Title: Collatz Conjecture Confirmed Through Connectivity of Odd and 8mod12 Positive Integers

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Abstract: In this paper, I will prove all orbits beginning with an odd positive integer link incrementally through connected orbits to a power of two orbit. Orbits connecting to a power of two orbit must decrease to one because connected orbits share values and finish the same way. These orbits connect because of a function I defined based on the connection between all odd and $8\text{mod}12$ positive integers. Applying this function infinitely many times to all odd positive integers always leads to a connection to power of two orbits. With these results, I'm able to prove the conjecture.

Keywords: Collatz function, orbit, trivial cycle, reverse Collatz relation, connecting orbits, 2-adic order, non-trivial cycle, bijection, cardinality, countably infinite, multiset

Introduction: The **Collatz function** is defined below where $n \in \mathbb{N}$:

$$C(n) = \left\{\begin{array}{|c|}\hline \frac{n}{2}, n \equiv 0\text{mod}(2) \\ \hline 3n + 1, n \equiv 1\text{mod}(2) \\ \hline\end{array}\right\}.$$

The Collatz conjecture starts with substituting any positive integer n into the function above. If the value n is even, the result is $\frac{n}{2}$. If the value n is odd, the result is $3n + 1$. The conjecture says continuing this process for any positive integer n always leads to one.

I define the **orbit** of a number n as the sequence $C^m(n)$ where $m \in \mathbb{N}_0$. Similarly, I define the set $R(n)$ by

$R_0(n) = C^0(n) = n$

$R_i(n) = C^i(n)$

where $i \in \mathbb{N}$, such that $R_a(n) \neq R_b(n)$ for $a, b \in \mathbb{N}$ removing all repeating cycles from the set $R(n)$. For example, when $n = 5$, the orbit is shown below.

$C(5) = 5 \cdot 3 + 1 = 16$

$C(16) = \frac{16}{2} = 8$

$C(8) = \frac{8}{2} = 4$

$C(4) = \frac{4}{2} = 2$

$C(2) = \frac{2}{2} = 1$

$C(1) = 1 \cdot 3 + 1 = 4$

$C(4) = \frac{4}{2} = 2$

$C(2) = \frac{2}{2} = 1$

$C(1) = 1 \cdot 3 + 1 = 4$

This 4, 2, 1 **trivial cycle** continues to repeat to infinity, so $R(5) = \{5, 16, 8, 4, 2, 1\}$. Since $1 \in R(5)$, every orbit containing 5 also contains 1. Similarly, $R(n)$ contains 1 for any positive integer n that satisfies the conjecture. "As of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \approx 2.95 \times 10^{20}$."[1]

Another way to study the Collatz conjecture is to follow orbits backwards. This **reverse Collatz relation**, where $n \in \mathbb{N}$, can be written as:

$$I(n) = \left\{ \boxed{\begin{array}{c} \{2n\}, n \equiv 0,1,2,3,5 \bmod(6) \\ \hline \left\{2n, \frac{n-1}{3}\right\}, n \equiv 4\bmod(6) \end{array}} \right\}.$$

For each case, I show $C(I(n)) = n$. $I(n) = 2n$ for all values of n, and $C(2n) = \frac{2n}{2} = n$ because the function divides its input by 2 if it is even. $I(n) = \frac{n-1}{3}$ for all $n \equiv 4\bmod(6)$, because $\frac{n-1}{3} \equiv 1\bmod(2)$ if and only if $3(\frac{n-1}{3}) + 1 = n \equiv 4\bmod(6)$. This means that $C(I(n)) = C(\frac{n-1}{3}) = 3(\frac{n-1}{3}) + 1 = n$ because the function multiplies its input by three and adds one if it's odd.

After studying many orbits, certain connections became apparent. The most important was the connection between odd and $8\bmod(12)$ positive integers. Two positive integers $q, s \in \mathbb{N}$ have **connecting orbits** if there exists at least one value $t \in \mathbb{N}$ such that $t \in R(q)$ and $t \in R(s)$.

Proposition: For $d, k \in \mathbb{N}_0$, if $a = 2k + 1$, then $b \in R(a)$ and $b \in R(2^d)$ for some value $b \in \mathbb{N}$ proving all orbits beginning with an odd positive integer must share values with a power of two orbit and decrease to one

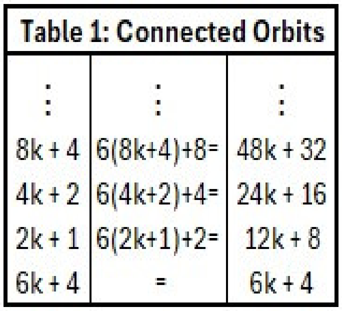


Table 1: Connected Orbits

⋮	⋮	⋮
8k + 4	6(8k+4)+8=	48k + 32
4k + 2	6(4k+2)+4=	24k + 16
2k + 1	6(2k+1)+2=	12k + 8
6k + 4	=	6k + 4

