal constructions; the scope is determined by which pairings admit an initial interpolant.

The bilateral factorization $\theta = \rho \star \gamma \star \lambda$ gives a common pattern for the constructions it covers: each is a factorization mediating between dual probing operations via an initial object. Which further pairings admit such an initial interpolant is the main open question; see Section 10.

Remark 6.10 (Bidirectional correspondence). The relationship between universal properties and canonical envelopes runs both ways.

Universal ⇒ envelope: A construction known to exist via a universal property can be expressed as a canonical envelope. Representability implies bilateral density (Proposition 4.5), and the existing universal object furnishes the initial interpolant. The prior existence of the object does the work; the presheaf construction alone does not guarantee it.

Envelope ⇒ universal: An initial object in $\mathrm{Fact}(\theta)$ automatically satisfies a universal property, by initiality.

The bilateral perspective makes the internal mediation structure explicit via γ, where the one-sided view does not.

6.5 Origin of the Envelope Construction: Riehl's Adjunction

The canonical envelope formalism arose directly from analyzing the adjunction displayed in Riehl's paper [28], itself based on lectures of Shulman. Let

$$J : A \otimes B^{\mathrm{op}} \to \mathcal{V}$$

be an A–B bimodule and let $D : B \to \mathcal{M}$ and $E : A \to \mathcal{M}$ be diagrams. Riehl's Theorem (5.1) asserts the natural adjunction

$$[A, \mathcal{M}](\mathrm{colim}^J D, E) \cong [A \otimes B^{\mathrm{op}}, \mathcal{V}](J, \mathcal{M}(D, E)) \cong [B, \mathcal{M}](D, \overset{J}{\lim} E). \tag{1}$$

The middle term is the hom-object of bimodules $J \to \mathcal{M}(D, E)$, while the outer terms express this bimodule hom in two "Kan directions" as either

- a morphism out of a J-weighted colimit, or
- a morphism into a J-weighted limit.

Corollary 6.7 shows how left Kan extensions arise from this structure. The remaining Kan directions follow similar patterns, all arising as canonical envelopes of appropriately constructed pairings from Riehl's adjunction (1).

The canonical envelope construction arises by taking (1) not as an isolated adjunction, but as *defining data.* A pairing consists exactly of such bimodule data (A, B, J, D, E) together with the transformation

$$\theta : J \Rightarrow \mathcal{M}(D, E),$$

and the canonical envelope is the initial factorization of θ into its dense–exact–compact components. Every example in this paper is a specialization of this situation.

Thus canonical envelopes may be read as a systematic abstraction of Riehl's adjunction (1): instead of taking either the left or the right adjoint side alone, we keep the entire bimodule interface and complete it universally.

6.6 Virtual Canonical Envelopes

When a classical completion does not exist within the ambient category C, the presheaf construction of Section 5 still produces the outer envelope objects Y_Q and X_Q in $\mathrm{Psh}(C)$, together with the canonical maps λ_Q and ρ_Q. These always exist, by the completeness of presheaf categories under coends and ends. Whether the full canonical envelope exists depends additionally on whether the interpolation problem has an initial solution — and for the examples below, this has not been established.

We call the quadruple $(Y_Q, X_Q, \lambda_Q, \rho_Q)$ a **virtual canonical envelope**. This is not a weakened or approximate version of a canonical envelope; it is precisely the outer presheaf envelope data that the construction of Section 5 always delivers. The qualifier "virtual" signals only that the middle interpolant is not yet known.

Definition 6.11 (Virtual Canonical Envelopes)**.** Let $\theta : Q \Rightarrow C(D, E)$ be a bilateral pairing. The **virtual canonical envelope** of θ is the quadruple $(Y_Q, X_Q, \lambda_Q, \rho_Q)$ consisting of the outer envelope objects

$$Y_Q(j) = \int^{i} Q(i, j) \cdot D(i), \qquad X_Q(i) = \int_{j} E(j)^{Q(i,j)}$$

in $\mathrm{Psh}(C)$, together with the canonical natural transformations $\lambda_Q : Q \Rightarrow \mathrm{Psh}(C)(D, Y_Q)$ and $\rho_Q : Q \Rightarrow \mathrm{Psh}(C)(X_Q, E)$ from Lemmas 5.4–5.6.

The virtual envelope always exists in $\mathrm{Psh}(C)$ under the size hypotheses of Section 2. A canonical envelope (in the full sense of Definition 3.9) exists if and only if the interpolation problem additionally has an initial solution $\gamma_Q : Q \Rightarrow \mathrm{Psh}(C)(Y_Q, X_Q)$ with $\theta = \rho_Q \star \gamma_Q \star \lambda_Q$.

The virtual envelope is not a weakened canonical envelope; it is the canonical outer presheaf envelope. The obstruction to an actual canonical envelope is precisely the absence, or non-initiality, of the middle interpolant γ_Q.

Example 6.12 (Outer Envelope for Non-Completely-Regular Spaces)**.** Let X be a T_1 space that is not completely regular, so the classical Stone-Čech compactification does not exist. The filter-ultrafilter bilateral pairing $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ is still well-defined, and the presheaf construction produces outer envelope objects Y_Q and X_Q in $\mathrm{Psh}(\mathbf{Top})$. These capture the universal structure of all continuous maps from X to compact Hausdorff spaces, but live in the presheaf category rather than in **Top**. There is no claim that a compactification object exists.

Whether the interpolation problem is solvable for this pairing — and hence whether a canonical envelope exists — is not established here.

Example 6.13 (Outer Envelope for Non-Residually-Finite Groups)**.** Let G be a group that is not residually finite. The bilateral pairing based on finite quotients is still well-defined, and the presheaf construction produces outer envelope objects in $\mathrm{Psh}(\mathbf{Grp})$ capturing all homomorphisms from G to finite groups. These live in the presheaf category; there is no claim that a profinite completion object exists.

Whether the interpolation problem has an initial solution in this setting is an open question, not a theorem of this paper.

Example 6.14 (Outer Envelope for Distributive Lattices)**.** For a distributive lattice L, the filter-ideal bilateral pairing is well-defined and the outer envelope objects Y_Q and X_Q in $\mathrm{Psh}(\mathbf{DLat})$ always exist. Recall from Section 7.2 that canonical extensions are not in general canonical envelopes: density and compactness do not produce an initial interpolant. Whether the interpolation problem is solvable — and hence whether a canonical envelope exists for this pairing — is the open question identified there.

Remark 6.15 (What virtual envelopes provide)*.* Even without the full canonical envelope, the outer objects Y_Q and X_Q have genuine content. By Lemmas 5.4 and 5.5, $Y_Q(j)$ represents the functor $\mathrm{Nat}(Q(-,j), \mathrm{Psh}(C)(D(-),-))$ and $X_Q(i)$ represents $\mathrm{Nat}(Q(i,-), \mathrm{Psh}(C)(-,E(-)))$. These are the universal left and right envelope objects in $\mathrm{Psh}(C)$; they exist regardless of whether the middle interpolant does. The virtual envelope thus provides a well-defined presheaf-category object that bounds any attempt at completion, even when the classical construction fails.

6.7 Completeness: When No Further Completion Is Needed (Interpretive Examples)

Remark 6.16 (Status of this section)*.* The examples below are interpretive: they identify classical completeness conditions as instances of the CE completeness notion, using the fact that classical completions are canonical envelopes (established in Section 7). They are heuristic illustrations, not independent theorems. The notation $\hat{C}$ refers informally to the canonical envelope of C; a fully general construction of $\hat{C}$ beyond the presheaf setting is not developed in this paper.

A bilateral pairing is **complete** when it already admits the required bilateral factorization within its ambient category—no external completion is necessary. Understanding completeness is crucial for determining when classical methods suffice versus when virtual extensions are required.

Definition 6.17 (Complete Bilateral Pairings)**.** A bilateral pairing $\theta : Q \Rightarrow C(D, E)$ is **complete** if the canonical envelope embedding $\varepsilon_C : C \to \hat{C}$ is an equivalence of categories. That is, C already contains all the structure required by the canonical envelope, so no external completion in $\mathrm{Psh}(C)$ is needed.

6.7.1 Complete Pairings: Examples

Example 6.18 (Compact Hausdorff Spaces Are Complete)**.** For the Stone-Čech compactification bilateral pairing applied to a compact Hausdorff space X:

Why it's complete:

- Every ultrafilter converges to a unique point (compactness + Hausdorff)
- Every bounded continuous function is already defined on the whole space
- The bilateral factorization exists entirely within **Top**: the embedding $X \hookrightarrow \beta X$ is the identity because $X = \beta X$
- No external completion is necessary

Domain-specific condition: Compact + Hausdorff is precisely the condition for Stone-Čech completeness.

Example 6.19 (Complete Boolean Algebras Are Complete)**.** For the canonical extension bilateral pairing applied to a *complete* Boolean algebra B:

Why it's complete:

- Complete Boolean algebras have all meets and joins, so the filter-ideal bilateral factorization exists within **DLat**
- Every filter meets every ideal in a complete Boolean algebra
- The canonical extension $B^\delta \cong B$ (complete Boolean algebras are their own canonical extensions)

Domain-specific condition: Being a complete Boolean algebra is precisely the completeness condition for the canonical extension pairing. An arbitrary Boolean algebra B is not complete as a lattice; its canonical extension B^δ is a proper extension unless B is already complete.

Example 6.20 (Profinite Groups Are Complete)**.** For the profinite completion bilateral pairing applied to a profinite group G:

Why it's complete:

- G is already the inverse limit of its finite quotients
- The bilateral factorization through finite quotients exists within **Grp**
- The completion $\hat{G} = G$ (profinite groups are their own completions)
- All required limits exist in the category of profinite groups

Domain-specific condition: Being profinite is precisely the completeness condition for profinite completion.

Example 6.21 (Complete Metric Spaces Are Complete)**.** For the Cauchy completion bilateral pairing applied to a complete metric space (X, d):

Why it's complete:

- Every Cauchy sequence converges to a point in X
- The bilateral factorization through Cauchy sequences exists within **Met**
- The Cauchy completion $\overline{X} = X$
- No additional points need to be added

Domain-specific condition: Completeness (every Cauchy sequence converges) is precisely the condition for Cauchy completion to be trivial.

6.7.2 Incomplete Pairings: Examples

Example 6.22 (Non-Regular Spaces Are Incomplete)**.** For a non-completely regular T_1 space X:

Why it's incomplete:

- X lacks sufficient continuous functions to separate closed sets from points
- The filter-ultrafilter bilateral pairing cannot be factored within **Top** using only X
- Additional points (and the compactified topology) are needed
- Must use virtual canonical envelope for universal property

Obstruction: Lack of complete regularity prevents bilateral factorization from existing in **Top**.

Example 6.23 (Non-Residually Finite Groups Are Incomplete)**.** For a group G that is not residually finite:

Why it's incomplete:

- The intersection of all finite-index normal subgroups is nontrivial
- Cannot recover G from its finite quotients alone
- The bilateral factorization through finite quotients requires external completion
- Virtual profinite completion provides approximation but not classical completion

Obstruction: Failure of residual finiteness means finite quotients don't "test" all of G.

Example 6.24 (Incomplete Metric Spaces Are Incomplete)**.** For an incomplete metric space (X, d) (e.g., $\mathbb{Q}$ with the usual metric):

Why it's incomplete:

- There exist Cauchy sequences with no limit in X
- The bilateral factorization through Cauchy sequences cannot be realized within X
- Additional "limit points" must be added (e.g., irrational numbers for $\mathbb{Q}$)
- The completion $\overline{X}$ properly extends X

Obstruction: Missing limits of Cauchy sequences prevent internal factorization.

6.7.3 Summary: Domain-Specific Completeness Conditions

Theorem 6.25 (Completeness Correspondence Principle)**.** *A bilateral pairing arising from a classical completion problem is complete if and only if the domain-specific completeness condition is satisfied:*

Completion Type	***Completeness Condition***
Stone-Čech compactification	*Compact Hausdorff*
Sobrification	*Sober*
Canonical extensions	*Complete Boolean algebra*
Profinite completion	*Already profinite*
Cauchy completion	*Complete metric space*
Ind-completion	*Category with all filtered colimits*
Kan extensions	*Existence of limits/colimits*
Isbell envelopes	*Adequate (Garner's sense [14])*

Proof sketch. Each correspondence follows from analyzing the bilateral density and compactness conditions:

Topology: Complete regularity is precisely the condition needed for the filter-ultrafilter bilateral pairing to be bilaterally dense—it ensures sufficient continuous functions exist to separate the bilateral structure.

Lattice Theory: For distributive lattices, being a complete Boolean algebra is exactly the condition for the filter-ideal bilateral pairing to admit internal factorization—every filter-ideal interaction can be resolved within the lattice.

Group Theory: A group is profinite if and only if it's the inverse limit of its finite quotients, which is precisely the condition for the finite quotient bilateral pairing to be complete.

Category Theory: Adequacy in Garner's sense [14] characterizes exactly when the presheaf-copresheaf bilateral pairing for Isbell envelopes is complete. ☐

Remark 6.26 (Practical Implication). The bilateral structure determines when classical completion methods succeed:

1. Formulate the completion problem as a bilateral pairing
2. Check the domain-specific completeness condition
3. If complete: use classical methods within the original category
4. If incomplete: use virtual canonical envelope for universal property

The bilateral framework unifies these diverse conditions under the common principle of bilateral density and compactness.

7 Applications

We now derive several classical completions as canonical envelopes:

Detailed examples:

- Structural comparison with classical and categorical canonical extensions (Section 7.2)
- Pratt's communes (Section 7.5)

Outlined examples:

- Ind- and pro-completions (Section 7.3)
- Isbell envelopes (Section 7.7)

Conceptual connections:

- Cauchy completions of Lawvere metric spaces (Section 7.4)

Summary: Factorization Components Across Examples

The following table shows how the bilateral pairing components and canonical envelope structure manifest in classical completion constructions.[2]

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[2] Entries marked † represent future work not yet fully developed within the present framework. See Section 10.2 for the complete list of characterizations.

Completion	**Left Index** I	**Right Index** J	**Source** D	**Target** E	**Weight** Q
Categorical Canonical Ext.	Filt(X) (filtered diagrams)	Cofilt(X) (cofiltered diagrams)	Filtered diagrams	Cofiltered diagrams	Incidence relation
Classical Canonical Ext.	Filt$(L)^{\mathrm{op}}$ (filters)	Idl(L) (ideals)	Filters	Ideals	Non-disjointness relation
Cylinder Factorization	$\mathcal{C}$ (objects)	$\mathcal{C}$ (objects)	Identity functor	Identity functor	Hom-functor $\mathcal{C}(-,-)$
Stone-Čech Compactification	UltraFilt$(X)^{\mathrm{op}}$ (ultrafilters)	Points in X	Ultrafilters	Points	Containment relation
Sobrification	Open sets in X (open subsets)	Points in X	Opens	Points	Membership relation
Cauchy Completion	$\mathbb{N}^{\mathrm{op}}$ (sequences)	$\mathbb{N}$ (sequences)	Forward Cauchy seq.	Backward Cauchy seq.	Eventual agreement
Ind-Completion	Filt$(C)^{\mathrm{op}}$ (filtered diagrams)	C	Filtered diagrams	Objects	Cocone relation
Pro-Completion	C^{op}	Cofilt(C) (cofiltered diagrams)	Objects	Cofiltered diagrams	Cone relation
Pratt's Communes	P (objects of P)	P (objects of P)	Identity functor	Identity functor	Identity pairing id_P
Avery-Leinster Adjunction Env.	A (left adjoint)	B (right adjoint)	Identity on A	Identity on B	$P(a,b) = B(F(a),b)$ for $F \dashv G$
Left Kan Extension	B (codomain of K)	A (domain of K)	$D : B \to C$	$F : A \to C$	$Q(b,a) = B(K(a),b)$ (representable)
Right Kan Extension	A (domain of K)	B (codomain of K)	$F : A \to C$	$E : B \to C$	$Q(a,b) = B(b,K(a))$ (representable)
Left Kan Lifting	A (domain of K)	B (codomain of K)	$F : A \to C$	$E : B \to C$	Slice category weight
Right Kan Lifting	B (codomain of K)	A (domain of K)	$D : B \to C$	$F : A \to C$	Slice category weight
Isbell Envelope	C^{op}	C	Contravariant functors	Covariant functors	Yoneda embedding
Tabular† Allegories	Rel	Rel	Relations	Relations	2-enriched structure
Regular† Categories	Cat	Cat	Categories	Categories	Cat-enriched structure
Dicategories	**Cat** (objects)	**CoCat** (objects)	Category structure	Cocategory structure	Duality relation

Completion	**Left Envelope Y**	**Right Envelope X**	**Interpolant γ**
Categorical Canonical Ext.	Filtered colimits of closed objects	Cofiltered limits of open objects	Natural factorization
Classical Canonical Ext.	Principal filters $\uparrow a$ for $a \in L$	Principal ideals $\downarrow a$ for $a \in L$	$\gamma(F, I) = F \cap I$ (filter-ideal meet)
Cylinder Factorization	Objects via left class L	Objects via right class R	Cylinder objects
Stone-Čech Compactification	Principal ultrafilters at points in X	Convergence points in βX	Ultrafilter convergence
Sobrification	Closure of points	Irreducible closed sets	Sober points
Cauchy Completion	Forward Cauchy sequences	Backward Cauchy sequences	Eventual agreement
Ind-Completion	Representable presheaves	Diagrams from filtered colimits	Cocone morphisms
Pro-Completion	Diagrams from cofiltered limits	Representable presheaves	Cone morphisms
Pratt's Communes	Objects via left didensity	Objects via right didensity	Identity interpolant
Avery-Leinster Adjunction Env.	Left adjoint $\hat{A}$	Right adjoint $\hat{B}$	Adjunction $\hat{F} \dashv \hat{G}$
Left Kan Extension	$Y(a) = K(a)$ (from domain)	$X(b) = \mathrm{colim}_{(K \downarrow b)} F$ (colimit over comma)	Universal cocone
Right Kan Extension	$X(b) = \lim_{(b \downarrow K)} F$ (limit over comma)	$Y(a) = K(a)$ (from domain)	Universal cone
Left Kan Lifting	Extension structure via slice category	Lifting structure via slice category	Universal factorization
Right Kan Lifting	Lifting structure via slice category	Extension structure via slice category	Universal factorization
Isbell Envelope	Covariant presheaves	Contravariant presheaves	Yoneda duality
Tabular† Allegories	Relational structure	Relational structure	Tabulation structure
Regular† Categories	Regular structure	Regular structure	Regularity structure
Dicategories	Categorical structure via $\circ$	Cocategorical structure via $\odot$	Duality operation $\dagger : \mathrm{Hom} \to \mathrm{CoHom}$

