On Search For The Largest Prime Island

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Abstract

There are several interesting ways to depict distribution of primes like Ulam Spiral [1], Klauber Triangle [2] or the Sacks Number Spiral [3]. In all cases, Prime Number Theorem describes the asymptotic distribution of such numbers among the positive integers. This work is devoted to illustration of primes of form $p \times q \pm C$ in a way that allows to search for clusters (so called prime islands). The direct goal of this experimental work is to locate prime islands with the largest surface area and potentially discover some further patterns in distribution of primes.

1 Introduction

Work presented in this paper consists of three major parts: illustration of primes with detection of interesting clusters, formulation of patterns and examination of correctness of formulated patterns in spirit of finding new primes.

Experiments are planned against various sequences of form $p \times q \pm C$, where p and q are integer variables (p, q>0) and C is constant. Primes in the sequences are depicted as a non-zero values in X-Y graph, creating a single tile of coordinates (p, q). If two tiles are adjacent to each other on both sides, they form an island of size 2. Discovery of big islands (at least of size 4) is the first goal of this work, depicting co-located primes.

For clarity, the following definitions are in charge in this work:

Definition 1. *A prime tile is a location of coordinates (x, y) on X-Y graph with prime tile value. Primality of tile is result of a function: isprime$(F(x, y, C))$ where $F()$ is a function of form $x \times y \pm C$.*

Definition 2. *A prime island is an area composed of at least two prime tiles that have a common side.*

Lemma 1. *Formula $p \times q + C$ does not produce prime islands if C is even and positive (p, q>1).*

Proof. Let a tile of coordinates (p, q) is a prime tile: $F(p, q, C) = p \times q + C$ is a prime number. $C \geq 2$ is even and the only even prime number is 2, then $p \times q + C$ needs to be odd. If $p \times q + C$ is odd and C is even, then $p \times q$ is odd, and a result both p and q are odd. Based on Definition 2, prime tiles that are a part of prime island, have a common side - for (p, q) it will be two horizontal neighbours (A): $(p+1, q)$, $(p-1, q)$, and two vertical neighbours (B): $(p, q+1)$, $(p, q-1)$. Let's analyze cases A and B. A) $F(p \pm 1, q, C) = (p \pm 1) \times q + C = p \times q \pm q + C$. Both $p \times q$ and q are odd. Difference or sum of two odd numbers $(p \times q \pm q)$ is even. After adding to even number even C, we still have even number. From assumption p>1 and q>1, thus $p \times q \pm q + C$>2 which cannot be prime because every even number >2 is complex. B) Analogously to A) - $p \times q$ is odd, p is odd, $p \times q \pm p$ is even, $p \times q \pm p + C$ is still even and cannot be prime because $p \times q \pm p + C$>2. Geometric conclusion: (p, q) is a prime tile surrounded by four non-prime tiles. By Definition 2, single tile does not produce prime island. □

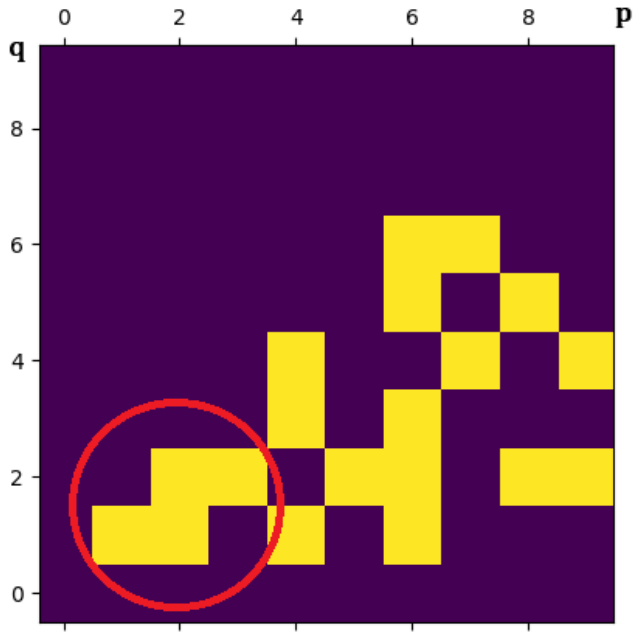


Figure 1: Illustration of prime tiles (prime islands of size 4 marked with red circle) for formula $p \times q + 1, p \geq q$.