Proof: Table 1 illustrates the odd and $8\bmod(12)$ positive integer connection and how their orbits are structured going backwards using only the $I(n) = 2n$ case of the reverse Collatz relation. Going forward in the orbits of $2k + 1$ and $12k + 8$ where $k \in \mathbb{N}_0$ gives $6k + 4$, which connects both orbits according to the definition of the Collatz function as seen in the equations below. Equation 1 shows how each positive odd integer connects to a distinct $8\bmod(12)$ integer.

$$C(2k + 1) = C(12k + 8)$$

We know that $2k + 1$ is odd, and $12k + 8$ is even, so applying the Collatz function once gives

$$3(2k + 1) + 1 = \frac{12k+8}{2} = 6k + 4.$$

Multiplying by two gives

$$6(2k + 1) + 2 = 12k + 8. \qquad (1)$$

Since all odd positive integers $2k + 1$ connect to $6k + 4$ as seen in the equations above and all even integers can be obtained from iterating $I(n) = 2n$ an infinite amount of times going backwards on each odd integer, all even integers obtained from these iterations must also connect to $6k + 4$. After applying the reverse Collatz relation to $2k + 1$ and $12k + 8$ integers v times where $v \in \mathbb{N}_0$, the Collatz function must be applied $v + 1$ times to connect to $6k + 4$. These connections are seen in the equations below. Equations 2, 3, and 4 show how each integer obtained by iterating $I(n) = 2n$ on $2k + 1$ a certain number of times must have a connecting orbit with the integer obtained by iterating $I(n) = 2n$ on $12k + 8$ the same amount of times.

Connecting orbits starting with $I(2k + 1) = 2(2k + 1) = 4k + 2$ and $I(12k + 8) = 2(12k + 8) = 24k + 16$ gives the equations below.

$$C^2(4k + 2) = C^2(24k + 16)$$

Because $4k + 2$ and $24k + 16$ are both divisible by two, iterating the Collatz function once gives

$$C(\frac{4k+2}{2}) = C(\frac{24k+16}{2}).$$

We know that $\frac{4k+2}{2} = 2k + 1$ is odd, and $\frac{24k+16}{2} = 12k + 8$ is even, so applying the Collatz function one more time on both orbits gives

$$3(\frac{4k+2}{2}) + 1 = \frac{24k+16}{4} = 6k + 4,$$

and multiplying by four gives

$6(4k + 2) + 4 = 24k + 16.$ (2)

Starting with $I^2(2k + 1) = 2^2(2k + 1) = 8k + 4$ and $I^2(12k + 8) = 2^2(12k + 8) = 48k + 32$, it follows that

$C^3(8k + 4) = C^3(48k + 32).$

Because $8k + 4$ and $48k + 32$ are both divisible by 2^2, iterating the Collatz function twice gives

$C(\frac{8k+4}{4}) = C(\frac{48k+32}{4}).$

We know that $\frac{8k+4}{4} = 2k + 1$ is odd, and $\frac{48k+32}{4} = 12k + 8$ is even, so applying the Collatz function one more time on both orbits gives

$3(\frac{8k+4}{4}) + 1 = \frac{48k+32}{8} = 6k + 4,$

and multiplying by eight gives

$6(8k + 4) + 8 = 48k + 32.$ (3)

Since $2k + 1$ and $12k + 8$ connect to $6k + 4$, every integer obtained from

$I^v(2k + 1) = 2^v(2k + 1) = 2^{v+1}k + 2^v$

and

$I^v(12k + 8) = 2^v(12k + 8) = 3k * 2^{v+2} + 2^{v+3}$

where $v \in \mathbb{N}_0$ must also connect to $6k + 4$. Connecting the values $2^{v+1}k + 2^v$ and $3k * 2^{v+2} + 2^{v+3}$ to $6k + 4$ gives the equations below.

$C^{v+1}(2^{v+1}k + 2^v) = C^{v+1}(3k * 2^{v+2} + 2^{v+3})$

Because $2^{v+1}k + 2^v$ and $3k * 2^{v+2} + 2^{v+3}$ are both divisible by 2^v, iterating the Collatz function v times gives

$C(\frac{2^{v+1}k+2^v}{2^v}) = C(\frac{3k*2^{v+2}+2^{v+3}}{2^v}).$

We know that $\frac{2^{v+1}k+2^v}{2^v} = 2k + 1$ is odd, and $\frac{3k*2^{v+2}+2^{v+3}}{2^v} = 12k + 8$ is even, so applying the Collatz function one more time on both orbits gives

$3(\frac{2^{v+1}k+2^v}{2^v}) + 1 = \frac{3k*2^{v+2}+2^{v+3}}{2^{v+1}} = 6k + 4$

Multiplying by 2^{v+1} gives

$6(2^{v+1}k + 2^v) + 2^{v+1} = 3k * 2^{v+2} + 2^{v+3}.$ (4)

The left side of Equation 4 gives the method for finding a connecting orbit for all positive integers written in the form $2^v(2k + 1) = 2^{v+1}k + 2^v$. As we have seen in Equations 1, 2, 3, and 4, finding a connecting orbit involves multiplying a positive integer by six and adding some power of two. According to the left side of Equation 4, the power of two added after multiplying by 6 would be 2^{v+1}. When a positive integer is written in the form

$2^v(2k + 1) = 2^{v+1}k + 2^v$

where $k, v \in \mathbb{N}_0$, the 2-adic order of that number is v. To calculate the number of times a positive integer is divisible by two before reaching an odd, denoted as the **2-adic order**, I can derive a function $f(x)$ for all $x \in \mathbb{N}$.