The following table shows how the factorization components (λ, γ, ρ) manifest in classical completion constructions:

Completion	**Left λ**	**Interpolant γ**	**Right ρ**
Categorical Canonical Ext.	Objects → Filtered colimits	Filtered → Cofiltered via factorization	Cofiltered → Limits
Classical Canonical Ext.	Elements → Principal filters	Filters → Ideals via $\gamma(F, I) = F \cap I$	Ideals → Principal ideals
Cylinder Factorization	Source → Cylinder via L	Left class → Right class via γ	Right class → Target
Sobrification	Points → Closures	Opens → Irreducibles	Irreducibles → Sober points
Cauchy Completion	Forward sequences	Forward → Backward via eventual agreement	Backward sequences
Ind-Completion	Objects → Representables	Filtered colimits → via cocone maps	Limits → Diagrams
Pro-Completion	Objects → Diagrams	Diagrams → via cone maps	Cofiltered limits → Representables
Pratt's Communes	Objects → Left didensity	Bilateral didensity	Objects → Right didensity
Avery-Leinster Adjunction Env.	$A \to \hat{A}$ via λ	Adjunction structure	$B \to \hat{B}$ via ρ
Left Kan Extension	$B \to B$ via $\lambda(b) = K(b)$	Colimits → representable	Colimits $\to C$ via $\mathrm{Lan}_K F$
Right Kan Extension	$A \to C$ via $\mathrm{Ran}_K F$	Representable → limits	$B \to B$ via $\rho(b) = K(b)$
Left Kan Lifting	Slice structure via λ	Lifting via slice category	Slice structure via ρ
Right Kan Lifting	Slice structure via λ	Lifting via slice category	Slice structure via ρ
Isbell Envelope	Covariant presheaves	Duality via Yoneda	Contravariant presheaves
Tabular† Allegories	Relations → Tabulated	Relational factorization	Relations → Tabulated
Regular† Categories	Categories → Regular structure	Regular factorization	Categories → Regular structure
Dicategories	Categories → Dual cocategories	Categorical ↔ Cocategorical via †	Cocategories → Dual categories

In each case:

- λ embeds the original objects into a left completion (covariant construction)
- ρ embeds into a right completion (contravariant construction)
- γ provides the unique mediating morphism between left and right completions

The bilateral perspective unifies these diverse constructions through the common pattern of factorization.

7.1 Worked Example: Two-Element Chain

Before developing the general theory, we illustrate the canonical envelope construction with a concrete finite example.

Example 7.1 (Canonical Envelope of a Two-Object Poset)**.** Let $\mathbf{2} = \{0 < 1\}$ be the two-element chain. We construct its canonical extension as a canonical envelope.

Setup: Define a pairing:

- $I = \mathrm{Filt}(\mathbf{2})^{\mathrm{op}} = \{\{1\}, \{0,1\}\}^{\mathrm{op}}$ (filters in $\mathbf{2}$)
- $J = \mathrm{Idl}(\mathbf{2}) = \{\emptyset, \{0\}, \{0,1\}\}$ (ideals in $\mathbf{2}$)
- $D(F) = \mathbf{2}$ for all filters F
- $E(I) = \mathbf{2}$ for all ideals I
- $Q(F, I) = \{*\}$ if $F \cap I = \emptyset$, otherwise $\emptyset$

Disjointness table:

$F \cap I$	$\emptyset$	$\{0\}$	$\{0,1\}$
$\{1\}$	$*$	$*$	—
$\{0,1\}$	$*$	—	—

Representing objects: The canonical extension $\mathbf{2}^\delta$ is the four-element lattice $\{\bot, 0, 1, \top\}$:

- $Y(\emptyset) = \bot$ (join of empty ideal)
- $Y(\{0\}) = 0$ (join of ideal $\{0\}$)
- $Y(\{0,1\}) = \top$ (join of entire lattice)
- $X(\{1\}) = 1$ (meet of filter $\{1\}$)
- $X(\{0,1\}) = \bot$ (meet of entire lattice)

Factorization: The pairing θ factors as:

$$\theta = \rho \star \gamma \star \lambda$$

where:

- $\lambda : Q \Rightarrow \mathbf{2}^\delta(D, Y)$ maps $(F, I, *)$ to the lattice homomorphism $\mathbf{2} \to Y(I)$
- $\gamma : Q \Rightarrow \mathbf{2}^\delta(Y, X)$ is the canonical interpolant
- $\rho : Q \Rightarrow \mathbf{2}^\delta(X, E)$ maps to morphisms $X(F) \to \mathbf{2}$

Verification of density: Each element of $\mathbf{2}^\delta$ is:

- Both a join of ideals: $\bot = \bigvee \emptyset$, $0 = \bigvee\{0\}$, $1 = 0 \vee 1$, $\top = \bigvee\{0,1\}$
- And a meet of filters: $\bot = \bigwedge\{0,1\}$, $0 = 0 \wedge 1$, $1 = \bigwedge\{1\}$, $\top = \bigwedge \emptyset$

This confirms bilateral density: representing objects exist for all ideals and filters.

Uniqueness: The compactness axiom ensures uniqueness. For instance, if $\bigvee\{0\} = \bigwedge\{1\}$ in any purported canonical extension, compactness would force $\{0\} \cap \{1\} \neq \emptyset$, which is false. Therefore no spurious identifications occur, and $\mathbf{2}^\delta$ is the unique canonical envelope.

7.2 Relation to Canonical Extensions

Canonical extensions—both classical Jónsson-Tarski extensions of distributive lattices [15, 20] and their categorical generalizations in the sense of Schoots [30]—are examples of completion phenomena characterized by density and compactness conditions. The canonical-envelope framework also centers on density (existence of factorizations) and compactness (uniqueness up to isomorphism), so a comparison is natural.

Remark 7.2 (Key distinction)*.* Canonical extensions and canonical envelopes differ in a structurally important way. A canonical extension is determined by density and compactness alone: existence and essential uniqueness of solutions suffice, without requiring a universal arrow into all other solutions. A canonical envelope, by contrast, is an *initial* object of $\mathrm{Fact}(\theta)$: a factorization that maps uniquely into every other factorization. Initiality is strictly stronger than essential uniqueness.

Consequently, canonical extensions do not in general arise as canonical envelopes. Density and compactness make $\mathrm{Fact}(\theta)$ a connected groupoid; they do not produce an initial object unless the interpolation problem of Theorem 5.9 additionally has an initial solution.

Remark 7.3 (Terminology)*.* The terms *bilateral density* and *bilateral compactness* used in this paper are motivated by the analogous density and compactness axioms in canonical extension theory (Schoots's CE2 and CE3). The structural parallel is:

- Schoots CE2 (P-density) $\leftrightarrow$ bilateral denseness: existence of a triple factorization
- Schoots CE3 (compactness) $\leftrightarrow$ bilateral compactness: any two factorizations are isomorphic

The compactness condition used here is formulated categorically, at the level of isomorphism of factorizations in $\mathrm{Fact}(\theta)$. Schoots's compactness is approximation-theoretic, concerned with how elements of the completion are approximated by elements of the original structure. While distinct, the latter appears to induce the former in admissible bilateral factorization settings: when Schoots's approximation axiom holds, the resulting factorizations become isomorphic in $\mathrm{Fact}(\theta)$.

The difference in sufficiency is that Schoots's framework treats density and compactness as sufficient for existence (in its specific presheaf setting), whereas the canonical envelope framework identifies a third condition—existence of an initial interpolant—as the genuine content of existence. In the classical setting, the interpolant is furnished by the concrete construction of L^δ; in the general categorical setting this must be established separately.

For these reasons, we treat canonical extensions as related completion phenomena sharing the density/compactness structure, rather than as instances of canonical envelopes. The bilateral pairing perspective does illuminate their structure: a classical canonical extension L^δ corresponds to the ideal-filter pairing with $I = \mathrm{Filt}(L)^{\mathrm{op}}$, $J = \mathrm{Idl}(L)$, and $Q(F, I) = \{*\}$ iff $F \cap I = \emptyset$, and the density and compactness axioms translate precisely into bilateral density and compactness for this pairing. Whether the ideal-filter pairing always admits an initial interpolant (and hence a canonical envelope in our sense) is a natural question we leave open.

7.3 Ind- and Pro-Completions

Ind- and pro-completions are canonical envelopes as direct corollaries of Theorem 6.5, which established that all weighted (co)limits are canonical envelopes.

Definition 7.4 (Ind-Completion [1, 10])**.** For a small category C, the ind-completion $\mathrm{Ind}(C)$ is the free cocompletion of C under filtered colimits. Equivalently, $\mathrm{Ind}(C)$ is the full subcategory of $\mathrm{Psh}(C)$ consisting of filtered colimits of representables.

Definition 7.5 (Pro-Completion [1, 10])**.** Dually, the pro-completion $\mathrm{Pro}(C)$ is the free completion of C under cofiltered limits. Equivalently, $\mathrm{Pro}(C)$ is the full subcategory of $\mathrm{Psh}(C^{\mathrm{op}})$ consisting of cofiltered limits of representables.

Corollary 7.6 (Ind- and Pro-Completions as Envelopes)**.** *Let C be a small category. Then:*

1. *The ind-completion* $\mathrm{Ind}(C)$ *is a canonical envelope*
2. *The pro-completion* $\mathrm{Pro}(C)$ *is a canonical envelope*

Proof. Ind- and pro-completions are instances of weighted (co)limits:

Ind-completion: Objects of $\mathrm{Ind}(C)$ are filtered colimits in $\mathrm{Psh}(C)$. Filtered colimits are weighted colimits where the weight $W : I \to \mathbf{Set}$ is constant at the singleton set and the diagram category I is filtered. By Corollary 6.6, weighted colimits are canonical envelopes. Therefore ind-completions are canonical envelopes.

Pro-completion: Dually, objects of $\mathrm{Pro}(C)$ are cofiltered limits, which are weighted limits with cofiltered diagram categories. By Theorem 6.5, weighted limits are canonical envelopes. Therefore pro-completions are canonical envelopes.

The universal properties of $\mathrm{Ind}(C)$ and $\mathrm{Pro}(C)$ (functors preserving filtered colimits/cofiltered limits extend uniquely) are instances of the general universal property for weighted (co)limits [10]. □

Remark 7.7. The weighted limit perspective handles ind- and pro-completions uniformly: both are special cases of the general weighted (co)limit construction rather than requiring separate treatment. The bilateral factorization structure for ind-completions involves:

- Left envelope: embedding $C \hookrightarrow \mathrm{Ind}(C)$
- Right envelope: inclusion $\mathrm{Ind}(C) \hookrightarrow \mathrm{Psh}(C)$
- Canonical interpolant: mediating via filtered cocones

The Gabriel-Ulmer construction [1, 10] of $\mathrm{Ind}(C)$ as filtered colimits of representables is precisely the canonical envelope construction in this special case.

7.4 Cauchy Completion of Categories

The Cauchy completion (also called Karoubi envelope or idempotent completion) provides a classical example of how canonical envelopes subsume well-known categorical constructions. The characterization of Cauchy completion as an adjunction is classical [24]; what we show here is that this adjunction naturally fits the canonical envelope framework through the Avery-Leinster theory.

Definition 7.8 (Semicategory)**.** A *semicategory* is like a category but without requiring that idempotents split. Equivalently, it is a category that may lack certain coequalizers. The forgetful functor $U : \mathbf{Cat} \to \mathbf{SemiCat}$ forgets the requirement that idempotents split.

Definition 7.9 (Cauchy Completion [24])**.** The *Cauchy completion* (or *Karoubi envelope*) of a category C is the universal category $\widehat{C}$ in which all idempotents of C split. Explicitly, $\widehat{C}$ has:

- Objects: pairs (c, e) where $c \in C$ and $e : c \to c$ is idempotent ($e \circ e = e$)
- Morphisms: $(c, e) \to (d, f)$ are morphisms $h : c \to d$ in C satisfying $h = f \circ h \circ e$

- The canonical functor $\eta_C : C \to \widehat{C}$ sends $c \mapsto (c, \mathrm{id}_c)$

Theorem 7.10 (Classical: Cauchy Completion as Adjunction [24]). *The Cauchy completion functor* $\widehat{(-)} : \mathbf{SemiCat} \to \mathbf{Cat}$ *is left adjoint to the forgetful functor* $U : \mathbf{Cat} \to \mathbf{SemiCat}$*:*

$$\widehat{(-)} \dashv U$$

That is, for any semicategory S and category C:

$$\mathbf{Cat}(\widehat{S}, C) \cong \mathbf{SemiCat}(S, U(C))$$

Remark 7.11 (All Adjunctions Give Canonical Envelopes). The Cauchy completion is an instance of a completely general phenomenon: *every adjunction gives rise to a canonical envelope* through its adjunction profunctor.

For any adjunction $F \dashv G : A \to B$, the profunctor $P(a, b) := B(F(a), b) \cong A(a, G(b))$ defines a bilateral pairing, and by Theorem 7.23, this pairing has a canonical envelope. The canonical envelope structure is:

- Left envelope: the free construction (left adjoint F)
- Right envelope: the forgetful functor (right adjoint G)
- Bilateral density: universal property of the adjunction (every morphism factors)
- Bilateral compactness: uniqueness of the factorization

Thus canonical envelopes subsume all free constructions arising from adjunctions. This includes:

- Free groups, monoids, rings, modules, etc.
- Stone-Čech compactification ($\beta \dashv U : \mathbf{CompHaus} \to \mathbf{Top}$)
- Cauchy completion ($\widehat{(-)} \dashv U : \mathbf{SemiCat} \to \mathbf{Cat}$)
- Ind-completion ($\mathrm{Ind} \dashv U : \mathbf{Cocomplete} \to \mathbf{Cat}$)
- Left/right Kan extensions (via their adjoint characterizations)

These adjunctions are classical; what canonical envelope theory adds is the bilateral factorization structure underlying all of them.

Construction 7.12 (Cauchy Completion as Canonical Envelope). Applying Theorem 7.23 (Avery-Leinster adjunction envelopes) to the classical adjunction $\widehat{(-)} \dashv U$, we obtain the bilateral pairing:

- $I = \mathbf{SemiCat}$ (semicategories as left index)
- $J = \mathbf{Cat}$ (categories as right index)
- $D : I \to \mathbf{Cat}$ is the identity on $\mathbf{SemiCat}$ (viewed as categories)
- $E : J \to \mathbf{Cat}$ is the identity on $\mathbf{Cat}$
- $Q : \mathbf{SemiCat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Set}$ is the adjunction profunctor $Q(S, C) := \mathbf{Cat}(\widehat{S}, C)$
- $\theta : Q \Rightarrow \mathbf{Cat}(D, E)$ is the natural transformation given by the adjunction isomorphism

The canonical envelope of this pairing is the Cauchy completion.

Theorem 7.13 (Cauchy Completion Fits Canonical Envelope Framework)**.** *The classical Cauchy completion arises as the canonical envelope of the adjunction profunctor (Construction 7.12). The bilateral envelope structure is:*

1. ***Bilateral Density:*** *Every functor $S \to C$ factors through $\widehat{S}$ by the universal property of the adjunction*
2. ***Bilateral Compactness:*** *Any two such factorizations are isomorphic*
3. ***Universality:*** *The embedding $\eta_S : S \to \widehat{S}$ is initial among functors that split all idempotents*
4. ***Left Envelope:*** *$Y(S) = \widehat{S}$ (the free idempotent-complete category)*
5. ***Right Envelope:*** *$X(C) = C$ (categories are already idempotent-complete)*
6. ***Interpolant:*** *The canonical factorization mediates via the unit-counit of the adjunction*

Proof. This follows immediately from Theorem 7.23: adjunctions give canonical envelopes. The bilateral structure arises from the adjunction:

- The left envelope $\widehat{S}$ is the free construction (left adjoint)
- The right envelope is $U(C) = C$ (forgetful functor, right adjoint)
- The interpolant γ is given by the unit-counit of the adjunction

Bilateral density and compactness are precisely the universal property and uniqueness of the adjunction. ☐

Remark 7.14 (Idempotency). The Cauchy completion is idempotent: $\widehat{\widehat{C}} \simeq \widehat{C}$. This follows immediately from the fact that the adjunction $\widehat{(-)} \dashv U$ gives rise to an idempotent monad $U \circ \widehat{(-)}$ on **SemiCat**. This is a special case of Theorem 8.7: the canonical envelope monad is idempotent. Categories that are already Cauchy-complete (i.e., in which all idempotents split) satisfy $C \simeq \widehat{C}$.

Remark 7.15 (Connection to Absolute Colimits). There is an alternative characterization of Cauchy completion through absolute colimits [1]: $\widehat{C}$ is the smallest full subcategory of $\mathrm{Psh}(C)$ containing C (via Yoneda) and closed under absolute colimits.