Lemma 2. *Formula $p \times q + C$ does not produce prime islands if C is 0 (p, q>1).*

Proof. If $C = 0$ then $p \times q + 0 = p \times q$ cannot be prime by definition (because neither p nor q is 1 and $p \times q$ is always a complex number). □

In further work C is odd - this allows to focus on the most interesting findings. Framework [4] is used in all experiments. Also to simplify further calculations, notation: $p \times q \pm C$ where $p \geq q$ will be denoted as: $S_{\pm C}$.

Figure 1 depicts sample prime islands found for sequence S_{+1}, found for the first values of p and q. For the first island we have the following set of contiguous primes/tiles/formula:

- $2/(1, 1)/1 \times 1 + 1$
- $3/(1, 2)/1 \times 2 + 1$

- $5/(2,2), 2 \times 2 + 1$
- $7/(2,3)/2 \times 3 + 1$

The island is marked with a red circle and its size is 4.

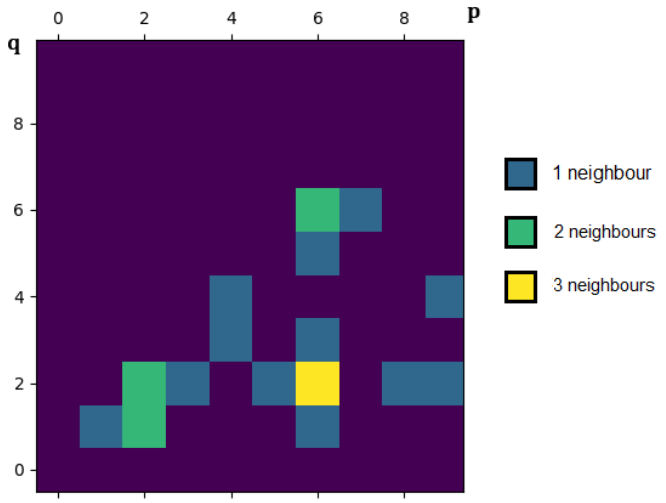


Figure 2: Illustration of prime tiles (skipping single tiles to reinforce prime islands) for S_{+1}. Additional coloring added to the picture, reflecting number of prime neighbours.

2 Search for prime islands (methodology)

Two methods to create X-Y matrices were considered and tested out.

Method A. Method faster. Each consecutive iteration equals study to one consecutive prime and its possible representation as a form of $S_{\pm C}$. For instance, if examined formula $F()$ is $p \times q + 1$ and we check 19 (the 8th prime), then 19 has two possible representations ($19 = 2 \times 9 + 1 = 3 \times 6 + 1$), meaning that 19 brings two prime tiles of coordinates $(2, 9)$ and $(3, 6)$ to the global picture.

Method B. Method slower but after acceptance tests was found as more accurate/reliable (mainly because we do not add new points to already examined area). In this approach we check every combination of p and q, where $p \geq q$, one-by-one.

3 Results of experiments and conclusions

Table 1 presents sample results for few formulas - a list of prime islands found with Method A. Generally, there are cases where there is just one example of largest island (ex. check of first $10k$ iterations for S_{-983} results in 1 islands of size 11) in the given set but there are also cases that largest prime island's surface is low and such list is long (ex. check of first $10k$ iterations for S_{-999} results in 13456 islands of size 3). Table 2 depicts how many islands of the given size were found for few selected values of C.