According to Legendre's formula, the number of times $x!$ is divisible by a prime p is:

$V_p(x!) = \sum_{n=1}^{\infty} \lfloor \frac{x}{p^n} \rfloor$. [2]

To calculate the 2-adic order, the value of p is two.

The expression $\lfloor \frac{x}{2^n} \rfloor$ counts the number of times x is divisible by 2^n without remainder, and $\frac{x}{2^n}$ counts the number of times x is divisible by 2^n with remainder.

Subtracting these expressions gives $\frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor$, which will output 0 when divisible by 2^n and a number in $(0, 1)$ when not divisible by 2^n.

Take the ceiling function so there can only be two outputs, $\lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 0$ when x is divisible by 2^n, and $\lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 1$ when x is not divisible by 2^n.

To switch these values, take one minus the ceiling function, so that $1 - \lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 1$ when x is divisible by 2^n and $1 - \lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 0$ when x is not divisible by 2^n.

Take the summation for $n = 1$ to $\log_2 x$, because any given value of x can be divisible by two at most $\log_2 x$ times before reaching an odd. Now, I define the function $f(x)$ which gives the 2-adic order of x for all $x \in \mathbb{N}$ as shown below.

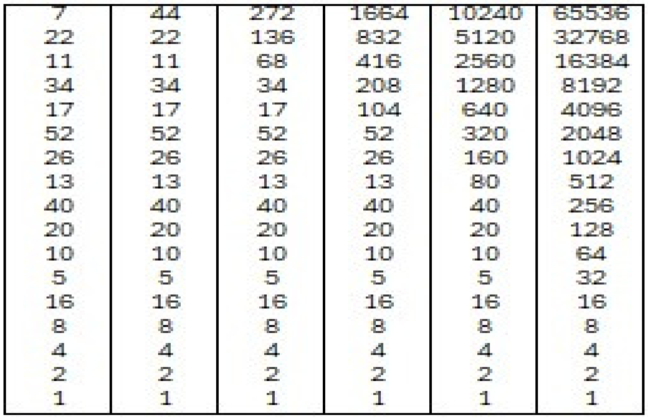
$f(x) = \sum_{n=1}^{\log_2 x} 1 - \lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil$ [3]

From the left side of equation 4, we know that finding a connected orbit for any positive integer involves multiplying by six and adding 2^{v+1}. We know v is the 2-adic order of this positive integer, so I will substitute the function f in for v giving the simplified connected orbits function g defined as

$g(x) = 6x + 2^{f(x)+1}$ where $x \in \mathbb{N}$.

The connected orbits created by the function g are seen below for the odd starting value seven. Figure 1 shows the orbit of seven and its connection to the first power of two orbit. The starting numbers of these connected orbits come from $g(7) = 44$, $g(44) = 272$, $g(272) = 1,664$, $g(1,664) = 10,240$, and $g(10,240) = 65,536$. These starting numbers contain the $8 \mathrm{mod}(12)$ integers used to replace the first five odd integers contained in the orbit of seven. The orbit of seven has six odd integers and each consecutive connecting orbit has one fewer odd than the previous orbit which eventually leads to a connection to an orbit with only one odd integer or power of two orbit. Since all connecting orbits must finish with the same values, the orbit of seven must decrease to one like the power of two orbit it has been connected to.

Figure 1: Orbit of 7 Connecting to One Power of 2 Orbit



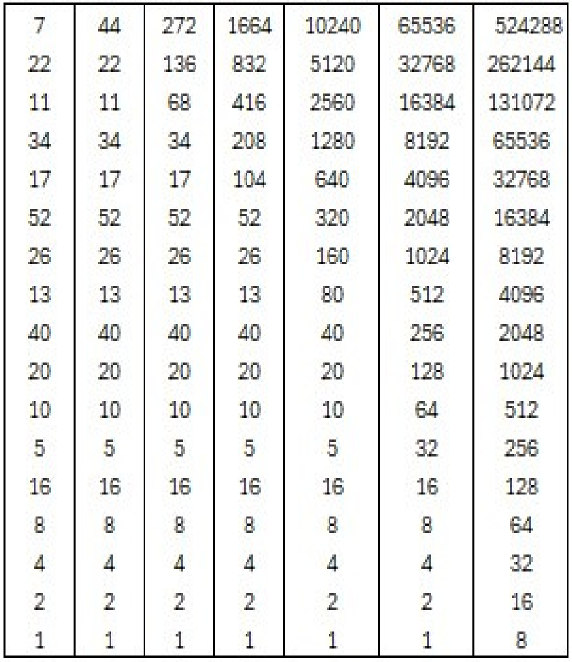
7	44	272	1664	10240	65536
22	22	136	832	5120	32768
11	11	68	416	2560	16384
34	34	34	208	1280	8192
17	17	17	104	640	4096
52	52	52	52	320	2048
26	26	26	26	160	1024
13	13	13	13	80	512
40	40	40	40	40	256
20	20	20	20	20	128
10	10	10	10	10	64
5	5	5	5	5	32
16	16	16	16	16	16
8	8	8	8	8	8
4	4	4	4	4	4
2	2	2	2	2	2
1	1	1	1	1	1