Absolute colimits are precisely the colimits that are preserved by all functors. The key examples are:

- Split coequalizers (which are exactly the colimits that split idempotents)
- Retracts of representables

The Cauchy completion $\widehat{C}$ is the free category with split idempotents, which is equivalent to being the free category with split coequalizers. Thus both characterizations—via the adjunction $\widehat{(-)} \dashv U$ and via closure under absolute colimits—describe the same classical completion. The adjunction approach makes the canonical envelope structure more transparent, while the absolute colimits approach connects to presheaf constructions and weighted (co)limit theory.

Remark 7.16 (Enriched Cauchy Completion). For categories enriched over a monoidal closed category V (such as Lawvere metric spaces enriched over $([0, \infty], \geq, +, 0)$), there is an analogous V-enriched Cauchy completion [23]. The adjunction approach extends naturally:

$$\widehat{(-)} \dashv U : V\text{-}\mathbf{SemiCat} \to V\text{-}\mathbf{Cat}$$

For Lawvere metric spaces specifically:

- The Cauchy completion $\widehat{X}$ adds limits of Cauchy sequences
- The adjunction structure gives the bilateral pairing between sequences approaching from different directions
- Left modules $\ell : X \to V$ and right modules $r : X^{\mathrm{op}} \to V$ arise from the profunctor structure
- The infimal convolution measures agreement through the canonical interpolant

The V-enriched theory requires extending the canonical envelope framework beyond **Set**-enrichment, but the organizational principle remains: Cauchy completion arises from the adjunction profunctor, with the bilateral envelope mediating between incomplete and complete structures.

Remark 7.17 (Relation to Other Completions). The Cauchy completion is related to other completion constructions:

- **Ind-completion**: Adds filtered colimits; contains Cauchy completion as a subcategory
- **Presheaf completion**: Adds all colimits; $\mathrm{Psh}(C) = \widehat{\mathrm{Ind}(C)}$
- **Free cocompletion**: The ind-completion $\mathrm{Ind}(C)$ is the free cocompletion under filtered colimits, while $\widehat{C}$ is the free completion under split idempotents (equivalently, split coequalizers)

Each of these can be characterized as a canonical envelope through appropriate adjunctions or weighted (co)limit structures.

7.5 Pratt's Communes

We establish equivalence between communes and canonical envelopes for identity pairings in the stated setting.

Definition 7.18 (Commune [25]). A commune on a profunctor $P : A^{\mathrm{op}} \times B \to \mathbf{Set}$ consists of functors $A_0 : A \to \mathbf{Set}$ and $X_0 : B \to \mathbf{Set}$ with a natural transformation

$$\rho^\sharp : A_0 \boxtimes X_0 \Rightarrow P$$

where $(A_0 \boxtimes X_0)(a, b) := A_0(a) \times X_0(b)$, satisfying:

1. **Didensity:** Every element of $P(a, b)$ factors through $A_0(a) \times X_0(b)$
2. **Extensionality:** The pairing $\rho^\sharp$ separates points

Construction 7.19 (Identity Pairing for Communes). For a profunctor $P : A^{\mathrm{op}} \times B \to \mathbf{Set}$, define:

- $I = A^{\mathrm{op}}$, $J = B$
- $D = \mathrm{id}_{A^{\mathrm{op}}}$, $E = \mathrm{id}_B$

- $Q = P$
- $\theta = \mathrm{id}_P$

Theorem 7.20 (Communes as Canonical Envelopes)**.** *The category* $\mathrm{Fact}(\theta)$ *of factorizations of* $\theta = \mathrm{id}_P$ *is equivalent to the category of communes on* P*. The canonical envelope (when it exists) corresponds to the universal commune* $\mathrm{Com}(P)$*. Moreover:*

1. *Pratt's didensity is equivalent to bilateral denseness*
2. *Pratt's extensionality is equivalent to bilateral compactness*

Proof. **Factorizations are communes.** A factorization (λ, γ, ρ) of id_P provides natural transformations $\lambda : P \Rightarrow C(\mathrm{id}, Y)$ and $\rho : P \Rightarrow C(X, \mathrm{id})$ for functors $Y : B \to C$ and $X : A^{\mathrm{op}} \to C$. By Yoneda, these determine functors $A_0 : A \to \mathbf{Set}$ and $X_0 : B \to \mathbf{Set}$ with $\rho^\sharp : A_0 \boxtimes X_0 \Rightarrow P$ given by composition through γ. This is a commune.

Communes are factorizations. Conversely, a commune $(A_0, X_0, \rho^\sharp)$ determines a factorization by taking $Y(b) = X_0(b)$, $X(a) = A_0(a)$, and defining λ, γ, ρ via the universal properties of A_0 and X_0 and the evaluation $\rho^\sharp$.

Morphisms correspond. A morphism of communes (α, β) satisfying $\rho'^\sharp \circ (\alpha \boxtimes \beta) = \rho^\sharp$ is precisely a morphism in $\mathrm{Fact}(\theta)$.

Didensity = bilateral denseness. Didensity states every $p \in P(a, b)$ factors through $A_0(a) \times X_0(b)$, which is exactly the condition $\theta = \rho \star \gamma \star \lambda$ (bilateral denseness).

Extensionality = bilateral compactness. Extensionality states $\rho^\sharp$ separates points, meaning factorizations are essentially unique, which is bilateral compactness.

Initial objects correspond. The universal commune $\mathrm{Com}(P)$ is the initial commune, corresponding to the canonical envelope as the initial factorization. ☐

Remark 7.21. Commune theory and canonical envelope theory thus agree for identity pairings in the stated setting. Pratt's didensity and extensionality axioms give the characterization conditions; the bilateral factorization perspective provides the categorical language.

7.6 Avery-Leinster Adjunction Envelopes

Avery and Leinster introduced a general notion of adjunction envelope that encompasses many classical constructions [2]. We show these are canonical envelopes as a direct consequence of Theorem 7.20, establishing a clean inclusion hierarchy.

Definition 7.22 (Adjunction Envelope [2])**.** Let $P : A^{\mathrm{op}} \times B \to \mathbf{Set}$ be a profunctor. An adjunction envelope of P consists of categories $\hat{A}$ and $\hat{B}$ with an adjunction

$$\hat{A} \rightleftarrows \hat{B}$$

and functors $A \to \hat{A}$, $B \to \hat{B}$ that are universal among adjunctions factoring P.

Theorem 7.23 (Avery-Leinster Envelopes are Canonical Envelopes)**.** *Every Avery-Leinster adjunction envelope is a canonical envelope. More precisely:*

1. *Avery and Leinster prove that adjunction envelopes satisfy Pratt's commune axioms (didensity and extensionality) [2, Proposition 3.2]*
2. *By Theorem 7.20, communes on a profunctor* P *are exactly canonical envelopes of the identity pairing* $\theta = \mathrm{id}_P$

3. *Therefore adjunction envelopes are canonical envelopes*

Proof. This is a direct application of the transitive inclusion:

$$\text{Adjunction envelopes} \subseteq \text{Communes} = \text{Canonical envelopes (for identity pairings)}$$

The first inclusion is established by Avery-Leinster's Proposition 3.2, which verifies that every adjunction envelope satisfies Pratt's didensity and extensionality axioms. The equality is Theorem 7.20, which establishes the equivalence between communes and canonical envelopes for identity pairings.

Since adjunction envelopes are communes, and communes are canonical envelopes, adjunction envelopes are canonical envelopes. □

Remark 7.24. The Avery-Leinster framework is thus a special case. Canonical envelopes handle pairings $\theta : Q \Rightarrow C(D, E)$ that are not identity maps, covering weighted limits, Kan extensions, and the other examples in Section 7. For identity pairings, communes, adjunction envelopes, and canonical envelopes all describe the same structure from different starting points.

7.7 Isbell Envelopes

The Isbell envelope is one of the most important classical constructions in category theory. We show it arises as a canonical envelope through the logical chain established in the previous sections.

Definition 7.25 (Isbell Envelope [17, 18])**.** For a small category C, the Isbell envelope consists of the adjunction between presheaves $\mathrm{Psh}(C) = [C^{\mathrm{op}}, \mathbf{Set}]$ and copresheaves $\mathrm{Psh}(C^{\mathrm{op}}) = [C, \mathbf{Set}]$ induced by the hom-profunctor $C(-, -) : C^{\mathrm{op}} \times C \to \mathbf{Set}$.

Explicitly, the Isbell adjunction is:

$$\mathrm{Psh}(C^{\mathrm{op}}) \rightleftarrows \mathrm{Psh}(C)$$

where the left adjoint sends a copresheaf $F : C \to \mathbf{Set}$ to the presheaf $c \mapsto \mathrm{Nat}(C(c, -), F)$, and the right adjoint is dual.

Theorem 7.26 (Isbell Envelope as Canonical Envelope)**.** *The Isbell envelope is a canonical envelope. The logical derivation proceeds through three established inclusions:*

$$\textit{Isbell} \subseteq \textit{Avery-Leinster} \subseteq \textit{Communes} = \textit{Canonical envelopes}$$

Proof. We establish each step of the inclusion chain:

Step 1: Isbell envelopes are Avery-Leinster adjunction envelopes. The Isbell envelope is defined as the adjunction envelope of the hom-profunctor $C(-, -) : C^{\mathrm{op}} \times C \to \mathbf{Set}$. This is precisely an instance of Avery-Leinster's adjunction envelope construction (Definition 7.22).

Step 2: Avery-Leinster envelopes are communes. By Avery-Leinster's Proposition 3.2 [2], adjunction envelopes satisfy Pratt's didensity and extensionality axioms, making them communes.

Step 3: Communes are canonical envelopes. By Theorem 7.20, communes on a profunctor P are exactly canonical envelopes of the identity pairing $\theta = \mathrm{id}_P$.

Combining these three steps: Isbell envelopes are Avery-Leinster envelopes (Step 1), Avery-Leinster envelopes are communes (Step 2), and communes are canonical envelopes (Step 3). Therefore Isbell envelopes are canonical envelopes.

Alternative direct route. It is classically known that Isbell envelopes satisfy Pratt's commune axioms, so Isbell envelopes are communes [25]. By Theorem 7.20, communes are canonical envelopes. Therefore Isbell envelopes are canonical envelopes.

Under the canonical envelope identification, the components correspond as follows:

- The left envelope λ is the left adjoint of the Isbell adjunction
- The right envelope ρ is the right adjoint
- The canonical interpolant γ mediates via the unit and counit of the adjunction

$\square$

Remark 7.27. This theorem shows that Isbell's classical construction [17, 18] is a special case of canonical envelopes. Recent work by Garner [14] has explored the monad structure on Isbell envelopes. The canonical envelope framework suggests this monad structure may extend to more general bilateral pairings, though we do not pursue this here.

Corollary 7.28 (Hom-Profunctor Envelope)**.** *For any small category C, the canonical envelope of the identity pairing on the hom-profunctor $\theta = \mathrm{id}_{C(-,-)}$ is the Isbell envelope. This follows from the logical chain:*

- *The hom-profunctor $C(-,-) : C^{\mathrm{op}} \times C \to \mathbf{Set}$ determines an identity pairing*
- *By Theorem 7.20, the canonical envelope of this identity pairing is the universal commune* $\mathrm{Com}(C(-,-))$
- *By Theorem 7.23, this commune is an Avery-Leinster adjunction envelope*
- *By classical results, this adjunction envelope is precisely the Isbell envelope*

Thus the Isbell envelope exemplifies all three frameworks: communes, adjunction envelopes, and canonical envelopes.

7.8 Topological Completions

The canonical envelope framework extends naturally to topological completions. We present two fundamental examples: Stone-Čech compactification and sobrification. These results appear in the author's companion paper [26]; we summarize the main theorems here.

7.8.1 Stone-Čech Compactification

The Stone-Čech compactification is the universal compactification of completely regular spaces. We show it arises as a canonical envelope with a bilateral structure involving filters and ultrafilters.

Theorem 7.29 (Stone-Čech Compactification as Canonical Envelope [26])**.** *Let X be a completely regular T_1 space. The Stone-Čech compactification βX arises as the canonical envelope of the bilateral pairing* $(\mathrm{Filt}(X), \mathrm{UF}(X), D, E, Q, \theta)$ *where:*

- $\mathrm{Filt}(X)$ *= category of proper filters on X with filter inclusions*
- $\mathrm{UF}(X)$ *= category of ultrafilters on X with inclusions*
- $D : \mathrm{Filt}(X) \to \mathbf{Top}$ *given by $D(F) = X$ for all filters F*
- $E : \mathrm{UF}(X) \to \mathbf{Top}$ *given by $E(U) = \{*\}$ (one-point space)*
- $Q : \mathrm{Filt}(X)^{\mathrm{op}} \times \mathrm{UF}(X) \to \mathbf{Set}$ *given by $Q(F, U) = \{*\}$ if $F \subseteq U$, $\emptyset$ otherwise*
- $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ *given by $\theta_{F,U}(*) = !_X : X \to \{*\}$ when $F \subseteq U$*

The canonical envelope yields βX with the Stone-Čech universal property: every continuous map $f : X \to K$ to a compact Hausdorff space K extends uniquely to $\bar{f} : \beta X \to K$.

Proof (Sketch). The proof appears in [26]. The key steps are:

Bilateral structure. The pairing captures the fundamental relationship between filters and ultrafilters in compactification:

- Filters F correspond to convergent nets in βX
- Ultrafilters U correspond to actual points in βX
- The condition $F \subseteq U$ captures the convergence relationship

Topological realization. The bilateral completion is realized topologically via the evaluation map $\mathrm{ev} : X \to \prod_{f \in C_b(X)} \mathbb{R}$ where $C_b(X)$ is the space of bounded continuous real-valued functions. The Stone-Čech compactification βX is the closure of the image.

Bilateral factorization. The factorization $\theta = \rho \star \gamma \star \lambda$ is realized as:

- $\lambda_{F,U} : Q(F,U) \to \mathbf{Top}(X, \beta X)$ via the Stone-Čech embedding $i : X \hookrightarrow \beta X$
- $\gamma_{F,U} : Q(F,U) \to \mathbf{Top}(\beta X, \beta X)$ via the identity $\mathrm{id}_{\beta X}$
- $\rho_{F,U} : Q(F,U) \to \mathbf{Top}(\beta X, \{*\})$ via the unique map $! : \beta X \to \{*\}$

Universal property. The universal property of Stone-Čech compactification [35] coincides with the universal property of canonical envelopes: any other compact Hausdorff completion must factor through βX uniquely.

Completeness. The pairing is complete (X is its own completion) if and only if X is already compact Hausdorff. $\square$

Remark 7.30. This result reveals the bilateral structure underlying Stone-Čech compactification: compactification mediates between filters (probing from the "inside") and ultrafilters (representing limit points). The canonical interpolant γ captures how filters converge to ultrafilter points in the compactification.

7.8.2 Sobrification

Sobrification is the completion operation for topological spaces that adds "generic points" for irreducible closed sets. It is the left adjoint to the forgetful functor from sober spaces to T_0 spaces.

Theorem 7.31 (Sobrification as Canonical Envelope [26])**.** *Let X be a T_0 space. Its sobrification* $\mathrm{Sob}(X)$ *arises as the canonical envelope of the bilateral pairing* $(\mathrm{Irr}(X), X_{\mathrm{disc}}, D, E, Q, \theta)$ *where:*

- $\mathrm{Irr}(X)$ = *category of irreducible closed subsets of X with inclusions*
- X_{disc} = *discrete category on the points of X*
- $D : \mathrm{Irr}(X) \to \mathbf{Top}$ *given by $D(Z) = Z$ with subspace topology*
- $E : X_{\mathrm{disc}} \to \mathbf{Top}$ *given by $E(x) = \{x\}$ with discrete topology*
- $Q : \mathrm{Irr}(X)^{\mathrm{op}} \times X_{\mathrm{disc}} \to \mathbf{Set}$ *given by $Q(Z, x) = \{*\}$ if $x \in Z$, $\emptyset$ otherwise*
- $\theta : Q \Rightarrow \mathbf{Top}(D, E)$ *given by $\theta_{Z,x}(*) = \iota_x : Z \to \{x\}$ when $x \in Z$*

The canonical envelope yields $\mathrm{Sob}(X)$ *with the sobrification universal property: every continuous map* $f : X \to Y$ *to a sober space* Y *extends uniquely to* $\tilde{f} : \mathrm{Sob}(X) \to Y$.

Proof (Sketch). The proof appears in [26]. The construction proceeds as follows:

Bilateral weighted (co)limits. For each point $x \in X$, the $Q(-, x)$-weighted colimit is:

$$Y(x) = \mathrm{colim}_{Q(-,x)} D = \mathrm{colim}_{\{Z : x \in Z\}} Z$$

This is the directed union of all irreducible closed sets containing x, which equals $\overline{\{x\}}$ (the closure of x).

For each irreducible closed set Z, the $Q(Z, -)$-weighted limit corresponds to the "generic point" of Z in the sobrification.

Sobrification construction. The sobrification $\mathrm{Sob}(X)$ adds generic points for irreducible closed sets lacking them. This is precisely the weighted completion structure where:

- Original points $x \in X$ correspond to objects $Y(x) = \overline{\{x\}}$
- Generic points correspond to objects Z^{sob} for irreducible closed sets Z
- The topology makes irreducible closed sets correspond bijectively to points

Completeness. The pairing is complete if and only if X is already sober—that is, every irreducible closed set has a generic point. □

Remark 7.32. Sobrification exhibits a different bilateral structure than Stone-Čech compactification: it mediates between irreducible closed sets (probing from the "space side") and points (the discrete structure). The completion adds precisely the missing generic points needed to make the space sober.

Remark 7.33 (Relationship to Stone Duality). Both Stone-Čech compactification and canonical extensions of distributive lattices are related through Stone duality. For a Boolean algebra B, the Stone-Čech compactification of its Stone space recovers the canonical extension of B via duality. The bilateral envelope perspective unifies these constructions: both arise from mediating between dual probing structures (filters/ideals for lattices, filters/ultrafilters for spaces).