Executed experiments allowed to gather interesting observations. The largest prime island found was of size 16 and was located with Method B for S_{+197} - $[(420, 2520), (421, 2520), (422, 2520), (423, 2520), (424, 2520), (425, 2520), (426, 2520), (427, 2520), (428, 2520), (429, 2520),$

Table 1: Number of prime islands of given size for each formula $p \times q + C$, results after $100k$ iterations, method A.

Size of island	-1	$+1$	-7	-9
3	1987	1920	1693	1541
4	207	179	212	1
5	42	29	43	0
6	9	4	10	0
7	7	0	3	0
8	0	1	2	0
9	0	0	1	0

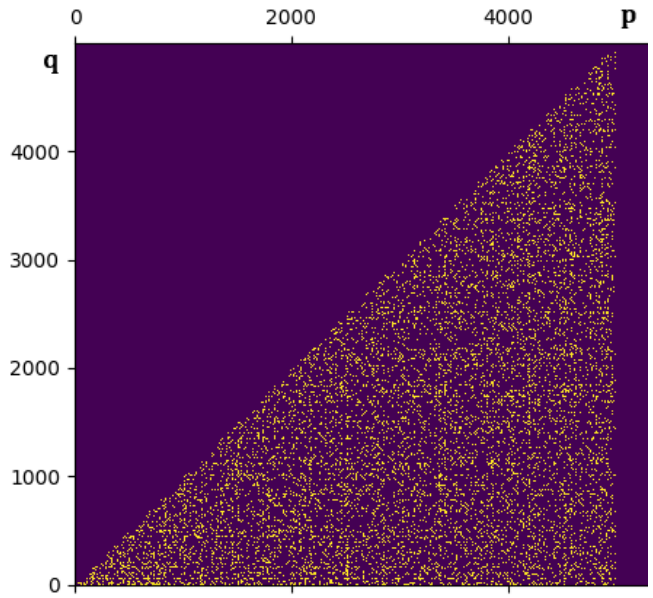


Figure 3: Prime tiles for S_{-293}, $10k$ iterations, method B in use.

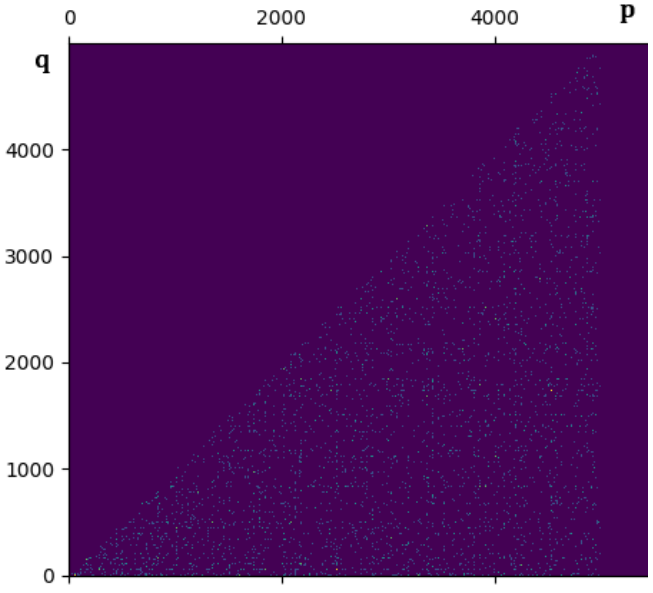


Figure 4: Prime islands for S_{-293}, $10k$ iterations, method B in use; this figure is result of post-processing of Figure 3 (single prime tiles were removed, all other prime tiles were colored, depending on number of prime neighbours).

$(424, 2519), (422, 2521), (420, 2521), (420, 2522), (421, 2522), (420, 2523)]$ (Figure 9).