Figures 2a and 2b seen below show how the connection to power of two orbits for the odd integer seven could continue because the function g can be applied to the starting values of each connected orbit to infinity. Running the function g on 65,536 gives

$$g(65,536) = 6(65,536) + 2^{16+1} = 6(65,536) + 2^{17} = 524,288$$

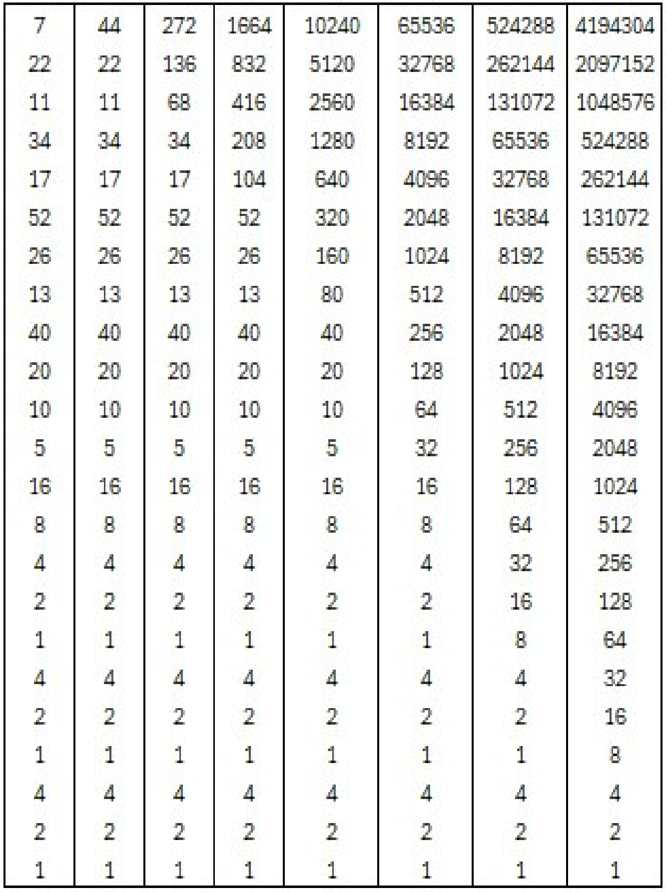
which is the next power of two orbit that seven is connected to. All previous connected orbits end with one, but this orbit would end with eight implying that it is an even only orbit as seen in Figure 2a. Since the previous orbit had one odd and the function g is replacing an odd with an $8\text{mod}12$ in each iteration, the next connecting orbit must have zero odd integers. Because orbits must contain at least one odd integer, trivial cycles have been added in Figure 2b, so all connecting orbits will have at least one odd and all finish at one. This is why the orbit of seven has two additional trivial cycles added to compensate for the two additional power of two orbits connected. Since all starting values up to 2^{68} have been proven by computer to go to one, they would all follow the same pattern as seven and connect to an infinite amount of power of two orbits through infinite iterations of g.

Figure 2a: Orbit of 7 Connecting to Orbit With No Odds



7	44	272	1664	10240	65536	524288
22	22	136	832	5120	32768	262144
11	11	68	416	2560	16384	131072
34	34	34	208	1280	8192	65536
17	17	17	104	640	4096	32768
52	52	52	52	320	2048	16384
26	26	26	26	160	1024	8192
13	13	13	13	80	512	4096
40	40	40	40	40	256	2048
20	20	20	20	20	128	1024
10	10	10	10	10	64	512
5	5	5	5	5	32	256
16	16	16	16	16	16	128
8	8	8	8	8	8	64
4	4	4	4	4	4	32
2	2	2	2	2	2	16
1	1	1	1	1	1	8

Figure 2b: More Power of 2 Orbits Connected With Trivial Cycle



7	44	272	1664	10240	65536	524288	4194304
22	22	136	832	5120	32768	262144	2097152
11	11	68	416	2560	16384	131072	1048576
34	34	34	208	1280	8192	65536	524288
17	17	17	104	640	4096	32768	262144
52	52	52	52	320	2048	16384	131072
26	26	26	26	160	1024	8192	65536
13	13	13	13	80	512	4096	32768
40	40	40	40	40	256	2048	16384
20	20	20	20	20	128	1024	8192
10	10	10	10	10	64	512	4096
5	5	5	5	5	32	256	2048
16	16	16	16	16	16	128	1024
8	8	8	8	8	8	64	512
4	4	4	4	4	4	32	256
2	2	2	2	2	2	16	128
1	1	1	1	1	1	8	64
4	4	4	4	4	4	4	32
2	2	2	2	2	2	2	16
1	1	1	1	1	1	1	8
4	4	4	4	4	4	4	4
2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1