7.9 Cylinder Factorization: Geometric Interpretation

The bilateral factorization $\theta = \rho \star \gamma \star \lambda$ admits a geometric interpretation through **cylinder factorization systems**. This perspective reveals how individual morphisms decompose through the completion structure, complementing the algebraic canonical envelope construction.

Definition 7.34 (Cylinder Classes from Bilateral Factorization)**.** Let $\theta : Q \Rightarrow C(D, E)$ be a bilateral pairing with canonical envelope $(\hat{C}, \varepsilon_C)$ and factorization $\varepsilon_C^* \theta = \rho \star \gamma \star \lambda$.

Define the **cylinder classes**:

- **Left cylinder class:**

$$\mathcal{L}_Q = \{\lambda(i, j, q) : \varepsilon_C D(i) \to Y(j) \mid q \in Q(i, j)\}$$

- **Right cylinder class:**

$$\mathcal{R}_Q = \{\rho(i, j, q) : Z(i) \to \varepsilon_C E(j) \mid q \in Q(i, j)\}$$

- **Bilateral interpolant class:**

$$\mathcal{B}_Q = \{\gamma(i,j,q) : Y(j) \to Z(i) \mid q \in Q(i,j)\}$$

where for each $q \in Q(i,j)$:

- $\lambda(i,j,q) : Q(i,j) \to \hat{C}(\varepsilon_C D(i), Y(j))$ evaluates to give λ_q
- $\gamma(i,j,q) : Q(i,j) \to \hat{C}(Y(j), Z(i))$ evaluates to give γ_q
- $\rho(i,j,q) : Q(i,j) \to \hat{C}(Z(i), \varepsilon_C E(j))$ evaluates to give ρ_q

Theorem 7.35 (Cylinder Factorization Property)**.** *Every morphism arising from the bilateral pairing factors uniquely through the cylinder diagram:*

$$\begin{tikzcd}
\varepsilon_C D(i) \arrow[r, "(\varepsilon_C^*\theta)_q"] \arrow[d, "\lambda_q"'] & \varepsilon_C E(j) \\
Y(j) \arrow[r, "\gamma_q"'] & Z(i) \arrow[u, "\rho_q"']
\end{tikzcd}$$

where the factorization satisfies:

$$(\varepsilon_C^*\theta)_q = \rho_q \circ \gamma_q \circ \lambda_q$$

Proof. This follows directly from the bilateral factorization $\varepsilon_C^*\theta = \rho \star \gamma \star \lambda$. For each (i,j) and $q \in Q(i,j)$:

Step 1: The natural transformation $\lambda : Q \Rightarrow \hat{C}(\varepsilon_C D, Y)$ provides, at component (i,j):

$$\lambda_{i,j} : Q(i,j) \to \hat{C}(\varepsilon_C D(i), Y(j))$$

Evaluating at q gives the morphism $\lambda_q : \varepsilon_C D(i) \to Y(j)$.

Step 2: Similarly, γ and ρ provide:

$$\gamma_q : Y(j) \to Z(i)$$
$$\rho_q : Z(i) \to \varepsilon_C E(j)$$

Step 3: The factorization equation $\theta = \rho\star\gamma\star\lambda$ means that at each component, the compositions match:

$$\theta_{i,j}(q) = (\rho \star \gamma \star \lambda)_{i,j}(q) = \rho_q \circ \gamma_q \circ \lambda_q$$

Step 4: Uniqueness follows from the initiality of the canonical envelope factorization. □

Example 7.36 (Cylinder Factorization for Canonical Extensions)**.** For the canonical extension of a distributive lattice L:

Left cylinder class $\mathcal{L}_Q$: Morphisms from elements of L to principal filters:

$$\lambda_a : a \mapsto \uparrow a \quad \text{(principal filter generation)}$$

Bilateral interpolant class $\mathcal{B}_Q$: Morphisms from filters to ideals when they meet:

$$\gamma_{F,I} : F \to I \quad \text{when } F \cap I \neq \emptyset$$

Right cylinder class $\mathcal{R}_Q$: Morphisms from ideals to principal ideals:

$$\rho_a : I \mapsto \downarrow a \quad \text{(principal ideal generation)}$$

The cylinder diagram shows how an element $a \in L$ factors through the filter-ideal structure:

$$a \xrightarrow{\lambda} \uparrow a \xrightarrow{\gamma} \downarrow a \xrightarrow{\rho} a$$

Example 7.37 (Cylinder Factorization for Stone-Čech Compactification)**.** For Stone-Čech compactification of a space X:

Left cylinder class: Morphisms from points to ultrafilter convergence:

$$\lambda_x : x \mapsto \text{principal ultrafilter at } x$$

Bilateral interpolant class: Morphisms from ultrafilters to convergence points:

$$\gamma_{\mathcal{U}} : \mathcal{U} \mapsto \lim \mathcal{U} \in \beta X$$

Right cylinder class: Morphisms from convergence structure back to points:

$$\rho_x : \beta X \to X \quad \text{(evaluation at basepoint)}$$

The cylinder diagram shows how a point $x \in X$ factors through the ultrafilter structure in βX.

Remark 7.38 (Orthogonality Properties)*.* The cylinder classes $(\mathcal{L}_Q, \mathcal{R}_Q)$ satisfy orthogonality-like properties inherited from the canonical envelope structure:

Unique lifting: For any $\ell \in \mathcal{L}_Q$ and $r \in \mathcal{R}_Q$, certain square diagrams admit unique diagonal fillers through the bilateral interpolant structure.

Closure properties: The classes $\mathcal{L}_Q$ and $\mathcal{R}_Q$ are closed under composition with isomorphisms and satisfy appropriate closure properties inherited from the weighted colimit/limit structure.

These properties echo the orthogonality conditions in weak factorization systems, though the bilateral context provides additional structure through the interpolant class $\mathcal{B}_Q$.

7.9.1 Connection to Garner's Cylinder Systems

Proposition 7.39 (Cylinder Factorization Systems as Canonical Envelopes)**.** *Every cylinder factorization system for a category $\mathcal{C}$ is the canonical envelope of the identity bilateral pairing. Specifically, a cylinder factorization system (L, R) on $\mathcal{C}$ corresponds exactly to the canonical envelope of the bilateral pairing:*

$$\begin{aligned}
I &= \mathcal{C} \quad \textit{(objects of } \mathcal{C}\textit{)} \\
J &= \mathcal{C} \quad \textit{(objects of } \mathcal{C}\textit{)} \\
D &= id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \\
E &= id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \\
Q &= \mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set} \quad \textit{(hom-functor)} \\
\theta &: \mathcal{C}(-,-) \Rightarrow \mathcal{C}(id_{\mathcal{C}}(-), id_{\mathcal{C}}(-)) \quad \textit{(identity)}
\end{aligned}$$

The canonical envelope factorization $\theta = \rho \star \gamma \star \lambda$ yields:

- ***Left class** L: Morphisms in the image of λ*
- ***Right class** R: Morphisms in the image of ρ*
- ***Cylinder objects:** The intermediate objects produced by the canonical interpolant γ*
- ***Factorization:** Every morphism $f : A \to B$ factors as $f = r \circ l$ where $l \in L$, $r \in R$, via the tri-partite factorization*
- ***Uniqueness:** Initiality of the factorization ensures essential uniqueness*

Conversely, given a cylinder factorization system (L, R) on $\mathcal{C}$, the factorization of morphisms through cylinder objects determines a bilateral pairing whose canonical envelope recovers (L, R).

Proof. The correspondence follows from the universal nature of both constructions:

(1) From canonical envelope to cylinder system: Given the canonical envelope of the identity pairing above, the factorization $\theta = \rho \star \gamma \star \lambda$ provides:

- For each morphism $f : A \to B$ in $\mathcal{C}$, we have $f \in Q(A, B) = \mathcal{C}(A, B)$
- The factorization produces intermediate objects X_f and Y_f with morphisms:

$$A \xrightarrow{\lambda_f} Y_f \xrightarrow{\gamma_f} X_f \xrightarrow{\rho_f} B$$

- Define $L = \{\lambda_f : f \in \mathcal{C}\}$ and $R = \{\rho_f : f \in \mathcal{C}\}$
- Initiality ensures that any two factorizations of f through the same intermediate objects are uniquely isomorphic
- This gives the essential uniqueness required for a cylinder factorization system

(2) From cylinder system to canonical envelope: Given a cylinder factorization system (L, R) on $\mathcal{C}$:

- Each morphism $f : A \to B$ factors as $f = r \circ l$ with $l \in L$, $r \in R$, through a cylinder object C_f
- This factorization determines natural transformations λ, γ, ρ forming a triple factorization of the identity pairing
- Essential uniqueness of the cylinder factorization translates to initiality in the factorization category
- Therefore the cylinder system determines the canonical envelope

The key observation is that both frameworks express the same universal property: the unique factorization of morphisms through a mediating structure. Cylinder systems express this geometrically (via lifting/orthogonality), while canonical envelopes express it algebraically (via initiality), but they characterize identical mathematical structures. □

Remark 7.40 (Relationship to Garner's Framework). Proposition 7.39 establishes that Richard Garner's cylinder factorization systems [11] are completely and simply characterized as canonical envelopes of identity bilateral pairings.

The translation is immediate:

- **Garner's left class** L = morphisms from left completion (via λ)
- **Garner's right class** R = morphisms to right completion (via ρ)
- **Garner's cylinder object** = canonical interpolant (via γ)
- **Garner's orthogonality** = bilateral density/compactness
- **Garner's uniqueness** = initiality in $\mathrm{Fact}(\theta)$

This makes explicit how Garner's geometric framework and the bilateral factorization approach describe the same categorical phenomena from complementary perspectives—geometric versus algebraic. Both are characterized by the same universal factorization property.

7.9.2 Geometric vs. Algebraic Perspectives

Remark 7.41 (Complementary Viewpoints). The cylinder factorization interpretation and the canonical envelope construction provide complementary perspectives on completion:

Canonical envelope (algebraic):

- Focuses on categorical completeness and universal properties
- Emphasizes initiality in factorization categories
- Provides existence and uniqueness results
- Natural for studying composition and monadic structure

Cylinder factorization (geometric):

- Focuses on how individual morphisms decompose
- Emphasizes orthogonality and lifting properties
- Provides geometric intuition through diagrams
- Natural for studying factorization systems and weak factorization systems

Both perspectives reveal important aspects of bilateral completion structure. The algebraic view via canonical envelopes captures the universal nature of completion, while the geometric view via cylinders shows how specific morphisms factor through the completion.

Example 7.42 (Both Perspectives for Ind-Completion)**.** For ind-completion of a category $\mathcal{C}$:

Algebraic perspective: $\mathrm{Ind}(\mathcal{C})$ is the canonical envelope of the bilateral pairing determined by filtered diagrams and cocone morphisms. Initiality ensures universal property.

Geometric perspective: Each morphism in $\mathrm{Ind}(\mathcal{C})$ factors uniquely through the cylinder diagram:

$$\operatorname{colim} F_i \xrightarrow{\lambda} Y(\operatorname{colim} F_i) \xrightarrow{\gamma} Z(C) \xrightarrow{\rho} C$$

where the cylinder reveals how filtered colimits mediate between objects.

Both perspectives illuminate the same completion phenomenon from different angles.

7.10 Dicategories as Bilateral Completions

Dicategories and dagger categories are mutually presentable: every dagger category admits a canonical dicategory presentation and vice versa. The dicategory presentation makes both categorical and cocategorical composition primitive and symmetric. This is an alternative axiomatization, not a generalization, and it arises naturally from the canonical envelope construction. The bilateral presentation makes explicit the symmetric structure that is implicit in the standard dagger axioms.

7.10.1 Foundations of Dicategorical Structure

We begin by recalling the definitions of category and cocategory objects following Clingman [7].

Definition 7.43 (Category Object)**.** Let $\mathcal{C}$ be a category with finite limits. A *category object* in $\mathcal{C}$ consists of objects C_0 (objects), C_1 (morphisms), C_2 (composable pairs), along with structure morphisms:

- $d, c : C_1 \to C_0$ (domain and codomain)
- $s_0, s_1 : C_0 \to C_1$ (source and target inclusions)
- $\mathrm{id} : C_0 \to C_1$ (identity assignment)
- $\mathrm{comp} : C_2 \to C_1$ (composition)

with pullback squares defining C_2 and axioms ensuring associativity and unitality.

Definition 7.44 (Cocategory Object [7])**.** Let $\mathcal{C}$ be a category with finite colimits. A *cocategory object* in $\mathcal{C}$ consists of objects C_0 (points), C_1 (arrows), C_2 (composable pairs), along with structure morphisms:

- $\bar{o}, \bar{t} : C_0 \to C_1$ (source and target)
- $i_0, i_1 : C_1 \to C_2$ (left and right inclusions)
- $\mathrm{pt} : C_1 \to C_0$ (point)
- $\mathrm{cc} : C_1 \to C_2$ (cocomposition)

with pushout squares and coassociativity/counitality axioms.

Definition 7.45 (Dicategory)**.** A *dicategory* $\mathcal{D}$ is a mathematical structure that simultaneously carries both category and cocategory structures, with these structures being compatible through a duality operation. Specifically, $\mathcal{D}$ consists of:

Data:

- A class of objects $\mathrm{Ob}(\mathcal{D})$
- For each pair (A, B) of objects:
  - A set $\mathrm{Hom}(A, B)$ of morphisms (written $f : A \to B$)
  - A set $\mathrm{CoHom}(A, B)$ of co-morphisms (written $h : A \rightsquigarrow B$)
- Categorical composition $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$
- Cocategorical composition $\odot : \mathrm{CoHom}(A, B) \times \mathrm{CoHom}(B, C) \to \mathrm{CoHom}(A, C)$
- Identity morphisms $\mathrm{id}_A \in \mathrm{Hom}(A, A)$
- Co-identities $\mathrm{coid}_A \in \mathrm{CoHom}(A, A)$
- Duality operation $\dagger : \mathrm{Hom}(A, B) \to \mathrm{CoHom}(B, A)$

Axioms:

1. $(\mathcal{D}, \circ, \mathrm{id})$ forms a category
2. Involution: $(f^\dagger)^\dagger = f$
3. Contravariance for $\circ$: $(g \circ f)^\dagger = f^\dagger \odot g^\dagger$
4. Contravariance for $\odot$: $(j \odot h)^\dagger = h^\dagger \circ j^\dagger$
5. Identity compatibility: $(\mathrm{id}_A)^\dagger = \mathrm{coid}_A$

Remark 7.46 (Bilateral Presentation). A dicategory presents a dagger category in bilateral form: morphisms and comorphisms are given as primitive data with an involutive contravariant isomorphism † mediating between them. As Theorem 7.51 establishes, every dagger category admits a canonical dicategory presentation and every dicategory determines a dagger category, with the constructions mutually inverse up to canonical isomorphism. The dicategory presentation makes explicit what is implicit in the standard dagger category axiomatization: the symmetric bilateral relationship between categorical and cocategorical composition.

Remark 7.47 (Derivable Structure). The cocategory axioms (associativity of $\odot$, co-identity laws) are derivable from the category axioms plus the duality axioms. Specifically:

- Associativity of $\odot$ follows from associativity of $\circ$ via contravariance and involution
- Co-identity laws follow from identity laws via compatibility
- $(\mathrm{coid}_A)^\dagger = \mathrm{id}_A$ follows from identity compatibility and involution

Thus the minimal complete axiomatization consists of: category axioms, involution, contravariance (both directions), and identity compatibility.

7.10.2 Dicategories as Canonical Envelopes

We now provide a universal characterization of dicategories as canonical envelopes.

Definition 7.48 (The Dicategory Bilateral Pairing)**.** Define the bilateral pairing for dicategories as follows:

Let **BiCat** be the category of "pre-categorical structures" with:

- Objects: Triples (C_0, C_1, σ) where C_0 is a set of "objects", C_1 is a set of "arrows", and $\sigma : C_1 \to C_0 \times C_0$ is a "source-target" map
- Morphisms: Pairs (F_0, F_1) preserving the source-target structure

Define embeddings:

- $D : \mathbf{Cat} \to \mathbf{BiCat}$ by $D(C) = (\mathrm{Ob}(C), \mathrm{Mor}(C), (\mathrm{dom}, \mathrm{cod}))$ (forgets composition)
- $E : \mathbf{CoCat} \to \mathbf{BiCat}$ by $E(C^{\mathrm{co}}) = (\mathrm{Ob}(C^{\mathrm{co}}), \mathrm{CoMor}(C^{\mathrm{co}}), (\mathrm{source}, \mathrm{target}))$ (forgets cocomposition)

Define the profunctor $Q : \mathbf{Cat}^{\mathrm{op}} \times \mathbf{CoCat} \to \mathbf{Set}$ by:

$$Q(C, D^{\mathrm{co}}) = \{\text{potential duality structures } (\dagger, \mathrm{coid}) \mid \mathrm{Ob}(C) = \mathrm{Ob}(D^{\mathrm{co}}),\ \text{compatibility holds}\}$$

where a potential duality structure consists of:

- A function $\dagger : \mathrm{Mor}(C) \to \mathrm{CoMor}(D^{\mathrm{co}})$
- A function $\mathrm{coid} : \mathrm{Ob}(C) \to \mathrm{CoMor}(D^{\mathrm{co}})$ assigning co-identities
- For $f : A \to B$ in C, we have $\dagger(f) : B \rightsquigarrow A$ in D^{co}
- $\dagger(\mathrm{id}_A) = \mathrm{coid}_A$

Define the pairing $\theta : Q \Rightarrow \mathbf{BiCat}(D, E)$ by sending each potential duality structure to the corresponding **BiCat**-morphism.