The initially first but eventually second largest prime island found was of size 15 and was located with Method B for the following formulas:

- S_{-641} - $[(4, 210), (5, 210), (6, 210), (7, 210), (8, 210), (9, 210), (10, 210), (11, 210), (12, 210), (13, 210), (12, 209), (8, 209), (6, 209), (6, 208), (6, 207)]$ (Figure 10)
- S_{-293} - $[(2, 148), (3, 148), (2, 149), (2, 150), (3, 150), (4, 150), (5, 150), (6, 150), (7, 150), (8, 150), (6, 149), (6, 151), (6, 152), (5, 152), (4, 151)]$ (Figure 11),
- S_{+851} - $[(838, 1092), (839, 1092), (840, 1092), (841, 1092), (840, 1091), (840, 1090), (840, 1089), (840, 1093), (840, 1094), (840, 1095), (840, 1096), (840, 1097), (840, 1098), (839, 1098), (841, 1098)]$ (Figure 12).
- S_{+1003} - $[(6, 840), (7, 840), (8, 840), (9, 840), (10, 840), (11, 840), (12, 840), (13, 840), (14, 840), (12, 839), (12, 838), (13, 838), (12, 837), (10, 841), (6, 839)]$ (Figure 13).
- S_{+1219} - $[(124, 420), (123, 420), (125, 420), (126, 420), (127, 420), (128, 420), (129, 420), (130, 420), (131, 420), (132, 420), (132, 419), (130, 421), (128, 419), (126, 419), (126, 418)]$ (Figure 14).

Appendix A contains results for $-1001 \leq C \leq 1301$ ($10k$ iterations, C is either negative or positive odd number), all gathered with the use of Method B. Figure 6 + Figure 7 visualize these results and provide foundation to the following hypothesis: there is periodic relationship (of period = 6) between odd positive parameter C and the size and number of the largest first islands found in $S_{\pm C}$, because the colors in the figures are arranged in clear, regular stripes, reinforced with color map (results are organized as follows: experiments for C from -1001 to $+1001$ (step 2) are arranged in consecutive numbering N (starting from 0); then on both Figure 5 and Figure 6 we have as follows: $b = N mod\ 30$ and $a = N - 30b$). This observation led to formulation of Lemma 3.

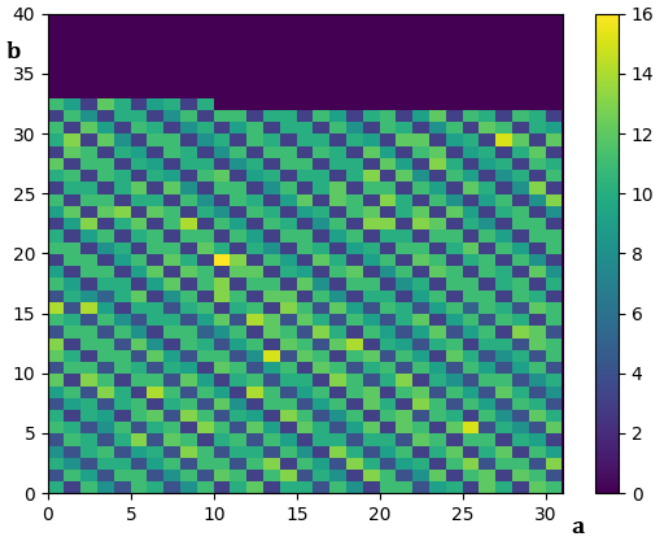


Figure 5: Size of largest island found for experiments depicted in appendix A: a set of results from S_{-1001} to S_{+1001} (step +2), method B, the first $10k$ iterations.