I will now summarize what the function g is doing in the connected orbit process. First, remember that the function g is only giving the initial values of each consecutive connecting orbit and the full orbits have been given for illustration purposes to show how each orbit is connected to the next and to show that all orbits would be the same length. This connecting orbit process would not be needed if we already had the first orbit beginning with an odd because we could just look and see if it goes to one. Since we always start this process with an orbit beginning with a odd positive integer, this first odd orbit would contain x odd integers and the 8mod12 orbit would contain $x - 1$ odd integers.This will always be true because these orbits are identical except for the initial values. The connecting orbit process can also be iterated to infinity because each initial value after the first odd integer will always be even. We know this is true because the second initial value will always be an $8\text{mod}12$, and each initial value after this is coming from multiplying the previous initial value by six and adding some power of two to obtain the next initial value. Since these values are always even, they will always divide to an odd integer that will connect to an $8\text{mod}12$ contained in the next connecting orbit. The function g gives the initial value of the next orbit containing the $8\text{mod}12$ used to replace the previous odd. Because this process can be iterated to infinity, an infinite number of odd integers can be replaced from any initial odd orbit in this connecting orbit process. Since the function g is using the odd and $8\text{mod}12$ connection to obtain the initial values of each consecutive connecting orbit, each connecting orbit will finish the same way and contain one less odd than the previous orbit. If a connection to a power of two orbit is established, the initial odd orbit must decrease to one like the power of two orbit it has been connected to. The last important thing to notice about this connecting orbits process is that it also gives an actual purpose for the existence of the repeating 4,2,1 trivial cycle as seen above in Figure 2b.

Another easier way to see that this connected orbit process works on the orbit of any odd integer is to just follow the odd and $8\text{mod}12$ positive integer connection the entire way down. I can start with any odd integer which will connect to a distinct $8\text{mod}12$ integer. Dividing this $8\text{mod}12$ integer by two enough times will give the second odd integer contained in the first odd orbit. This second odd integer can be connected to an $8\text{mod}12$ integer that will be divided by two enough times to give the third odd integer contained in the first odd orbit, and this can continue to infinity because each odd integer must connect to an $8\text{mod}12$ and each $8\text{mod}12$ integer divided by two enough times will always give an odd integer contained in the first odd orbit. The function g is giving the initial values of the connecting orbits that would contain each of the odd and $8\text{mod}12$ integers in this process. Since an odd integer is being replaced with an $8\text{mod}12$ with each iteration either way you do it and both processes can be iterated to infinity, an infinite number of odd integers can be replaced with $8\text{mod}12$ integers from any orbit. Replacing all odd integers from a beginning odd orbit with $8\text{mod}12$ integers would connect it to a second power of two orbit containing no odd integers as seen in Figure 2a confirming this beginning odd orbit must decrease to one.

Showing all orbits decrease to one would disprove the non-trivial cycle and infinite increasing odd counter examples. A **non-trivial cycle** is an orbit with a repeating cycle that does not contain one. The infinite increasing odd orbit contains distinct odd integers that increase to infinity instead of decreasing to one. To prove these orbits could not exist, I must show that all odd positive integers contained in every orbit can be replaced with $8\text{mod}12$ integers. To do this, I will use a bijection. The bijection is used to show that all odd positive integers can be replaced with $8\text{mod}12$ integers in the connecting orbit process, and the function g can be established as a bijective function by showing that it has an inverse g^{-1}, such that

$g^{-1}(x) = \frac{x-2}{6}$ where $x \equiv 8\text{mod}(12)$.

The **bijection** must also be injective and surjective. Injective means that no odd can be paired with more than one $8\text{mod}(12)$, and no $8\text{mod}(12)$ can be paired with more than one odd. Surjective means that every odd pairs with at least one $8\text{mod}(12)$, and every $8\text{mod}(12)$ pairs with at least one odd. Applying the function g to an odd $2j+1$ where $j \in \mathbb{N}_0$ gives

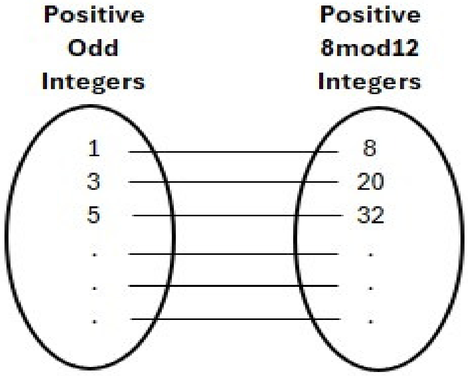
$g(2j+1) = 6(2j+1) + 2^{f(2j+1)+1} = 6(2j+1) + 2 = 8 + 12j \equiv 8\text{mod}(12)$.