Theorem 7.49 (Bilateral Presentation as Canonical Envelope)**.** *The canonical envelope of the bilateral pairing* $\theta : Q \Rightarrow \mathbf{BiCat}(D, E)$ *(Definition 7.48) exists and is the bilateral dicategory presentation of dagger categories. This gives a universal characterization of the dicategory structure within the CE framework. Moreover:*

1. ***Bilateral Density:*** *Every potential duality structure* $(C, D^{\mathrm{co}}, \dagger)$ *factors through the bilateral presentation via the triple factorization* $\theta = \rho \star \gamma \star \lambda$
2. ***Bilateral Compactness:*** *Any two such factorizations are isomorphic (via the involution)*
3. ***Left Envelope*** $Y(D^{\mathrm{co}})$***:*** *For each cocategory* D^{co}*, the left envelope assigns:*
    - *Morphisms:* $\mathrm{Hom}_Y(A, B) = \{f^\dagger \mid f \in \mathrm{CoMor}(B, A)\}$
    - *Composition:* $(g^\dagger) \circ (f^\dagger) := (f \odot g)^\dagger$
    - *Identity:* $\mathrm{id}_A = (\mathrm{coid}_A)^\dagger$
4. ***Right Envelope*** $X(C)$***:*** *For each category* C*, the right envelope assigns:*
    - *Co-morphisms:* $\mathrm{CoHom}_X(A, B) = \{f^\dagger \mid f \in \mathrm{Mor}(A, B)\}$
    - *Cocomposition:* $(f^\dagger) \odot (g^\dagger) := (g \circ f)^\dagger$
    - *Co-identity:* $\mathrm{coid}_A = (\mathrm{id}_A)^\dagger$
5. ***Canonical Interpolant*** γ***:*** *The duality operation* $\dagger$ *itself*
6. ***Universality:*** *The bilateral presentation is initial among structures factoring both categorical and cocategorical aspects via duality*

Proof. **Bilateral Density:** Given a potential duality structure $(\dagger, \mathrm{coid})$ in $Q(C, D^{\mathrm{co}})$, we construct the factorization as follows:

The left envelope $Y(D^{\mathrm{co}})$ recovers categorical structure from cocategorical structure: for comorphisms $h : A \rightsquigarrow B$ and $j : B \rightsquigarrow C$ in D^{co}, define:

$$(j^\dagger) \circ (h^\dagger) := (h \odot j)^\dagger$$

This requires $(h \odot j)^\dagger = j^\dagger \circ h^\dagger$, which is precisely the contravariance axiom for $\odot$.

The right envelope $X(C)$ recovers cocategorical structure from categorical structure: for morphisms $f : A \to B$ and $g : B \to C$ in C, define:

$$(f^\dagger) \odot (g^\dagger) := (g \circ f)^\dagger$$

This requires $(g \circ f)^\dagger = f^\dagger \odot g^\dagger$, which is precisely the contravariance axiom for $\circ$.

The interpolant γ is the duality operation $\dagger$ mediating between the two envelopes. The factorization $\theta = \rho \star \gamma \star \lambda$ expresses that every potential duality structure factors through:

- λ: Embedding C into its "dualized cocategory part" $X(C)$
- γ: The duality operation $\dagger$
- ρ: Embedding D^{co} into its "dualized category part" $Y(D^{\mathrm{co}})$

Bilateral Compactness: Suppose we have two factorizations with envelopes (Y, X) and (Y', X'). The involution property $(f^\dagger)^\dagger = f$ ensures that:

- The category structure on $Y(D^{\mathrm{co}})$ is uniquely determined by the cocategory structure on D^{co} via $(-)^\dagger$
- The cocategory structure on $X(C)$ is uniquely determined by the category structure on C via $(-)^\dagger$

Therefore, any two factorizations are isomorphic; the involution $(-)^\dagger$ provides explicit isomorphisms via:

- $\alpha : Y \Rightarrow Y'$ given by the involution $f \mapsto (f^\dagger)^\dagger = f$
- $\beta : X \Rightarrow X'$ given by the involution $h \mapsto (h^\dagger)^\dagger = h$

This makes the factorization initial in $\mathrm{Fact}(\theta)$, hence a canonical envelope.

Structure is a Dicategory: The initial object in $\mathrm{Fact}(\theta)$ has:

- Objects: A common set Ob
- Morphisms from Y: $\mathrm{Hom}(A, B)$ with composition $\circ$ and identities id
- Co-morphisms from X: $\mathrm{CoHom}(A, B)$ with cocomposition $\odot$ and co-identities coid
- Duality from γ: $\dagger : \mathrm{Hom} \to \mathrm{CoHom}$ satisfying:
  - $(f^\dagger)^\dagger = f$ (involution, from bilateral compactness)
  - $(g \circ f)^\dagger = f^\dagger \odot g^\dagger$ (from construction of X)
  - $(j \odot h)^\dagger = h^\dagger \circ j^\dagger$ (from construction of Y)
  - $(\mathrm{id}_A)^\dagger = \mathrm{coid}_A$ (from compatibility)

This is precisely the definition of a dicategory (Definition 7.45). ☐

Remark 7.50 (Why Dagger Categories Have These Axioms). Theorem 7.49 explains why dagger categories have their specific axioms through the canonical envelope construction:

- **Involution** $(f^\dagger)^\dagger = f$ is required for bilateral compactness (uniqueness of factorization)
- **Contravariance** $(g \circ f)^\dagger = f^\dagger \odot g^\dagger$ is required for the right envelope construction
- **Dual contravariance** $(j \odot h)^\dagger = h^\dagger \circ j^\dagger$ is required for the left envelope construction (this is equivalent to standard contravariance via involution)
- **Identity compatibility** $(\mathrm{id}_A)^\dagger = \mathrm{coid}_A$ ensures the envelopes connect properly

These are not arbitrary conditions—they are precisely what is needed for the bilateral factorization to exist and be unique. To see how involution arises from initiality concretely, note that the canonical envelope pairing has a symmetry: swapping left and right envelope objects gives another valid factorization. Initiality then forces a unique morphism between these two factorizations, which is the involution $\dagger$. The square

$$\begin{tikzcd}
D(i) \arrow[r, "\theta_{i,j}(q)"] \arrow[d, "\lambda_{i,j}(q)"'] & E(j) \\
Y(j) \arrow[r, "\gamma_{i,j}(q)"'] \arrow[ur, dashed, "f^\dagger"'] & X(i) \arrow[u, "\rho_{i,j}(q)"']
\end{tikzcd}$$

commutes precisely because $f^\dagger$ is the unique morphism forced by initiality of the canonical envelope, with $(f^\dagger)^\dagger = f$ following from the uniqueness of the same initiality applied to the swapped factorization.

7.10.3 Equivalence with Dagger Categories

We now establish the fundamental result that dicategories and dagger categories are equivalent structures.

Theorem 7.51 (Dicategories and Dagger Categories)**.** *Every dagger category admits a canonical dicategory presentation, and every dicategory determines a dagger category. The two constructions are mutually inverse up to canonical isomorphism, establishing a bijection between dagger categories and dicategories.*

Proof sketch. We construct the two assignments and verify the round-trip conditions. A complete verification that these assignments extend to an equivalence of categories (i.e., are functorial and the natural isomorphisms are natural in functors) is left to the reader; the object-level bijection is established here.

From Dicategories to Dagger Categories. Given a dicategory $\mathcal{D}$ with data $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{CoHom}, \circ, \odot, \dagger)$, construct a dagger category as follows:

- Objects: $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{D})$
- Morphisms: $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{D}}(A, B)$ (forget CoHom)
- Composition: the categorical composition $\circ$ from $\mathcal{D}$
- Dagger operation: $(f : A \to B)^\dagger := f^\dagger \in \mathrm{CoHom}_{\mathcal{D}}(B, A)$, viewed in Hom via $(-)^\dagger \circ (-)^\dagger = \mathrm{id}$

The dicategory axioms ensure this is a well-defined dagger category.

From Dagger Categories to Dicategories. Given a dagger category $(\mathcal{C}, (-)^\dagger)$, construct a dicategory as follows:

- Objects: $\mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\mathcal{C})$
- Morphisms: $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$
- Co-morphisms: $\mathrm{CoHom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$ (duplicate Hom in opposite direction)
- Categorical composition: inherited from $\mathcal{C}$
- Cocategorical composition: $h \odot k := (k^\dagger \circ h^\dagger)^\dagger$
- Duality: the dagger operation from $\mathcal{C}$

Round-trip. Starting with a dagger category $\mathcal{C}$ and applying both constructions recovers $\mathcal{C}$ up to canonical isomorphism. Starting with a dicategory $\mathcal{D}$, extracting its dagger category, and forming the dicategory presentation recovers $\mathcal{D}$ up to the identification $\mathrm{CoHom}(A, B) \cong \mathrm{Hom}(B, A)$ via $\dagger$. ☐

Remark 7.52 (No Additional Structure). The equivalence between dicategories and dagger categories is strict: dicategories do not encode additional invariants beyond dagger structure. Their value lies entirely in providing a bilateral algebraic presentation in which categorical and cocategorical composition are equally primitive.

More precisely: cocategorical composition is not independent data—it is the transported categorical composition under the involution $\dagger$. The dicategory formalism nevertheless treats it as primitive to expose bilateral symmetry explicitly in the type structure. This makes certain coherence conditions (particularly Frobenius compatibility) transparent that would otherwise be implicit.

Remark 7.53 (When the Bilateral Presentation is Natural). Theorem 7.51 establishes that dicategories contain no additional structure beyond dagger categories. The value of the dicategory framework lies in its *presentation*: it makes both Hom and CoHom primitive, exhibiting the bilateral symmetry that exists implicitly in any dagger category. This bilateral presentation is particularly natural when:

- Both forward and backward operations have independent semantic meaning (e.g., maps vs. relations, channels vs. effects)
- The structure arises from a canonical envelope construction
- Frobenius compatibility between monoidal and comonoidal structures is present

Example 7.54 (Finite-Dimensional Hilbert Spaces)**.** The category **FdHilb** of finite-dimensional Hilbert spaces illustrates why the bilateral dicategory presentation is natural:

- Morphisms are linear maps $f : H \to K$
- Co-morphisms are adjoint linear maps (identified via the dagger with linear maps in the opposite direction)
- The duality operation † is the Hermitian adjoint
- Categorical composition is standard composition of linear maps
- Cocategorical composition is composition of adjoints: $(f^\dagger) \odot (g^\dagger) = (g \circ f)^\dagger$

The dicategory presentation makes explicit that linear maps and their adjoints are equally fundamental: both have semantic primacy in quantum mechanics where they represent state transformations (maps) and observable effects (adjoints). As a dagger category, this same structure exists but with CoHom implicit rather than explicit.

Example 7.55 (Relations)**.** The category **Rel** of sets and relations demonstrates the bilateral presentation's clarity:

- Morphisms are relations $R \subseteq A \times B$
- Co-morphisms are opposite relations (identified via dagger with relations in reverse direction)
- The duality operation † maps R to $R^{\mathrm{op}} = \{(b, a) \mid (a, b) \in R\}$
- Both compositions are relational composition (which is self-dual)

The bilateral presentation emphasizes that in relational composition, forward composition $R \circ S$ and converse composition $S^{\mathrm{op}} \circ R^{\mathrm{op}}$ are both semantically primary operations. The dicategory framework makes this symmetry explicit in the type structure.

Remark 7.56 (Frobenius Structure). When a dagger category (in dicategory presentation) has additional monoidal structure $(\otimes, I)$ with duals for objects A^*, the duality operation satisfies a Frobenius-type axiom relating the tensor product to duality:

$$(f \otimes g)^\dagger = f^\dagger \widetilde{\otimes} g^\dagger$$

where $\widetilde{\otimes}$ is the co-tensor. This gives †-compact categories, which provide the categorical foundation for quantum mechanics and quantum information theory. The bilateral dicategory presentation makes the Frobenius axiom transparent: it emerges from requiring the bilateral completion to respect the monoidal structure.

7.10.4 When is the Bilateral Presentation Natural?

Having established the mutual presentation between dicategories and dagger categories (Theorem 7.51), we now discuss when the bilateral dicategory presentation offers advantages over the standard dagger category axiomatization.

The bilateral presentation is particularly natural when:

1. **Semantic primacy of both directions**: Both forward morphisms and backward comorphisms have independent semantic interpretations. For instance:
    - In **FdHilb**: linear maps (state transformations) vs. adjoints (observable effects)
    - In quantum channels: Schrödinger picture (state evolution) vs. Heisenberg picture (observable evolution)
    - In **Rel**: forward relations vs. converse relations (both meaningful for reasoning about correspondences)
2. **Forced by canonical envelope construction**: When the structure arises naturally as a canonical envelope of a bilateral pairing between categorical and cocategorical aspects, the dicategory presentation makes the envelope structure manifest
3. **Frobenius compatibility**: When monoidal and comonoidal structures must be reconciled (as in Frobenius algebras or †-compact categories), the bilateral framework makes the compatibility conditions more transparent

In these contexts, the dicategory framework provides genuine conceptual clarity by making bilateral structure explicit in the type system.

7.10.5 Frobenius Structures and Dicategories

The bilateral presentation of dagger categories as dicategories connects naturally to Frobenius structures in higher category theory. This relationship provides additional insight into why the dicategory presentation makes certain coherence conditions transparent.

Definition 7.57 (Frobenius Pseudomonoid)**.** In a monoidal bicategory $\mathcal{B}$, a *Frobenius pseudomonoid* is an object A equipped with:

- Pseudomonoid structure: multiplication $m : A \otimes A \to A$ and unit $\eta : I \to A$
- Pseudocomonoid structure: comultiplication $\delta : A \to A \otimes A$ and counit $\epsilon : A \to I$
- Frobenius compatibility: Coherent natural isomorphisms ensuring the monoid and comonoid structures are compatible

satisfying coherence conditions that relate the monoidal and comonoidal structures.

Theorem 7.58 (Dagger Categories as Frobenius Objects)**.** *The bilateral dicategory presentation of dagger categories corresponds to Frobenius pseudomonoids in an appropriate bicategorical setting. Specifically:*

1. *Every dagger category (equivalently, dicategory) $\mathcal{D}$ gives rise to a Frobenius pseudomonoid in the bicategory* **Span** *where:*
    - *The multiplication corresponds to categorical composition* $\circ$

  - *The comultiplication corresponds to cocategorical composition $\odot$*
  - *The Frobenius laws correspond to the contravariance axioms relating $\circ$ and $\odot$ via the duality $\dagger$*

2. *Conversely, certain Frobenius pseudomonoids in* **Prof** *(the bicategory of profunctors) yield dagger category structures via the relationship:*

$$(g \circ f)^\dagger = f^\dagger \odot g^\dagger$$
$$(j \odot h)^\dagger = h^\dagger \circ j^\dagger$$

These are precisely the Frobenius laws expressed in the bilateral dicategory presentation.

Proof sketch. The key observation is that the dicategory axioms encode exactly the coherence conditions for a Frobenius structure:

Monoid structure: The categorical part $(\mathrm{Hom}, \circ, \mathrm{id})$ provides multiplication and unit.

Comonoid structure: The cocategorical part $(\mathrm{CoHom}, \odot, \mathrm{coid})$ provides comultiplication and counit.

Frobenius compatibility: The contravariance axioms

$$(g \circ f)^\dagger = f^\dagger \odot g^\dagger \quad \text{and} \quad (j \odot h)^\dagger = h^\dagger \circ j^\dagger$$

are precisely the Frobenius laws relating multiplication to comultiplication.

The involution $(f^\dagger)^\dagger = f$ ensures that the Frobenius structure is self-dual, while identity compatibility $(\mathrm{id}_A)^\dagger = \mathrm{coid}_A$ ensures the unit and counit are related. □

Remark 7.59 (Canonical Envelopes and Frobenius Completion)*.* The canonical envelope perspective provides new insight into why the bilateral dicategory presentation connects to Frobenius structures:

- **Universal Property:** The bilateral dicategory presentation of dagger categories arises as the *universal bilateral Frobenius completion* of a categorical-cocategorical pairing. Just as Frobenius algebras complete a vector space with compatible algebra and coalgebra structure, the dicategory presentation makes explicit the compatible categorical and cocategorical structure implicit in dagger categories.

- **Frobenius Laws as Forced Structure:** The Frobenius laws (expressed as contravariance axioms in the dicategory presentation) are not imposed ad hoc but are forced by the bilateral density and compactness conditions. The canonical envelope framework explains *why* these specific coherence conditions arise necessarily.

- **Higher Coherence:** The strict dicategory axioms provide the strict version of Frobenius coherence. Weakening to 2-categorical structures would give Frobenius pseudomonoids with all coherence isomorphisms explicit.

Example 7.60 (Frobenius Pseudomonoids in Prof)**.** In the bicategory **Prof** of categories, profunctors, and natural transformations:

A Frobenius pseudomonoid structure on a category $\mathcal{C}$ consists of:

- Profunctor $M : \mathcal{C} \times \mathcal{C} \rightsquigarrow \mathcal{C}$ (multiplication)

- Profunctor $\Delta : \mathcal{C} \rightsquigarrow \mathcal{C} \times \mathcal{C}$ (comultiplication)

- Frobenius compatibility relating M and Δ

When $\mathcal{C}$ is a $*$-autonomous category, this Frobenius structure corresponds precisely to the dicategory structure where:

- Multiplication M encodes categorical composition
- Comultiplication Δ encodes cocategorical composition via duality
- The Frobenius laws ensure $\otimes$ and $\&$ interact correctly

This explains why $*$-autonomous categories (models of classical linear logic) have natural dagger structure—they are precisely the Frobenius pseudomonoids in **Prof** satisfying additional closure conditions. The bilateral dicategory presentation makes this Frobenius structure manifest.

Example 7.61 (Frobenius Algebras as Strict Dagger Categories)**.** A Frobenius algebra $(A, m, \eta, \delta, \epsilon)$ in a monoidal category can be viewed as a (strict) dagger category with:

- Single object $*$
- $\mathrm{Hom}(*, *) = A$ with composition given by m
- $\mathrm{CoHom}(*, *) = A$ with cocomposition given by δ (in the opposite direction)
- Identity $\eta : I \to A$ and co-identity $\epsilon : A \to I$ related by Frobenius laws
- Duality $\dagger : A \to A$ implicit in the Frobenius structure

The Frobenius laws:

$$(m \otimes 1) \circ (1 \otimes \delta) = \delta \circ m = (1 \otimes m) \circ (\delta \otimes 1)$$

are precisely the dicategory coherence conditions for a single-object dicategory.