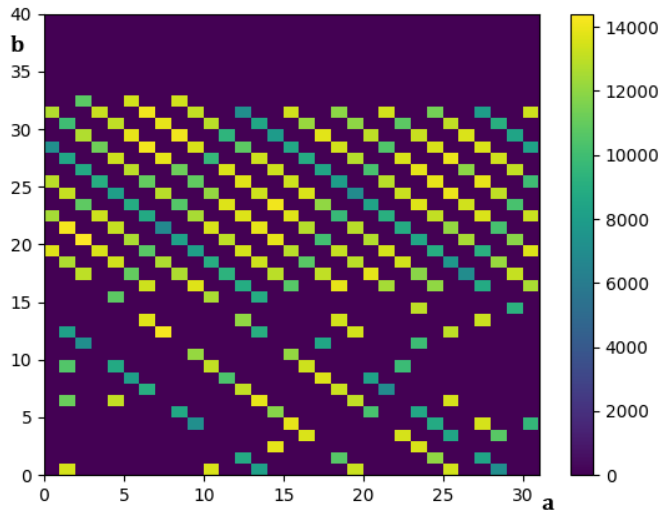


Figure 6: Number of islands of largest size found for experiments depicted in appendix A: a set from S_{-1001} to S_{+1001} (step +2), method B, the first $10k$ iterations.

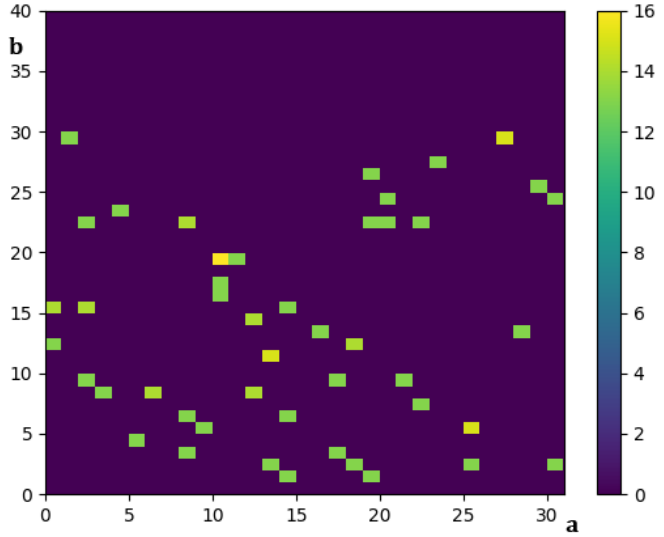


Figure 7: Islands of the greatest size only (>12) based on experiments in appendix A: a set from S_{-1001} to S_{+1001} (step +2), method B, the first $10k$ iterations.

Lemma 3. *If $C = 3+6a$ (a is integer ≥ 0) then the largest size of prime island in S_{+C} is 3.*

Proof. To prove Lemma 3, we must demonstrate that for any constant of the form $C = 3 + 6a$ (where $a \in \mathbb{Z}$), the maximum size of a prime island within the grid defined by the function $F(p, q, C) = p \times q + C$ is exactly 3.

Any prime number >3 leaves a remainder of either 1 or 2 when divided by 3. If any cell in the grid yields a result that is divisible by 3 (and greater than 3), that result is necessarily a composite number, acting as a "block" that prevents the island from expanding further. Constant C can be written as: $C \equiv 0 \pmod 3$). Based on that we can simplify: $F(p, q, C) = p \times q + C \equiv p \times q + 0 \equiv p \times q \pmod 3$. Conclusion: the value for tile (p, q) is divisible by 3 if and only if $p \times q$ is divisible by 3, thus at least one of p or q must be divisible by 3.

All tiles for which p is a multiple of 3 (i.e., $p \equiv 0 \pmod 3$) or q is a multiple of 3 (i.e., $q \equiv 0 \pmod 3$) yield

a result divisible by 3. Since we are examining large parameter values ($p \geq q > 1$), and for large islands the values of $F(p, q, C)$ are significantly greater than 3, each such tile is a composite number. Consequently: every third column of the grid ($p = 3, 6, 9, \dots$) consists entirely of composite numbers and every third row of the grid ($q = 3, 6, 9, \dots$) consists entirely of composite numbers (see: Figure 5). These lines form a rigid grid of barriers (marked as light red in Figure 5) across the plot, which divide the entire space into isolated 2×2 squares of tiles that can be a prime tiles. A prime island within such a zone could, in theory, reach a size of 4 if all four tiles in the square were prime numbers. Let us prove that this is impossible.