Likewise, applying g^{-1} to a value $8 + 12j$, where $j \in \mathbb{N}_0$ gives

$g^{-1}(8+12j) = \frac{8+12j-2}{6} = 2j+1$,

so it is surjective. Assume a and b are distinct odd integers. If they were paired with the same $8\text{mod}(12)$ value through the function g, $6a + 2 = 6b + 2$, which could only be true if $a = b$. Likewise, if $6a + 2$ and $6b + 2$ are distinct $8\text{mod}(12)$ values paired with the same odd through g^{-1}, $a = b$, so it is injective as well.

To disprove the infinite increasing odd counter example, I use the function g as a bijection on the set of odd positive integers as seen below.



Remember, the infinite increasing odd orbit could only contain distinct odd integers greater than 2^{68}, and the bijection above clearly shows that the entire set of odd positive integers can be replaced with $8\bmod 12$ integers in the connecting orbit process. Since these infinite sets are mapped with bijective functions g and g^{-1}, they must have the same **cardinality** which means these infinite sets have the same number of elements. We also know these sets have the same cardinality from equation 1 that shows $6(2k+1)+2=12k+8$ for all $k \in \mathbb{N}_0$. These sets are also subsets of $\mathbb{N}$, and "any subset of the natural numbers is countable…A countable set that is not finite is said to be **countably infinite**."[4] Since both sets are the same size due to the bijection and are countably infinite, all odd positive integers contained in all orbits would be replaced with $8\bmod 12$ positive integers using the connected orbit process at infinity. Replacing all odd integers with $8\bmod 12$ integers using the connecting orbit process guarantees a connection to a power of two orbit at infinity, so the infinite increasing odd counter example could not exist.

To disprove the non-trivial cycle counter example, I will use the function g as a bijection on the set of odd integers contained in this orbit. When creating the set of odd integers to be replaced from a non-trivial cycle orbit, we need to know how to handle the infinite amount of duplicates. There are two ways of dealing with repeated values entered into a set. The first involves only entering a repeated value once because the number of times an element appears does not matter. The other way involves entering all repeated values because the number of times an element appears does matter. This is called a **multiset**[5]. An example of a multiset would be a set containing the prime factors of an integer. The prime factorization of $36=2^2*3^2$ which gives the multiset {2,2,3,3}. Creating the set of odd integers to be replaced for the non-trivial cycle counter example must have finite cardinality because the number of times these elements appear does not matter. For example, running an orbit on seven finishes after one instance of the trivial cycle. Repeating the trivial cycle more than once does not change the orbit. As we saw earlier, the only time the trivial cycle needs repeated is when we are connecting orbits to additional power of two orbits as seen in Figure 2b. In fact, a Mathematics Stack Exchange webpage mentions the idea of total stopping time, which measures the number of steps from the initial value to the first instance of one in a Collatz sequence[6]. This confirms that further repetitions of the trivial cycle do not matter and the non-trivial cycle would be treated the same way. Since I am only running the function g as a bijection on one cycle of the non-trivial cycle, the set of odd integers would be finite, so the function g must connect the non-trivial cycle to a power of two orbit. Even if we consider infinite repetitions of the non-trivial cycle, all odd integers must be replaced at infinity since all odd integers would be replaced with $8\bmod 12$ integers in each finite cycle. Since I can replace all odd integers with $8\bmod 12$ integers in the first cycle and each cycle would contain the same odd integers, I can replace them in all cycles. This proves that the trivial cycle is the only cycle that could exist which confirms that all odd positive integers must connect to a power of two orbit and decrease to one. ☐

Now that we know all orbits go to one going forward, I wanted to show how these orbits could be structured going backwards from one. To do this, I will use the propositions below.

Proposition: For $n \in \mathbb{N}$***, if*** $3 \nmid n$***, then*** $n \in R((2x+1)*3)$ ***for*** $x \in \mathbb{N}_0$

Proof: Examining integers that are elements of set X in $\mathrm{mod}(9)$, where $X=\mathbb{Z}^+ - 3\mathbb{Z}^+$ gives residue classes of $1, 2$

$, 4, 5, 7,$ and $8 \bmod(9)$. From the division algorithm, we can rewrite all numbers of these modulo classes as $9z+1$, $9z+2, 9z+4, 9z+5, 9z+7,$ or $9z+8$ where $z \in \mathbb{N}_0$. Using the reverse Collatz relation to go backwards, we are able to multiply these values by powers of two of the form 2^y, where $y \in \mathbb{N}$. The powers of two will repeat $\bmod(9)$ with a period of 64 and are listed below.

$2 \equiv 2 \bmod(9)$

$4 \equiv 4 \bmod(9)$

$8 \equiv 8 \bmod(9)$

$16 \equiv 7 \bmod(9)$

$32 \equiv 5 \bmod(9)$

$64 \equiv 1 \bmod(9)$

The powers of two and elements of X contain the same residues $\bmod(9)$, and are both the set of residue classes relatively prime to 9. This means each set will contain the other's inverses $\bmod(9)$. If we multiply a residue class $\bmod(9)$ that is an element of X by its inverse, a power of two, we get an integer that is $1 \bmod(9)$ as listed below.