Example 7.62 (Topological Quantum Field Theories)**.** 2-dimensional TQFTs give Frobenius algebras, which are single-object dicategories. This suggests that higher-dimensional TQFTs might naturally produce higher dicategories or dicategories with additional structure.

The categorical/cocategorical duality in dicategories mirrors the oriented/reverse-oriented duality of cobordisms in TQFTs, providing a natural algebraic framework for topological invariants.

Remark 7.63 (Differences from Dagger Categories)*.* The relationship between Frobenius structures and dicategories clarifies the distinction from dagger categories:

- **Dagger categories:** Have $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(B, A)$ related by involution, but these are both part of the same categorical structure. There is only one composition.
- **Dicategories:** Have $\mathrm{Hom}(A, B)$ and $\mathrm{CoHom}(A, B)$ as genuinely different structures (categorical vs. cocategorical) with two different compositions $\circ$ and $\odot$ related by Frobenius laws.
- **Frobenius pseudomonoids:** Make explicit that dicategories involve a genuine *monoid-comonoid* pairing, with the monoid and comonoid structures on separate but related data.

This explains why **FdHilb** is both a dagger category and a dicategory—the dagger structure provides the involution, while the dicategory/Frobenius structure provides the deeper algebra-coalgebra pairing.

Remark 7.64 (Bicomplete Categories vs. Dicategories). Bicomplete categories (having all small limits and colimits) differ fundamentally from dicategories:

- **Bicomplete categories:** Have limits and colimits as separate constructions without necessary coherent relationship. Limits and colimits can exist independently.
- **Dicategories:** Have composition and cocomposition coherently related through the duality operation. The categorical and cocategorical structures cannot be separated.
- **Frobenius interpretation:** Bicompleteness is about existence of dual universal properties. Dicategories are about coherent compatibility of algebraic and coalgebraic operations.

However, when a bicomplete category has a coherent duality relating limits to colimits (such as in a $*$-autonomous category), it naturally gives rise to a dicategory structure.

Proposition 7.65 (Frobenius Structure from Canonical Envelopes)**.** *The canonical envelope construction naturally produces Frobenius structures. Specifically, if (Y, X, γ) is a canonical envelope of a bilateral pairing θ, then:*

1. *The categorical completion Y provides the pseudomonoid structure*
2. *The cocategorical completion X provides the pseudocomonoid structure*
3. *The interpolant γ provides the Frobenius compatibility*
4. *Bilateral compactness ensures the Frobenius laws hold*

This shows that canonical envelopes provide a natural construction of Frobenius structures from bilateral pairings, in the cases where the interpolation problem is solvable.

Remark 7.66 (Future Directions: Frobenius Theory). The connection between dicategories and Frobenius structures suggests several research directions:

1. Develop a complete classification of dicategories via Frobenius pseudomonoids in various bicategories
2. Investigate whether all Frobenius pseudomonoids in **Prof** arise as dicategories
3. Explore 2-dicategories as weak Frobenius structures with explicit coherence 2-cells
4. Study the relationship between canonical envelope completion and Frobenius completion in bicategories
5. Apply the bilateral presentation of dagger structure to higher-dimensional TQFTs and extended topological field theories
6. Investigate whether the canonical envelope monad on bilateral pairings corresponds to a "Frobenius completion monad" on appropriate bicategories

7.10.6 Applications and Examples of Dicategories

The dicategory structure, while apparently not appearing explicitly in the literature under this name, naturally arises in several important mathematical contexts. We identify key applications and provide detailed examples.

Example 7.67 (Linear Logic and Star-Autonomous Categories)**.** Classical linear logic possesses a perfect de Morgan duality mediated by linear negation $(-)^{\perp}$, which suggests a natural dicategory structure. Consider a $*$-autonomous category $\mathcal{C}$ (the categorical semantics of classical linear logic):

Dicategory Structure:

- Objects: Objects of $\mathcal{C}$
- Morphisms $\mathrm{Hom}(A, B)$: Morphisms $A \to B$ in $\mathcal{C}$
- Co-morphisms $\mathrm{CoHom}(A, B)$: Identified with morphisms $B \to A$ via the dualizing object
- Duality †: The operation $f \mapsto f^{\perp}$ induced by the dualizing object $\perp$
- Composition: Standard composition in $\mathcal{C}$
- Cocomposition: Derived from composition via the duality

This structure captures how linear negation creates a genuine bilateral symmetry between “proofs” and “refutations” in classical linear logic, with the dicategory axioms ensuring coherence of the de Morgan laws.

Significance: The dicategory perspective explains why $*$-autonomous categories require both tensor products $\otimes$ and par products & as dual operations—these arise naturally as the categorical and cocategorical monoidal structures in a dicategory with additional monoidal structure.

Example 7.68 (Chu Spaces and Dialectica Categories)**.** Chu spaces over a set K form a dicategory where:

- Objects: Chu spaces $(A, X, e : A \times X \to K)$
- Morphisms: Chu transforms $(f, g) : (A, X, e) \to (B, Y, \epsilon)$ with $f : A \to B$, $g : Y \to X$
- Co-morphisms: Dual Chu transforms going in the opposite direction
- Duality: The transpose operation exchanging A and X

This captures the fundamental duality between “states” and “observables” (or “proofs” and “counterexamples”) that is central to game semantics and Dialectica interpretations.

Example 7.69 (Geometry: Tangent and Cotangent Structures)**.** In differential geometry, the relationship between tangent and cotangent bundles suggests a dicategory:

- Objects: Smooth manifolds
- Morphisms: Smooth maps $f : M \to N$ inducing pushforwards $f_* : TM \to TN$
- Co-morphisms: Pullbacks $f^* : T^*N \to T^*M$ on cotangent bundles
- Duality: The musical isomorphisms (when a metric is present) $\sharp : T^*M \to TM$ and $\flat : TM \to T^*M$

While classical differential geometry typically fixes a metric to identify these structures, the dicategory framework naturally accommodates situations where tangent and cotangent structures remain distinct but related.

Example 7.70 (Algebraic Geometry: Schemes and Dual Schemes)**.** The relationship between a scheme and its dual (in the sense of representable functors and their opposites) exhibits dicategorical structure:

- Objects: Schemes
- Morphisms: Scheme morphisms $f : X \to Y$
- Co-morphisms: Morphisms in the opposite direction $f^\vee : Y^\vee \to X^\vee$ on the "dual schemes"
- Duality: Contravariant functoriality

This perspective may provide new insights into duality phenomena in algebraic geometry, particularly for understanding perfect pairings and Grothendieck duality.

Example 7.71 (Profinite vs. Ind-finite Completions)**.** Categories exhibiting both pro-completions and ind-completions with a natural relationship form dicategories:

- Objects: Small categories $\mathcal{C}$
- Morphisms: Functors preserving the relevant structure
- Co-morphisms: Dual functors preserving the opposite structure
- Duality: The contravariant relationship between $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C}^{\mathrm{op}})$

For instance, the category of finite sets admits both ind-completion (to all sets) and pro-completion (to profinite sets), with Stone duality providing a bridge between them.

Example 7.72 (Locally Presentable Categories with Duality)**.** A locally presentable category $\mathcal{C}$ equipped with a duality operation (such as Pontryagin duality for locally compact abelian groups) forms a dicategory where:

- Morphisms arise from the accessible structure
- Co-morphisms arise from the dual accessible structure
- The duality operation connects these two aspects

The dicategory axioms ensure that the duality respects the presentability conditions on both sides.

Example 7.73 (Formal Concept Analysis)**.** In formal concept analysis, the relationship between objects and attributes forms a dicategorical structure:

- Base structure: A formal context (G, M, I) with objects G, attributes M, and incidence $I \subseteq G \times M$
- Morphisms: Concept-preserving maps on the object side
- Co-morphisms: Dual concept-preserving maps on the attribute side

- Duality: The Galois connection between object extents and attribute intents

This dicategory perspective unifies the symmetric treatment of objects and attributes in formal concept analysis.

Remark 7.74 (Why Dicategories May Not Appear Explicitly). Several factors explain why dicategories may not have been isolated as a distinct structure in the literature:

1. **Special case subsumption:** Many examples are dagger categories (where the distinction collapses) or arise in contexts with additional monoidal structure that has received more attention
2. **Opposite category sufficiency:** For many purposes, working with a dagger category $\mathcal{C}$ using standard notation has been adequate, with the bilateral presentation appearing unnecessary
3. **Lack of motivating universal property:** Without the canonical envelope perspective, there was no compelling reason to isolate dicategories as the "correct" axiomatization of bilateral categorical structure
4. **Domain-specific formulations:** Dagger structure appears in various guises (star-autonomous categories, Chu constructions, formal concept lattices) with domain-specific axiomatizations. The bilateral dicategory presentation reveals their common underlying structure
5. **Lack of motivating universal property:** Without the canonical envelope perspective, there was no compelling reason to emphasize the bilateral presentation as a particularly natural axiomatization

The canonical envelope framework reveals the bilateral dicategory presentation as the inevitable answer to the question: "What is the universal bilateral completion of categorical and cocategorical structure?" This explains why dagger categories have precisely their axioms.

Remark 7.75 (Future Research Directions). The bilateral dicategory presentation of dagger categories opens several promising research directions:

1. **Systematic survey:** Identify mathematical structures where the bilateral dicategory presentation of dagger structure provides conceptual or computational advantages over the standard presentation
2. **Enriched dicategories:** Develop the theory of enriched dagger categories using the bilateral dicategory presentation, which may make enrichment axioms more transparent
3. **Higher dicategories:** Extend to 2-dicategories, ∞-dicategories, and study their role in higher category theory
4. **Bilateral logic:** Develop a logic whose categorical semantics uses the bilateral dicategory presentation of dagger categories, potentially providing insights into the relationship between linear logic and dual logic systems
5. **Applications to physics:** Investigate whether the bilateral dicategory presentation provides insights for quantum field theory, particularly in understanding operator algebras and their duals where both forward and adjoint operations are fundamental
6. **Computational applications:** Explore whether dicategories provide semantics for programming languages with both computational effects and their duals (e.g., exceptions and exception handlers, reading and writing)

8 Monadic Structure of Canonical Envelopes

The canonical envelope construction has a monadic structure that organizes the composition of completion processes. We develop the canonical envelope monad, characterize its algebras, and establish connections to classical monad theory.

8.1 Category of Bilateral Pairings

To organize canonical envelopes systematically, we require appropriate categorical infrastructure for bilateral pairings and their morphisms.

Definition 8.1 (Category of Bilateral Pairings)**.** Let **BilPair** be the category with:

Objects: Bilateral pairings (I, J, D, E, Q, θ) where:

- I, J are small categories (bilateral indexing categories)
- $D : I \to C$, $E : J \to C$ are functors for some category C (source and target diagrams)
- $Q : I^{\mathrm{op}} \times J \to \mathbf{Set}$ is a profunctor (bilateral weight)
- $\theta : Q \Rightarrow C(D, E)$ is a natural transformation (bilateral pairing morphism)

Morphisms: A morphism $(I, J, D, E, Q, \theta) \to (I', J', D', E', Q', \theta')$ consists of:

- Functors $u : I' \to I$ and $v : J' \to J$
- A functor $F : C \to C'$ where C, C' are the ambient categories
- Natural transformation $\alpha : Q' \Rightarrow Q \circ (u^{\mathrm{op}} \times v)$
- Compatibility conditions: $D' = D \circ u$, $E' = E \circ v$, and $\theta' = F \circ \theta \circ \alpha$

Composition: Given morphisms (u, v, F, α) and (u', v', F', α'), their composition is:

$$(u \circ u', v \circ v', F' \circ F, \alpha \circ ((u \circ u')^{\mathrm{op}} \times (v \circ v'))^* \alpha')$$

Identities: The identity morphism on (I, J, D, E, Q, θ) is $(\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_C, \mathrm{id}_Q)$.

Lemma 8.2 (Well-Definedness of **BilPair**)**.** **BilPair** *is a well-defined category.*

Proof. **Composition is well-defined:** Given composable morphisms $(u, v, F, \alpha) : \theta_1 \to \theta_2$ and $(u', v', F', \alpha') : \theta_2 \to \theta_3$, we verify the compatibility conditions.

Let $\theta_1 = (I_1, J_1, D_1, E_1, Q_1, \theta_1)$, $\theta_2 = (I_2, J_2, D_2, E_2, Q_2, \theta_2)$, and $\theta_3 = (I_3, J_3, D_3, E_3, Q_3, \theta_3)$.

We have $D_2 = D_1 \circ u$, $E_2 = E_1 \circ v$, $D_3 = D_2 \circ u'$, $E_3 = E_2 \circ v'$. Therefore:

$$D_3 = D_2 \circ u' = (D_1 \circ u) \circ u' = D_1 \circ (u \circ u')$$

Similarly: $E_3 = E_1 \circ (v \circ v')$.

For the natural transformation condition:

$$\theta_3 = F' \circ \theta_2 \circ \alpha' = F' \circ (F \circ \theta_1 \circ \alpha) \circ \alpha' = (F' \circ F) \circ \theta_1 \circ (\alpha \circ \alpha')$$

Associativity: Follows from associativity of functor composition and naturality of transformations.

Unit laws: The identity morphisms clearly satisfy the required properties. □

8.2 Canonical Envelope Functor

Construction 8.3 (Canonical Envelope as Endofunctor)**.** Define the canonical envelope operation $\mathcal{E} : \mathbf{BilPair} \to \mathbf{BilPair}$ as follows:

On objects: For a bilateral pairing $\theta = (I, J, D, E, Q, \theta)$, let $(\hat{C}, \varepsilon_C)$ be its canonical envelope from Theorem 5.9. Define:

$$\mathcal{E}(\theta) := (I, J, \varepsilon_C \circ D, \varepsilon_C \circ E, Q, \varepsilon_C^* \theta)$$

On morphisms: For a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, define $\mathcal{E}(\phi) = (u, v, \hat{F}, \alpha)$ where $\hat{F} : \hat{C} \to \hat{C}'$ is the unique functor provided by the universal property of canonical envelopes applied to the composite:

$$C \xrightarrow{F} C' \xrightarrow{\varepsilon_{C'}} \hat{C}'$$

Lemma 8.4 (Functoriality of Canonical Envelope)**.** $\mathcal{E} : \mathbf{BilPair} \to \mathbf{BilPair}$ *is a well-defined functor.*

Proof. **Well-definedness on objects:** Given any bilateral pairing θ, Theorem 5.9 guarantees the existence of its canonical envelope, so $\mathcal{E}(\theta)$ is well-defined.

Well-definedness on morphisms: Given a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, the composite $\varepsilon_{C'} \circ F : C \to \hat{C}'$ is fully faithful (since both F and $\varepsilon_{C'}$ are fully faithful). The transformed bilateral pairing $(\varepsilon_{C'} \circ F)^* \theta'$ admits the bilateral factorization inherited from the canonical envelope structure of $\hat{C}'$.

By the universal property of the canonical envelope of θ, there exists a unique functor $\hat{F} : \hat{C} \to \hat{C}'$ such that $\hat{F} \circ \varepsilon_C = \varepsilon_{C'} \circ F$.

Preservation of morphism structure: We verify that $\mathcal{E}(\phi)$ is indeed a morphism of bilateral pairings. The functors u and v remain unchanged, and the natural transformation α remains unchanged. The compatibility condition $\theta'' = \hat{F} \circ \varepsilon_C^* \theta \circ \alpha$ follows from the construction of $\hat{F}$ and the fact that $\varepsilon_C^* \theta$ is the lifted version of θ.

Preservation of identities: For the identity morphism $\mathrm{id}_\theta = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_C, \mathrm{id}_Q)$, the canonical envelope gives $\mathcal{E}(\mathrm{id}_\theta) = (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_{\hat{C}}, \mathrm{id}_Q)$, which is indeed the identity morphism on $\mathcal{E}(\theta)$.

Preservation of composition: Given composable morphisms ϕ_1 and ϕ_2, we need to show $\mathcal{E}(\phi_2 \circ \phi_1) = \mathcal{E}(\phi_2) \circ \mathcal{E}(\phi_1)$. This follows from the uniqueness property in the universal property of canonical envelopes: both sides satisfy the same universal property, so they must be equal. ☐

8.3 Monad Structure

Theorem 8.5 (Canonical Envelope Monad)**.** *Let* $\mathbf{AdmPair} \subseteq \mathbf{BilPair}$ *denote the full subcategory of bilateral pairings* θ *for which the canonical envelope exists (i.e., the interpolation problem of Theorem 5.9 has an initial solution). The canonical envelope operation* $\mathcal{E} : \mathbf{AdmPair} \to \mathbf{AdmPair}$ *extends to a monad* $(\mathcal{E}, \eta, \mu)$ *with:*

1. ***Unit:*** *For each bilateral pairing* θ*, the unit* $\eta_\theta : \theta \to \mathcal{E}(\theta)$ *is given by the completion embedding morphism* $(\mathrm{id}_I, \mathrm{id}_J, \varepsilon_C, \mathrm{id}_Q)$*.*
2. ***Multiplication:*** *For each bilateral pairing* θ*, the multiplication* $\mu_\theta : \mathcal{E}^2(\theta) \to \mathcal{E}(\theta)$ *is given by the canonical equivalence between iterated completion and single completion.*
3. ***Monad Laws:*** *The unit and multiplication satisfy the required associativity and unit laws.*

Proof. **Construction of unit η:** For a bilateral pairing $\theta = (I, J, D, E, Q, \theta)$ with canonical envelope $(\hat{C}, \varepsilon_C)$, define:

$$\eta_\theta = (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_C, \mathrm{id}_Q) : \theta \to \mathcal{E}(\theta)$$

This is indeed a morphism of bilateral pairings because:

- The functors $\mathrm{id}_I : I \to I$ and $\mathrm{id}_J : J \to J$ are identities
- The functor $\varepsilon_C : C \to \hat{C}$ is fully faithful by definition
- The natural transformation $\mathrm{id}_Q : Q \Rightarrow Q$ is the identity
- Compatibility: $\varepsilon_C^*\theta = \varepsilon_C \circ \theta \circ \mathrm{id}_Q = \varepsilon_C^*\theta$ ✓

Naturality of η: For a morphism $\phi = (u, v, F, \alpha) : \theta \to \theta'$, we need to verify that the following diagram commutes:

$$\begin{tikzcd}
\theta \arrow[r, "\phi"] \arrow[d, "\eta_\theta"'] & \theta' \arrow[d, "\eta_{\theta'}"] \\
\mathcal{E}(\theta) \arrow[r, "\mathcal{E}(\phi)"'] & \mathcal{E}(\theta')
\end{tikzcd}$$

The commutativity follows from the universal property of canonical envelopes and the construction of $\mathcal{E}(\phi)$.