Consider any such 2×2 zone. Within it, the p coordinates take on two consecutive values that are not divisible by 3 - values of remainder either 1 or 2. Similarly, q coordinates have remainders of 1 or 2. Let us examine the remainders of the product $pq \pmod 3$ for the four tiles within this zone (A-D). A) For $p \equiv 1$, $q \equiv 1 \implies p \times q \equiv 1 \times 1 = 1$). B) For $p \equiv 1$, $q \equiv 2 \implies p \times q \equiv 1 \times 2 = 2$). C) For $p \equiv 2$, $q \equiv 1 \implies p \times q \equiv 2 \times 1 = 2$). D) For $p \equiv 2$, $q \equiv 2 \implies p \times q \equiv 2 \times 2 = 4 \equiv 1$). By adding a constant $C \equiv 0 \pmod 3$ to these, we obtain the following remainders for the entire 2×2 block: 2 tiles yield a remainder of 1 (mod 3) and 2 tiles yield a remainder of 2 (mod 3). From the previously proven Lemma 1, we know that for even C, no islands are formed at all; therefore, in Lemma 3, we consider the case where C is odd. Given that $C = 3 + 6a$ is odd, the mechanics of parity - as described in the paper - imply the following: 1) Within each 2×2 block, tiles with coordinates of the same parity (e.g., both odd or both even) yield even values for the formula $F(p, q, C)$. 2) Exactly two out of the four tiles in each 2×2 block will be even, and thus composite numbers (acting as parity-based "blockers"). Since the grid of divisibility-by-3 blockers restricts us to isolated 2×2 cells, and the additional parity rules "carve out" at least two tiles from each such cell as composite numbers, prime-valued tiles cannot physically connect to form a structure larger than 3 tiles (in practice, they typically form paths with a maximum length of 2 or 3, depending on how the parity blockers are positioned relative to the divisibility-by-3 blockers). This conclusively proves that the size of any prime islands for $C = 3 + 6a$ cannot exceed 3.

□

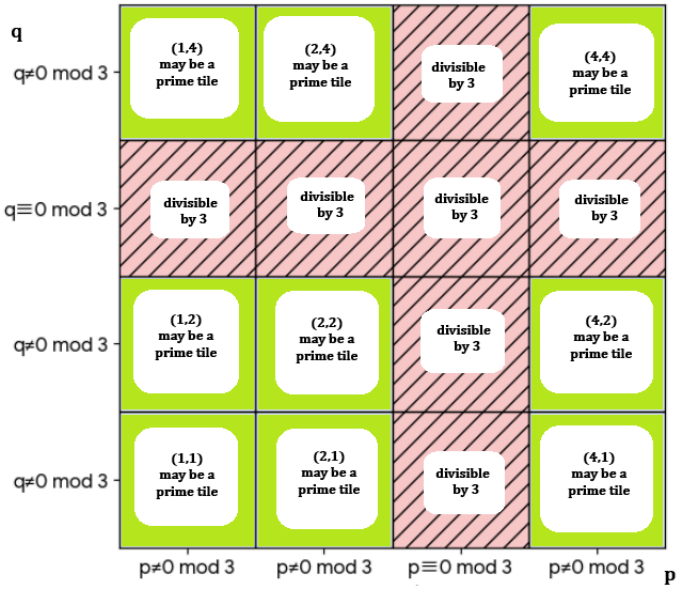


Figure 8: Grid used in proof for Lemma 3

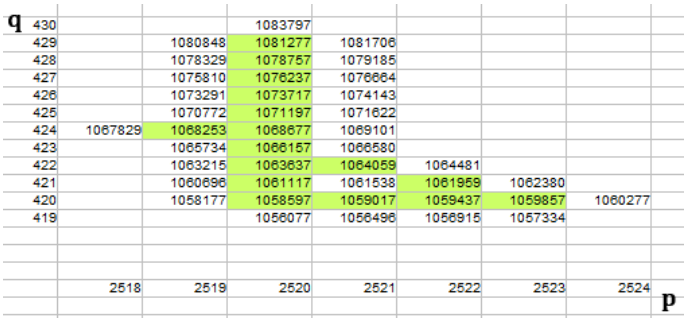


Figure 9: The largest prime island found (of size 16, S_{+197}).