$(9z+1) \cdot 64 \equiv 1 \cdot 1 \equiv 1 \bmod(9)$

$(9z+2) \cdot 32 \equiv 2 \cdot 5 \equiv 1 \bmod(9)$

$(9z+4) \cdot 16 \equiv 4 \cdot 7 \equiv 1 \bmod(9)$

$(9z+5) \cdot 2 \equiv 5 \cdot 2 \equiv 1 \bmod(9)$

$(9z+7) \cdot 4 \equiv 7 \cdot 4 \equiv 1 \bmod(9)$

$(9z+8) \cdot 8 \equiv 8 \cdot 8 \equiv 1 \bmod(9)$

The value following an odd integer $3s$ in its orbit, according to the Collatz function, is $3(3s)+1 = 9s+1$. Since $3s$ is odd, s is odd, so s can be written as $s = 2p+1$ where $p \in \mathbb{N}_0$. This means $9s+1$ is even and can be written as

$9s+1 = 9(2p+1)+1 = 18p+10 = 6(3p+1)+4$, so $9s+1 \equiv 4 \bmod(6)$.

The reverse Collatz relation can now be used with the $I(n) = \frac{n-1}{3}$ case to obtain $\frac{9s+1-1}{3} = 3s$, which is an odd multiple of three. As a result, we can trace a number in set X back to an even number that is $1 \bmod(9)$, and that value can be traced back to an odd multiple of three. This confirms all elements of set X are contained in the orbits of odd multiples of three. $\square$

To see how this process works, choose a starting value that is an element of X. For example, we use the starting value 61. Since 61 is $7 \bmod(9)$, that means it will be multiplied by it's modular inverse $\bmod(9)$, which is 4. This gives $j = 61 \cdot 4 = 244$. Now, we have $3s = \frac{244-1}{3} = 81$. Thus, 61 will be contained in the orbit of 81. Assume the starting value is 128 which is $2 \bmod(9)$, so we multiply it by it's inverse $\bmod(9)$, which is 32. This gives $j = 128 \cdot 32 = 4096$. Finally, $3s = \frac{4096-1}{3} = 1365$, and 128 is contained in the orbit of 1365. Thus, all values in X can be traced back to an even $1 \bmod(9)$ value, which can be traced back to an odd multiple of three.

Proposition: For $m, n \in \mathbb{N}$ ***and*** $k \in \mathbb{N}_0$***, if*** $n = (2k+1)(3)$, ***then*** $n \in R(6m)$ ***where*** $6m = n \cdot 2^q$ ***for all*** $q \in \mathbb{N}$

Proof: If $n = (2k+1)(3)$, then $n = 6k+3$, and $n \bmod(6) = 3$. Using the reverse Collatz relation, $I(n) = 2n = 2(6k+3) = 12k+6$. Iterating the reverse Collatz relation on $12k+6 = 6(2k+1) \equiv 0 \bmod(6)$ gives $I(2n) = 4n = 4(6k+3) = 24k+12 = 6(4k+2) \equiv 0 \bmod(6)$. Any multiple of three multiplied by a power of two will be a multiple of six, and thus could not be equivalent to $4 \bmod(6)$. This means the only numbers preceding n in their orbit will be n times a power of two, which are multiples of six. Thus, the orbits of all possible values of n are contained in the orbits of all multiples of six. ☐

The previous two propositions confirm that all natural numbers not divisible by three can be traced back to an odd multiple of three going backwards. Continuing to go backwards from this odd multiple of three will give multiples of six to infinity. Following this process would give orbits that contain a finite number of distinct odd integers going backwards from one.

These propositions and some additional modular arithmetic also show there could be orbits containing an infinite number of distinct odd integers going backwards from one. Every odd integer not divisible by three multiplied by two going backwards will generate a 2 or $4\text{mod}(6)$ because these integers are all even. There would never be a $0\text{mod}(6)$ integer in this process because those only come from multiplying the odd multiple of three integers by two. Multiplying these 2 and $4\text{mod}(6)$ integers by two going backwards will give an alternating pattern of 2 and $4\text{mod}(6)$ integers to infinity as seen below for $k \in \mathbb{N}_0$.

$$2(6k+2) = 12k+4 = (6*2)k+4 = 6(2k)+4 \equiv 4\text{mod}(6)$$

$$2(6k+4) = 12k+8 = (6*2)k+8 = 6(2k)+8 = 6(2k+1)+2 \equiv 2\text{mod}(6)$$

This tells us that there will be a $4\text{mod}(6)$ every other time we multiply by two going backwards. Now all we have to do is make sure this $4\text{mod}(6)$ integer is not also a $1\text{mod}(9)$, so we can subtract one from this integer and divide that result by three giving a distinct odd integer that is not an odd multiple of three. As we learned in the first proposition above, multiplying the odd integers by two going backwards will have a repeating $2, 4, 8, 7, 5$, and $1\text{mod}(9)$ sequence because of the pattern of powers of two in $\text{mod}(9)$. Because the repeating cycle in $\text{mod}(9)$ contains six distinct $\text{mod}(9)$ values and we know half of these will be $4\text{mod}(6)$, there will always be a $4\text{mod}(6)$ that is not $1\text{mod}(9)$ in each repeating $\text{mod}(9)$ cycle going backwards. Subtracting one from these integers and dividing this result by three will always give a distinct odd integer that is not divisible by three. Because these $\text{mod}(6)$ and $\text{mod}(9)$ patterns continue to infinity going backwards, orbits could also contain an infinite number of distinct odd integers going backwards from one.