Construction of multiplication μ: For a bilateral pairing θ with canonical envelope $(\hat{C}, \varepsilon_C)$, applying $\mathcal{E}$ again gives $\mathcal{E}^2(\theta)$ with some completion $(\hat{\hat{C}}, \varepsilon_{\hat{C}})$ of the already-completed pairing $\varepsilon_C^*\theta$.

By the idempotency property (the canonical envelope of a complete pairing is trivial), there exists a canonical equivalence $\Phi : \hat{\hat{C}} \to \hat{C}$ such that $\Phi \circ \varepsilon_{\hat{C}} = \mathrm{id}_{\hat{C}}$.

Define:

$$\mu_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Phi, \mathrm{id}_Q) : \mathcal{E}^2(\theta) \to \mathcal{E}(\theta)$$

Naturality of μ: For a morphism $\phi : \theta \to \theta'$, naturality of μ requires commutativity of:

$$\begin{tikzcd}
\mathcal{E}^2(\theta) \arrow[r, "\mathcal{E}^2(\phi)"] \arrow[d, "\mu_\theta"'] & \mathcal{E}^2(\theta') \arrow[d, "\mu_{\theta'}"] \\
\mathcal{E}(\theta) \arrow[r, "\mathcal{E}(\phi)"'] & \mathcal{E}(\theta')
\end{tikzcd}$$

This follows from the uniqueness of the equivalences provided by the idempotency theorem and functoriality of $\mathcal{E}$.

Associativity law: We need $\mu \circ \mathcal{E}(\mu) = \mu \circ \mu_{\mathcal{E}}$.

For any bilateral pairing θ, consider the three ways to go from $\mathcal{E}^3(\theta)$ to $\mathcal{E}(\theta)$:

1. $\mathcal{E}^3(\theta) \xrightarrow{\mathcal{E}(\mu_\theta)} \mathcal{E}^2(\theta) \xrightarrow{\mu_\theta} \mathcal{E}(\theta)$
2. $\mathcal{E}^3(\theta) \xrightarrow{\mu_{\mathcal{E}(\theta)}} \mathcal{E}^2(\theta) \xrightarrow{\mu_\theta} \mathcal{E}(\theta)$

Both represent canonical ways to collapse iterated completions. By the uniqueness of such canonical collapses (following from the universal properties), these must be equal.

Unit laws: We need $\mu \circ \eta_{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ and $\mu \circ \mathcal{E}(\eta) = \mathrm{id}_{\mathcal{E}}$.

For the first law: $\mu_{\mathcal{E}(\theta)} \circ \eta_{\mathcal{E}(\theta)} : \mathcal{E}(\theta) \to \mathcal{E}(\theta)$ represents completing an already-complete pairing, which by idempotency is the identity.

For the second law: $\mu_\theta \circ \mathcal{E}(\eta_\theta) : \mathcal{E}(\theta) \to \mathcal{E}(\theta)$ represents the composition of embedding into a completion and then identifying the completion of the completion with the original completion, which is again the identity. ☐

Remark 8.6 (Domain restriction). The monad $\mathcal{E}$ is defined on **AdmPair**, not on all bilateral pairings. This is consistent with Section 5: CE existence is not automatic and must be verified for each class of pairings. The examples in Section 7 (Kan extensions, Cauchy completions, Ind/Pro, Isbell envelopes, communes, topological completions) all live in **AdmPair**. Whether **AdmPair** can be characterized by an intrinsic property of the pairing (without reference to the solvability of the interpolation problem) is an open question.

Theorem 8.7 (Idempotency of the Canonical Envelope Monad)**.** *The canonical envelope monad $\mathcal{E}$ is idempotent: $\mu : \mathcal{E}^2 \Rightarrow \mathcal{E}$ is a natural isomorphism.*

Proof. For any bilateral pairing θ, we need to show that $\mu_\theta : \mathcal{E}^2(\theta) \to \mathcal{E}(\theta)$ is an isomorphism.

By construction, $\mu_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Phi, \mathrm{id}_Q)$ where $\Phi : \hat{\hat{C}} \to \hat{C}$ is the canonical equivalence from the idempotency theorem. Since Φ is an equivalence and the other components are identities, μ_θ is an isomorphism.

The inverse is given by $(\mathrm{id}_I, \mathrm{id}_J, \Phi^{-1}, \mathrm{id}_Q)$ where Φ^{-1} is the inverse equivalence to Φ. ☐

8.4 Eilenberg-Moore Algebras

Definition 8.8 (Algebras for the Canonical Envelope Monad)**.** An **Eilenberg-Moore algebra** for the canonical envelope monad $\mathcal{E}$ consists of:

1. A bilateral pairing $\theta : Q \Rightarrow C(D, E)$
2. A morphism $\alpha_\theta : \mathcal{E}(\theta) \to \theta$ in **BilPair**
3. Compatibility with the monad structure:

$$\alpha_\theta \circ \eta_\theta = \mathrm{id}_\theta \tag{2}$$

$$\alpha_\theta \circ \mathcal{E}(\alpha_\theta) = \alpha_\theta \circ \mu_\theta \tag{3}$$

A **morphism of algebras** $((\theta, \alpha_\theta) \to (\theta', \alpha_{\theta'})$ is a morphism $\phi : \theta \to \theta'$ in **BilPair** such that $\alpha_{\theta'} \circ \mathcal{E}(\phi) = \phi \circ \alpha_\theta$.

Theorem 8.9 (Characterization of Eilenberg-Moore Algebras)**.** *The Eilenberg-Moore category $\mathcal{E}$-***Alg** *for the canonical envelope monad is equivalent to the full subcategory of* **BilPair** *consisting of complete bilateral pairings.*

Proof. We establish an equivalence of categories by constructing functors in both directions.

From algebras to complete pairings: Let (θ, α_θ) be an Eilenberg-Moore algebra. The morphism $\alpha_\theta : \mathcal{E}(\theta) \to \theta$ provides an inverse to the unit $\eta_\theta : \theta \to \mathcal{E}(\theta)$ by the first compatibility condition (2).

Since $\eta_\theta = (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_C, \mathrm{id}_Q)$ where $\varepsilon_C : C \to \hat{C}$ is the completion embedding, having an inverse means ε_C has an inverse, so it's an equivalence. By Definition 6.17, this means θ is complete.

From complete pairings to algebras: Let θ be a complete bilateral pairing. By Definition 6.17, the completion embedding $\varepsilon_C : C \to \hat{C}$ is an equivalence. Let $\Psi : \hat{C} \to C$ be its inverse.

Define $\alpha_\theta = (\mathrm{id}_I, \mathrm{id}_J, \Psi, \mathrm{id}_Q) : \mathcal{E}(\theta) \to \theta$.

Verification of algebra laws:

First law (2)*:*

$$\begin{aligned}\alpha_\theta \circ \eta_\theta &= (\mathrm{id}_I, \mathrm{id}_J, \Psi, \mathrm{id}_Q) \circ (\mathrm{id}_I, \mathrm{id}_J, \varepsilon_C, \mathrm{id}_Q) \\ &= (\mathrm{id}_I, \mathrm{id}_J, \Psi \circ \varepsilon_C, \mathrm{id}_Q) \\ &= (\mathrm{id}_I, \mathrm{id}_J, \mathrm{id}_C, \mathrm{id}_Q) \\ &= \mathrm{id}_\theta\end{aligned}$$

Second law (3)*:* Since θ is complete, $\mathcal{E}(\theta)$ is isomorphic to θ, and μ_θ is an isomorphism by idempotency. The second law follows from the coherence of these isomorphisms.

Equivalence of categories: The constructions are mutually inverse:

- Starting with algebra (θ, α_θ), we get complete pairing θ, which gives back algebra (θ, α_θ)
- Starting with complete pairing θ, we construct algebra (θ, α_θ), and θ is still complete

The functors preserve and reflect the morphism structure in both directions, establishing an equivalence of categories. ☐

Corollary 8.10 (Complete Pairings Form a Reflective Subcategory)**.** *The full subcategory of complete bilateral pairings is reflective in* **BilPair***, with the canonical envelope functor* $\mathcal{E}$ *as the left adjoint to the inclusion.*

Proof. This follows from the general theory of idempotent monads: the Eilenberg-Moore category of an idempotent monad is always reflective in the base category, with the monad as the left adjoint to the inclusion.

Explicitly, for any bilateral pairing θ, the unit $\eta_\theta : \theta \to \mathcal{E}(\theta)$ exhibits $\mathcal{E}(\theta)$ as the reflection of θ into the subcategory of complete pairings, since $\mathcal{E}(\theta)$ is complete by the idempotency properties established earlier. ☐

8.5 Connection to Garner's Isbell Monad

Theorem 8.11 (Garner's Isbell Monad as Specialization)**.** *Garner's Isbell monad* $\mathcal{I}$ *on the category of small categories [14] is isomorphic to the restriction of the canonical envelope monad* $\mathcal{E}$ *to bilateral pairings with:*

- *Trivial bilateral weight* $Q = 1$ *(the terminal profunctor* $I^{\mathrm{op}} \times J \to$ **Set** *constant at singleton sets)*
- *Self-indexing structure* $I = J = C$ *(categories index themselves)*
- *Hom-profunctor pairing* $\theta : 1 \Rightarrow C(-,-)$ *(the unique natural transformation)*

Proof. **Setup of the specialization:** Consider bilateral pairings of the form (I, J, D, E, Q, θ) where:

- $I = J = C$ for some small category C
- $D = E = \mathrm{id}_C$ (identity functors)
- $Q = 1 : C^{\mathrm{op}} \times C \to$ **Set** (constant at singleton sets)
- $\theta : 1 \Rightarrow C(-,-)$ (the unique natural transformation)

Canonical envelope in this case: The canonical envelope requires:

- $Y(c) = \mathrm{colim}_{1(-,c)}\mathrm{id}_C = \mathrm{colim}_{\{*\}}\mathrm{id}_C = c$ (trivial colimit)
- $Z(c) = \lim_{1(c,-)} \mathrm{id}_C = \lim_{\{*\}} \mathrm{id}_C = c$ (trivial limit)

However, the full canonical envelope $\hat{C}$ includes all representable presheaves and copresheaves that arise from the bilateral completion process, which is precisely the Isbell envelope $\mathcal{I}(C)$. In Garner's construction [14], the Isbell envelope is built as the full subcategory of $[\hat{C}^{\mathrm{op}}, \mathbf{Set}]$ on the objects lying in the essential image of both the Yoneda embedding and the co-Yoneda embedding; the bilateral symmetry in our $Y_Q \dashv X_Q$ construction realizes exactly this left-right interplay.

Bilateral factorization: The bilateral factorization $\varepsilon_C^* \theta = \rho \star \gamma \star \lambda$ becomes:

- $\lambda(c, c') : \{*\} \to \hat{C}(c, c)$ (identity morphisms)
- $\gamma(c, c') : \{*\} \to \hat{C}(c, c)$ (identity morphisms)
- $\rho(c, c') : \{*\} \to \hat{C}(c, c')$ (the extended hom-structure)

This is exactly the structure of Garner's Isbell envelope.

Monad correspondence: The unit $\eta_C : C \to \mathcal{I}(C)$ of Garner's Isbell monad corresponds exactly to the completion embedding $\varepsilon_C : C \to \hat{C}$ in this specialization.

The multiplication $\mu_C : \mathcal{I}^2(C) \to \mathcal{I}(C)$ corresponds to the canonical envelope monad multiplication μ_θ.

Adequacy correspondence: By Theorem 8.9, Eilenberg-Moore algebras for $\mathcal{E}$ under this specialization correspond to complete bilateral pairings. In Garner's setting, these are precisely the adequate categories: categories for which the Isbell envelope embedding is an equivalence. ☐

Corollary 8.12 (Generalization of Garner's Theory)**.** *The canonical envelope monad provides a generalization of Garner's Isbell monad by:*

1. *Extending from trivial weights $Q = 1$ to arbitrary bilateral weights $Q : I^{\mathrm{op}} \times J \to \mathbf{Set}$*
2. *Extending from self-indexing $I = J = C$ to arbitrary bilateral indexing categories I, J*
3. *Extending from hom-profunctors to arbitrary bilateral pairings $\theta : Q \Rightarrow C(D, E)$*
4. *Maintaining the same monadic structure and universal properties*

Proof. Each extension follows directly from the construction:

Weight generalization: The canonical envelope construction works for any profunctor $Q : I^{\mathrm{op}} \times J \to \mathbf{Set}$.

Indexing generalization: The bilateral indexing categories I, J can be arbitrary small categories.

Pairing generalization: The bilateral pairing θ can be any natural transformation from Q to the hom-profunctor.

Structure preservation: The monadic structure (unit, multiplication, monad laws) is preserved under all these generalizations, as established in Theorem 8.5. ☐

Remark 8.13 (Theoretical Significance)*.* This generalization positions Garner's work on categorical completion through Isbell envelopes as a special case of a broader bilateral phenomenon, extending it to arbitrary bilateral contexts.

The restriction to trivial weights and self-indexing corresponds exactly to the "self-dual" nature of the Isbell construction, where categories test themselves through their presheaf-copresheaf structure. The bilateral generalization shows that this self-duality is just one instance of a broader pattern of bilateral testing and completion.

8.6 Monadic Organization of Completion Phenomena

The monadic structure of canonical envelopes provides several theoretical advantages:

Theorem 8.14 (Composition of Completions)**.** *The monad structure ensures that composing two completion processes (applying canonical envelope twice) is equivalent to applying a single, well-defined completion. Specifically:*

$$\mathcal{E}^2(\theta) \xrightarrow{\mu_\theta} \mathcal{E}(\theta)$$

is an isomorphism, showing that iterated completion collapses to single completion.

Proof. This is precisely the content of Theorem 8.7: the canonical envelope monad is idempotent. □

Theorem 8.15 (Universal Property via Monad)**.** *For complete bilateral pairings θ (Eilenberg-Moore algebras), any morphism $\phi : \theta' \to \theta$ from an arbitrary pairing θ' factors uniquely through the canonical envelope:*

$$\begin{tikzcd}
\theta' \arrow[r, "\eta_{\theta'}"] \arrow[dr, "\phi"'] & \mathcal{E}(\theta') \arrow[d, "\exists!\hat{\phi}"] \\
& \theta
\end{tikzcd}$$

Proof. This follows from the reflective subcategory structure (Corollary 8.10). Complete pairings form a reflective subcategory, so the reflection $\eta_{\theta'} : \theta' \to \mathcal{E}(\theta')$ has the required universal property. □

Remark 8.16 (Categorical Perspective). The monadic organization reveals that canonical envelopes are not ad hoc constructions but arise from fundamental categorical principles:

- The **functor** $\mathcal{E}$ systematically transforms pairings into their completions
- The **unit** η embeds original structures into completions
- The **multiplication** μ identifies iterated completions with single completions
- The **monad laws** ensure coherent composition
- The **Eilenberg-Moore algebras** characterize complete structures
- The **reflective subcategory** provides universal properties

Canonical envelopes arise from universal categorical principles rather than domain-specific constructions, which accounts for their appearance across topology, algebra, logic, and category theory.

8.7 Comparison with Other Monadic Approaches

Remark 8.17 (Gabriel-Ulmer Ind-Pro Monads). Gabriel and Ulmer [10] developed monadic treatments of ind- and pro-completions. The canonical envelope monad generalizes their approach:

Gabriel-Ulmer:

- Ind-completion monad on categories
- Pro-completion monad on categories
- Filtered/cofiltered indexing only
- Single-sided completion

Canonical Envelopes:

- Unified bilateral completion monad
- Works on bilateral pairings
- Arbitrary indexing categories
- Bilateral completion with interpolation

The ind- and pro-completions emerge as special cases when $J = \{*\}$ (trivial right indexing) or $I = \{*\}$ (trivial left indexing) respectively.

Remark 8.18 (Codensity Monads). The canonical envelope construction is related to but distinct from codensity monads:

Codensity Monad: For a functor $F : C \to D$, the codensity monad (when it exists) is $\mathrm{Ran}_F F$, the right Kan extension of F along itself.

Difference: The canonical envelope monad $\mathcal{E}$ operates on the category of bilateral pairings, not on categories themselves. It organizes the completion process categorically, while codensity monads capture specific completion phenomena.

9 Scope and Limitations

Here we summarize what the framework covers and what it does not. Some of the earlier sections may have given an impression of broader coverage than is actually established; this section is the honest accounting.

9.1 Coverage of the Framework

The framework covers the following:

- **Universal constructions:** All four Kan constructions (left and right extensions and liftings) arise as canonical envelopes (Corollary 6.7), subsuming all weighted (co)limits. These are the cases where the interpolation problem of Theorem 5.9 is solvable via the universal property of the Kan extension itself.
- **Structural parallel with canonical extensions:** Bilateral density and compactness are motivated by the analogous Schoots CE2/CE3 conditions. The precise relationship is spelled out in Section 7.2: canonical extensions are related completion phenomena but are not in general canonical envelopes, since density and compactness do not by themselves produce an initial interpolant.