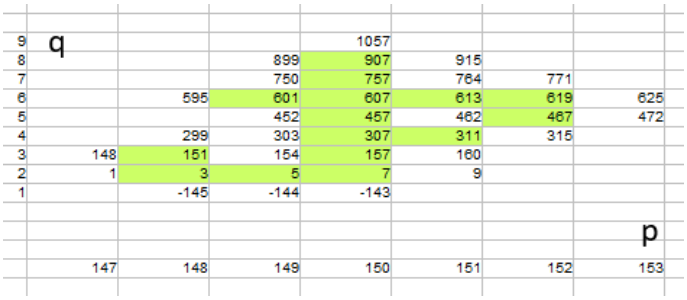


Figure 10: Big prime island found (of size 15, S_{-641}).

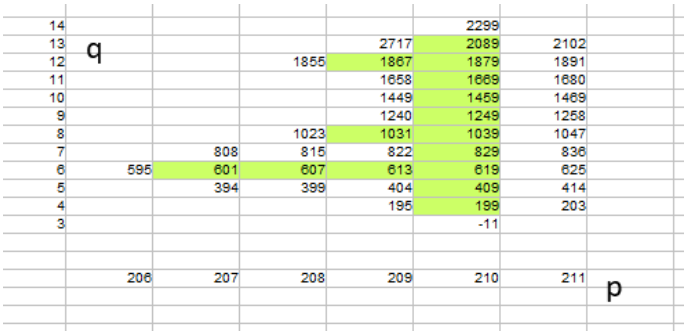


Figure 11: Big prime island found (of size 15, S_{-293}).

References

[1] Weisstein, Eric W., *Prime Spiral.* From MathWorld – A Wolfram Resource. https://mathworld.wolfram.com/PrimeSpiral.html

[2] *The Klauber Triangle, accessed at 07/19/2025.* https://scipython.com/blog/the-klauber-triangle/

[3] *The Sacks Number Spiral, accessed at 07/19/2025.* https://www.naturalnumbers.org/sparticle.html

[4] *Library for various operations on primes.* https://github.com/mbarylsk/primes

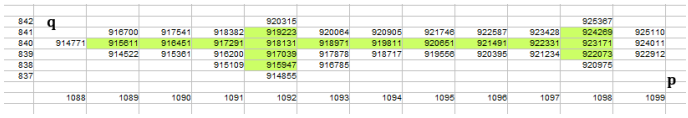


Figure 12: Big prime island found (of size 15, S_{+851}).

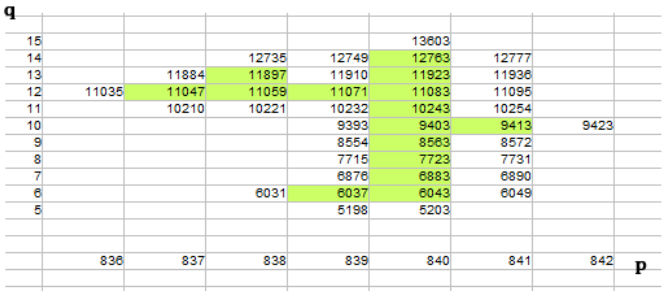


Figure 13: Big prime island found (of size 15, S_{+1003}).

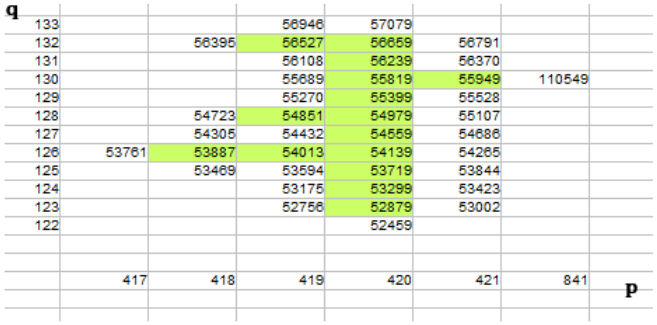


Figure 14: Big prime island found (of size 15, S_{+1219}).