Conclusion: Since all odd positive integers connect to powers of two through orbits connected by the function g, all odd positive integers reach one when applying the Collatz function. Because all odd positive integers are contained in the orbits of all even positive integers, all even positive integers must reach one as well, which confirms the Collatz Conjecture is true.

The purpose of this paper was to prove all positive integers decrease to one when put into the Collatz function, but this connecting orbit process also works for all negative integers after making some minor changes to the Collatz and g functions. The modified Collatz and g functions are defined below for n and $x \in \mathbb{Z}^-$.

$$T(n) = \left\{ \boxed{\begin{array}{c} \frac{n}{2}, n \equiv 0\text{mod}(2) \\ \hline 3n-1, n \equiv 1\text{mod}(2) \end{array}} \right\}$$

$$m(x) = 6x - 2^{f(x)+1}$$

The connected orbits created by the T and m functions are seen below for the odd starting value negative seven. Figures 1 and 3 are identical except for the negative signs proving all negative integers would connect to negative one using the modified functions for negative integers.

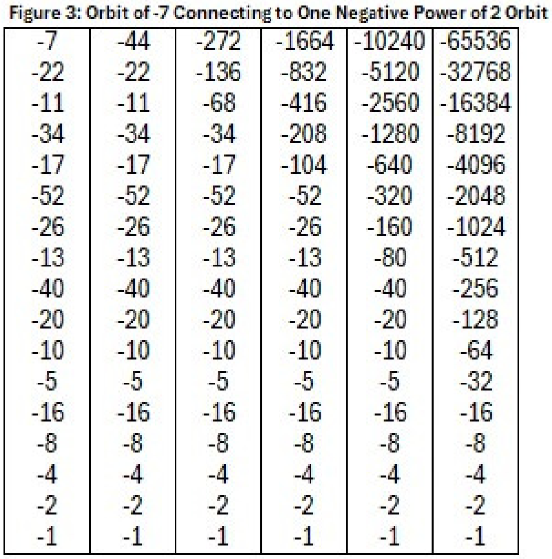


Figure 3: Orbit of -7 Connecting to One Negative Power of 2 Orbit

-7	-44	-272	-1664	-10240	-65536
-22	-22	-136	-832	-5120	-32768
-11	-11	-68	-416	-2560	-16384
-34	-34	-34	-208	-1280	-8192
-17	-17	-17	-104	-640	-4096
-52	-52	-52	-52	-320	-2048
-26	-26	-26	-26	-160	-1024
-13	-13	-13	-13	-80	-512
-40	-40	-40	-40	-40	-256
-20	-20	-20	-20	-20	-128
-10	-10	-10	-10	-10	-64
-5	-5	-5	-5	-5	-32
-16	-16	-16	-16	-16	-16
-8	-8	-8	-8	-8	-8
-4	-4	-4	-4	-4	-4
-2	-2	-2	-2	-2	-2
-1	-1	-1	-1	-1	-1

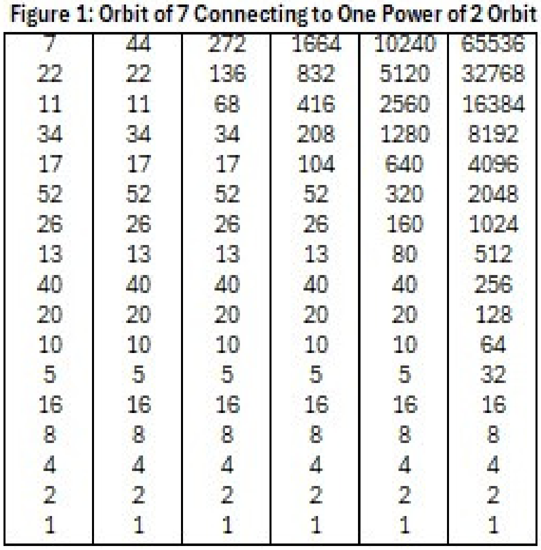


Figure 1: Orbit of 7 Connecting to One Power of 2 Orbit

7	44	272	1664	10240	65536
22	22	136	832	5120	32768
11	11	68	416	2560	16384
34	34	34	208	1280	8192
17	17	17	104	640	4096
52	52	52	52	320	2048
26	26	26	26	160	1024
13	13	13	13	80	512
40	40	40	40	40	256
20	20	20	20	20	128
10	10	10	10	10	64
5	5	5	5	5	32
16	16	16	16	16	16
8	8	8	8	8	8
4	4	4	4	4	4
2	2	2	2	2	2
1	1	1	1	1	1

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