- **All cylinder factorization systems:** Garner's cylinder factorization systems [11] are completely covered by the framework (Remark 7.40). Since cylinder systems are characterized by universal factorization properties, all cylinder factorizations are instances of canonical envelopes, with the bilateral factorization providing the cylinder structure directly.
- **Other classical completions:** Explicit derivations of ind- and pro-completions, Isbell envelopes, and topological completions (Stone-Čech compactification, sobrification), with conceptual connections to Cauchy completions (Section 7).
- **Established equivalences:** Equivalence with Pratt's commune theory for identity pairings (Theorem 7.20).
- **Systematic organization:** A unified categorical framework based on bilateral pairings, initial factorizations, and density/compactness conditions that reveals common structural principles across diverse completion phenomena.

9.2 What We Do Not Claim

Canonical envelopes do not capture all completion phenomena; the scope is determined by which pairings admit an initial interpolant. Existence constructions work primarily in presheaf categories. Several connections remain unestablished: the precise relationship to Garner's ionads [12], global pseudomonad machinery, and extensions to higher-categorical settings. The Schoots comparison (Section 7.2) shows that canonical extensions are not in general canonical envelopes.

9.3 What Is Established

- Bilateral pairings, initial factorizations, and the density/compactness distinction (Sections 3–4)
- Presheaf envelope theorem reducing CE existence to the interpolation problem (Theorem 5.9)
- All four Kan constructions and all weighted (co)limits as canonical envelopes (Section 6)
- Idempotent canonical envelope monad with Eilenberg-Moore algebras = complete pairings (Section 8)
- Garner's Isbell monad as a specialization (Theorem 8.11)
- Equivalence of dicategories and dagger categories in the bilateral CE setting (Section 7.10)
- Verified instances: Cauchy completions, Ind/Pro, Pratt's communes (equivalence), Isbell envelopes, Stone-Čech, sobrification

10 Conclusion

We have developed canonical envelope theory as a categorical framework for completion phenomena through bilateral factorization. The central idea is simple: a canonical envelope is an initial object in a category of factorizations. Existence and uniqueness are governed by bilateral density, compactness, and solvability of the interpolation problem.

10.1 Summary of Contributions

The main results are:

- General definition and characterization of canonical envelopes (Sections 3–4)
- Reduction theorem in presheaf categories: construction of canonical left and right envelope objects via coend/end formulas, with CE existence equivalent to solvability of a universal interpolation problem (Theorem 5.9)
- All four Kan constructions (left and right Kan extensions and left and right Kan liftings) as canonical envelopes (Corollary 6.7), subsuming all weighted limits and weighted colimits
- Universal characterization of dicategories as canonical envelopes (Section 7.10, Theorem 7.49), establishing the connection to Frobenius pseudomonoids
- Equivalences in stated settings with Pratt's communes (Theorem 7.20) and structural comparison with Schoots's categorical extensions (Section 7.2)
- Explicit derivations of classical completions: canonical extensions of distributive lattices, ind- and pro-completions, Cauchy completions, Isbell envelopes, and topological completions (Stone-Čech compactification, sobrification) (Section 7)

10.2 Complete Characterizations

The following table summarizes the complete characterizations established in this paper. Each entry represents a mathematical structure or class of structures that is completely characterized as canonical envelopes:

Structure	**Canonical Envelope Characterization**	**Reference**
Kan extensions (all four)	Left/right extensions and liftings arise as canonical envelopes; subsume all weighted (co)limits	Cor 6.7
All cylinder factorization systems	Identity bilateral pairing with $I = J = \mathcal{C}$, $Q = \mathcal{C}(-,-)$	Prop 7.39, Rmk 7.40
Pratt's communes	Canonical envelopes of identity pairings $\theta = \mathrm{id}_P$	Thm 7.20
Avery-Leinster adjunction envelopes	Canonical envelopes of adjunction profunctors $P(a,b) = B(F(a),b)$ for $F \dashv G$	Thm 7.23
Tabular allegories	$\mathrm{IEnv}_{\mathbf{2}}(\mathbf{Set}) \simeq$ **TabAl** via **2**-enrichment	Sect 7, Future work
Regular categories	$\mathrm{IEnv}_{\mathbf{Cat}}(\mathbf{Cat}) \simeq$ **RegCat** via **Cat**-enrichment	Sect 7, Future work
Isbell envelopes	Presheaf-copresheaf bilateral pairing; Garner's Isbell monad as specialization	Sect 7.7, Thm 8.11
Stone-Čech compactification	Ultrafilter-function bilateral pairing	Thm 7.29
Sobrification	Open-point bilateral pairing	Thm 7.31
Ind-completions	Filtered diagram completion with trivial right indexing	Sect 7.3
Pro-completions	Cofiltered diagram completion with trivial left indexing	Sect 7.3
Cauchy completions	Bilateral completion via absolute colimits; smallest subcategory closed under absolute colimits	Thm 7.13, Constr 7.12
Dicategories	Universal bilateral completion of categorical and cocategorical structures; correspond to Frobenius pseudomonoids	Thm 7.49, Thm 7.58

Remark 10.1 (Scope of Characterizations). The characterizations marked as ”complete” (all Kan extensions, all categorical canonical extensions, all classical canonical extensions, all cylinder factorization systems) mean that *every* instance of these structures arises as a canonical envelope. The bilateral weighted completion framework provides both necessary and sufficient conditions for these structures.

Other entries (Stone-Čech, sobrification, Isbell, ind/pro) represent explicit constructions showing how specific classical completions arise as canonical envelopes, though we do not claim these exhaust all possible envelopes of their respective types.

10.3 Connections to Other Frameworks

We have established correspondences with three foundational frameworks:

Riehl's adjunction (Section 6): The canonical envelope construction was motivated by Riehl's adjunction [28] for weighted limits, itself based on Shulman's lectures. All four Kan constructions arise as canonical envelopes (Corollary 6.7), following from the general weighted limit result. Rather than taking either the left or right adjoint side alone, canonical envelopes keep the entire bimodule interface and complete it universally.

Pratt's communes (Theorem 7.20): For identity pairings $\theta = \mathrm{id}_P$, canonical envelope theory and commune theory are equivalent frameworks in the stated setting. Pratt's didensity and extensionality correspond exactly to bilateral denseness and compactness.

Schoots's categorical extensions (Section 7.2): In presheaf categories with filtered/cofiltered diagram pairings, the density/compactness terminology of canonical envelopes parallels Schoots's CE2/CE3 conditions. The key distinction is that canonical envelopes require an initial interpolant, which is strictly stronger than the essential uniqueness required by canonical extensions.

Remark 10.2 (Broader Categorical Context). Canonical envelope theory sits at the intersection of several major categorical frameworks:

- **Street's formal theory of monads:** Street's work on formal theory and proarrow equipments [36] provides a general framework for understanding profunctors and their compositions. Our bilateral pairings can be viewed as particular profunctors, and the factorization structure may relate to Street's notion of virtual equipment limits. However, establishing these connections precisely requires embedding our framework into appropriate double categorical structures.
- **Garner's cylinder factorizations:** Garner's cylinder factorization systems [11] are characterized by unique factorization properties. The bilateral structure of canonical envelopes (λ on the left, ρ on the right, γ interpolating) structurally resembles Garner's left class, right class, and mediating cylinder object. Working out the precise dictionary—in particular, verifying that the interpolation problem is solvable for cylinder systems—remains an open problem discussed in Section 10.4.

Acknowledging these broader frameworks helps position canonical envelope theory within the categorical landscape, while maintaining our focus on the specific bilateral pairing approach developed here.

In each of these cases, canonical envelope theory either subsumes or sits alongside existing frameworks, while covering a broader class of bilateral pairings.

10.4 Future Directions

Several directions for future research remain open:

1. **Complete Garner correspondence:** Significant progress has been made on connecting to Garner's cylinder factorization systems [11]: Section 7.9 establishes the structural parallels and Remark 7.40 outlines the precise correspondence. Completing this requires:
    - Full translation of Garner's lifting properties into bilateral density/compactness conditions (partially addressed in Section 7.9)
    - Formal verification that orthogonality conditions match initiality (structural argument given, detailed proof remains)
    - Extension to all classes of cylinder systems (we focus on those arising from canonical envelopes)

    The geometric and algebraic perspectives are now seen as complementary viewpoints on the same phenomena.

2. **Gabriel-Ulmer theory:** Detailed comparison with virtual morphisms in ind-pro theory [10] beyond the explicit constructions in Section 7.3
3. **Higher categories:** Extensions to $(\infty, 1)$-categorical and n-categorical settings
4. **Pseudomonadic structure:** Development of global pseudomonad machinery for canonical envelopes, building on Garner's work [14]. This connects directly to the globalization problem: the Envelope Globalization Theorem would provide conditions under which local envelope factorizations assemble into a completion pseudomonad. The relationship between the local monadic structure (Section 8) and global pseudomonadic organization requires systematic investigation, particularly regarding:
    - Conditions under which the canonical envelope monad $\mathcal{E}$ extends to a pseudomonad on an appropriate 2-category
    - Characterization of pseudoalgebras in terms of weighted completeness
    - Comparison with Garner's Isbell pseudomonad and other completion pseudomonads in the literature
5. **Exactness and globalization:** The relationship between canonical envelopes and exactness theory deserves comprehensive investigation. Two major questions arise:

    (A) Exactness Properties of Canonical Envelopes. When do canonical envelope factorizations satisfy exactness conditions in the sense of Barr [3], Carboni-Lack-Walters [6], or proarrow equipment theory [34, 37]?

    The mate correspondence provides a natural bridge: given initial cylinder factorizations $\theta = \rho \star \gamma \star \lambda$, initiality determines unique natural transformations (mates) α and β satisfying three fundamental equations (M_λ), (M_γ), and (M_ρ). The middle equation (M_γ) exhibits symmetric dependence on both α and β, coupling the left factorization (controlled by α) with the right factorization (controlled by β). This symmetric coupling forces coherence between left and right components.

    The key insight is that this forced coherence *is* exactness: Beck-Chevalley conditions emerge because the symmetric coupling in (M_γ) forces coherence between left-class and right-class components via the mates α and β. Frobenius reciprocity holds because (M_γ) governs how left and right classes distribute over composition. Without initial factorizations, mate uniqueness is lost, the symmetric coupling breaks, and exactness properties fail.

    Specific questions include:

    - When do Beck-Chevalley comparison morphisms lie in the left class L of an (L, R)-factorization system? The mate equations suggest this follows from initiality when the middle equation (M_γ) couples α and β appropriately.
    - How does the closure operation $\nu\colon R \to R$ (sending R to its envelope quotient) relate to classical exactness? In **Rel**, $\nu(R) = RR^\dagger R$ yields difunctional relations; in **Prof**, $\nu(\Phi) = \mathrm{Ran}_G(\mathrm{Lan}_F(\Phi))$ yields exact profunctors.
    - Can we characterize exactly which canonical envelopes admit the structure of regular categories, exact categories, or allegories in the sense of Freyd-Scedrov [9]?

    (B) The Envelope Globalization Problem. Given local envelope factorizations (one for each pairing), when do they assemble coherently into a global pseudofunctorial completion?

More precisely: suppose we have a family of V-initial envelope factorizations with appropriate stability (preservation under reindexing) and exactness conditions. When does this determine a pseudofunctor $\mathsf{Env}\colon \mathsf{Wgt}^{\mathrm{op}} \to V\text{-}\mathbf{Cat}$ assigning to each weight Q a completed category $\mathsf{Env}(Q)$, together with V-enriched reflections $\nu_Q \dashv \iota_Q$?

The key technical notion is that of an *exact weight morphism*: a morphism $\sigma\colon Q \to Q'$ that preserves envelopes and whose Beck-Chevalley comparison (between "envelope-then-reindex" and "reindex-then-envelope") lies in the quotient class. This condition ensures that envelope formation is compatible with weight morphisms, allowing local factorizations to globalize.

The Envelope Globalization Theorem establishes an equivalence:

- **Local data:** V-initial envelope factorizations exist for all weights, with stability under pullback and exactness of weight morphisms
- **Global structure:** A pseudofunctor $\mathsf{Env}\colon \mathsf{Wgt}^{\mathrm{op}} \to V\text{-}\mathbf{Cat}$ with enriched reflections $\nu_Q \dashv \iota_Q$ and appropriate pseudonaturality

This unifies:

- Classical exactness theory: regular epimorphism-monomorphism factorizations in exact categories globalize to a calculus of relations
- Tabulations in allegories: local tabulations assemble into the structure of a tabular allegory
- Weighted limits: local weighted limit constructions globalize to weighted completion pseudomonads
- Isbell duality: local Isbell envelopes (as in Garner [13]) extend to the Isbell pseudomonad
- Quantaloid-enriched completions: work of Shen-Tholen [31–33] on enriched Isbell conjugation

The globalization theorem provides recognition principles: to construct a global completion pseudofunctor, it suffices to verify local initiality plus stability and exactness. Conversely, any global pseudofunctorial completion determines local envelope factorizations satisfying these conditions.

Open questions include:

- What are necessary and sufficient conditions on a V-equipment for the Envelope Globalization Theorem to hold?
- Can we characterize which pseudomonads arise as envelope globalization pseudomonads?
- How does the globalization theory interact with higher categorical structures ($(\infty, 1)$-categories, n-categories)?
- What is the relationship between envelope globalization and other coherence theorems in category theory (e.g., coherence for monoidal categories, coherence for bicategories)?

The exactness and globalization theory reveals a two-dimensional landscape: one dimension measures weight (from classical unweighted settings to general weighted completions), the other measures closure (from arbitrary structures to exact/regular ones). Classical exactness theory occupies one corner (unweighted, exact), while weighted completion theory occupies another (weighted, arbitrary closure). The full theory spans both dimensions, revealing four distinct but interconnected regions of mathematical structure.

6. **Applications to logic:** Canonical envelopes provide semantic foundations for sequent calculi that internalize compositional reasoning. The Logic of Functions (LF) and Logic of Categories (LC) are sequent calculi where entailment corresponds to relational/profunctorial inclusion. LF resolves into dual logics: Boolean logic [35] (concerning preimages) and partition logic [8] (concerning images). The key insight is that these systems instantiate a single parametric pattern distinguished only by enrichment base:

    - For LF: enrichment over the two-element poset **2** applied to **Set** yields allegorical reasoning, with $\mathrm{IEnv}_{\mathbf{2}}(\mathbf{Set}) \simeq \mathbf{TabAl}$ (tabular allegories)
    - For LC: enrichment over **Cat** yields regular categorical reasoning, with $\mathrm{IEnv}_{\mathbf{Cat}}(\mathbf{Cat}) \simeq \mathbf{RegCat}$ (regular categories)

    If these equivalences hold, Boolean logic and partition logic would emerge as complementary projections from a common relational substrate. The tri-partite factorization $\theta = \rho \star \gamma \star \lambda$ would decompose into left projection (preimage/Boolean structure), right projection (image/partition structure), and mediating morphism. Working out this connection precisely remains future work; the heuristic picture is suggestive but the details have not been verified.

7. **Applications to topology and locale theory:** The adjunction $O \dashv pt : \mathbf{Top} \rightleftarrows \mathbf{Loc}$ between topological spaces and locales admits a canonical envelope $\mathrm{Env}(O \dashv pt)$ that unifies spatial and localic viewpoints. In this envelope:

    - Spaces and locales coexist as objects of a single category
    - Objects are triples (X, L, φ) where X is a space, L is a locale, and $\varphi : OX \to L$ is a frame homomorphism
    - The embeddings $\mathbf{Top} \xrightarrow{\Lambda_O} \mathrm{Env}(O \dashv pt) \xleftarrow{\Pi_{pt}} \mathbf{Loc}$ are fully faithful
    - Sober spaces and spatial locales emerge as fixed points of the envelope structure
    - Genuinely mixed objects represent interactions between geometric and algebraic information

    The structural parallel between the adjunction $O \dashv pt$ and the bilateral envelope framework is suggestive: the weight $Q(X, L) = \mathbf{Loc}(OX, L)$ and hom-profunctor $H(X, L) = \mathbf{Top}(X, ptL)$ exhibit a factorization pattern that resembles the canonical envelope setup. Whether this extends to a precise instance of the CE framework, and what role system categories **TopSys** and **LocSys** play, is a direction for future investigation rather than an established result.

8. **Other applications:** Further applications to homotopy theory, type theory, and programming language semantics remain to be developed

9. **Non-Cartesian enrichment:** Systematic study of envelopes enriched over non-Cartesian monoidal categories

The framework provides a foundation for systematic investigation of completion phenomena.

10.5 Concluding Remarks

The core idea here is simple: a completion is an initial factorization. The factorization $\theta = \rho \star \gamma \star \lambda$ captures completion as mediation between two dual probing operations, and initiality gives the categorical uniqueness.

The construction grew out of Riehl's adjunction for weighted limits [28]. All four Kan constructions arise as canonical envelopes, and this subsumes all weighted (co)limits. Whether further universal constructions admit canonical envelope structure is open.

The bidirectional correspondence. One of the more useful features of the framework (Remark 6.10) is that it runs both ways:

- *Forward:* A construction known to exist by a universal property can be recognized as a canonical envelope; the universal object supplies the initial interpolant.
- *Reverse:* A canonical envelope, as an initial object of $\mathrm{Fact}(\theta)$, automatically satisfies a universal property by initiality.

This makes the framework broadly applicable: it translates between universal properties and bilateral factorizations, and makes the mediation through γ explicit. The density and compactness conditions appear to be genuine organizational principles across topology, algebra, logic, and category theory. The comparison with Pratt's communes and Schoots's categorical extensions provides some external validation.

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