Table 2: Formulas S subjected to experiments, results after $100k$ iterations - example results, method A.

i	S	List of largest prime islands found.
1	S_{+1}	[(5, 6872), (6, 6872), (6, 6871), (6, 6870), (5, 6870), (4, 6870), (3, 6870), (7, 6870)]
2	S_{-1}	[(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (4, 5), (2, 7)], [(2, 120), (3, 120), (4, 120), (5, 120), (6, 120), (7, 120), (2, 121)], [(3, 18), (4, 18), (5, 18), (6, 18), (6, 17), (6, 19), (4, 17)], [(3, 798), (4, 798), (5, 798), (6, 798), (6, 799), (6, 800), (4, 797)], [(4, 587), (4, 588), (5, 588), (6, 588), (6, 589), (6, 590), (7, 590)], [(4, 4452), (5, 4452), (6, 4452), (6, 4453), (6, 4454), (7, 4454), (6, 4455)], [(4, 9300), (5, 9300), (6, 9300), (7, 9300), (6, 9299), (6, 9298), (5, 9298)]
3	S_{-3}	[(1, 5), (2, 5), (2, 4), (2, 3), (1, 6)]
4	S_{+3}	A lot (one of examples: [(1093, 1120), (1094, 1120), (1094, 1121)])
5	S_{-5}	[(2, 4), (3, 4), (4, 4), (2, 5), (2, 6), (3, 6), (4, 6), (4, 7)]
6	S_{-7}	[(1, 60), (2, 60), (3, 60), (4, 60), (5, 60), (6, 60), (6, 59), (6, 61), (4, 59)]
7	S_{-9}	[(1, 11), (2, 11), (2, 10), (1, 12)]

A

Set of results from $C = -1001$ to $C = 1301$ (step 2), method B

Set is of form of: { C : (greatest size of prime island found, number of such islands found), ... }. Calculations performed for the first $10k$ iterations of $p \times q + C$ ($p \geq q > 0$) with method B. Bolded the greatest ever islands found.

{$-1001 : (10, 3), -999 : (3, 13456), -997 : (11, 2), -995 : (8, 6), -993 : (4, 1), -991 : (11, 3), -989 : (10, 2), -987 : (4, 2), -985 : (8, 11), -983 : (11, 1), -981 : (3, 13586), -979 : (10, 4), -977 : (12, 1), -975 : (3, 8124), -973 : (10, 2), -971 : (10, 3), -969 : (4, 1), -967 : (11, 1), -965 : (8, 9), -963 : (3, 13271), -961 : (11, 1), -959 : (11, 1), -957 : (4, 3), -955 : (9, 1), -953 : (10, 3), -951 : (3, 13430), -949 : (12, 1), -947 : (10, 2), -945 : (3, 7073), -943 : (12, 1), -941 : (11, 1), -939 : (4, 1), -937 : (11, 2), -935 : (9, 3), -933 : (4, 1), -931 : (11, 1), -929 : (11, 2), -927 : (4, 1), -925 : (8, 8), -923 : (11, 2), -921 : (4, 2), -919 : (12, 1), -917 : (11, 1), -915 : (3, 8776), -913 : (11, 1), -911 : (13, 1), -909 : (4, 2), -907 : (10, 2), -905 : (8, 11), -903 : (3, 10613), -901 : (13, 1), -899 : (11, 2), -897 : (4, 2), -895 : (8, 7), -893 : (12, 1), -891 : (3, 12019), -889 : (11, 1), -887 : (12, 2), -885 : (3, 8439), -883 : (12, 1), -881 : (11, 1), -879 : (4, 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