Skill in Backgammon: Cubeful vs Cubeless

Tilemachos Zoidis, May 2026
bloarootz@yahoo.gr

Abstract. This paper presents GNU Backgammon (GNUbg) rollouts between unequally skilled players, indicating that the use of the doubling cube does not confer a systematic advantage to the stronger player. The implications of these findings for the Elo rating system are also examined.

Keywords: Backgammon, Cubeful, Cubeless, Portes, Tavli, Skill, Elo

1. Introduction

Luck is perhaps the most frequent source of frustration among backgammon players. The introduction of the doubling cube—relatively recent in comparison to the game's long history—added a strategic dimension that has been viewed as a mechanism for mitigating the role of chance. But is this actually true? And if so, does it hold in both money and match play? To address these questions, we used GNUbg to perform rollouts between unequally skilled players. The results indicate that the use of the doubling cube does not, in fact, confer a systematic advantage to the stronger player in either format. These findings have direct implications for the Elo rating system, which are explored in depth.

2. Money Play

We consider 3 types of money games: *cubeful games* with the Jacoby rule, *cubeless games* with gammons and backgammons counting and *gammonless games* without the doubling cube. To compare the skill in these formats, two quantities are required for each: the equity (E) of the stronger player and the expected value (V) of the winner (assuming optimal play).

2.1. Expected Value of Money Games

Using GNUbg 2ply we rolled out 19,440 cubeless games with *Variance Reduction* (VR). These simulations produced a gammon rate of 27.74% and a backgammon rate of 1.23% for an expected value of **1.2897ppg**. An additional rollout was performed under the same parameters for cubeful games which produced an expected value of **2.385ppg**.

2.2. Equity of the Stronger Player

For each format **1,399,680 games** were rolled out with GNUbg playing one side at 1ply and the other at 0ply. These simulations were conducted without VR. VR works by subtracting the sum of the equity differences between two consecutive plies from the final result. This procedure is often described as canceling out the estimated luck but it's equivalent to think about it as using subsequent evaluations to estimate the error in previous ones. However, that error is precisely what we want to measure. The results are presented in the table below.

Table 1. Outcome Equities (1ply vs 0ply)

Format	Lose BG	Lose G	Win	Win G	Win BG	**E**	SE
Gammonless	-	-	0.5147	-	-	**0.0293**	0.0008
Cubeless	0.0058	0.1318	0.5108	0.1426	0.0063	**0.0329**	0.0011
Cubeful	0.0062	0.1353	0.5090	0.1420	0.0063	**0.0565**	0.0025

2.3. Comparing Money Formats

If we just compared the equities of the stronger player, cubeful games would appear to be the most skillful. However, this conclusion might simply reflect the greater number of points at stake. A more appropriate comparison therefore adjusts these equities by the expected value of the format—that is, by considering the proportion of the average points at stake that the stronger player is able to capture in each format. This normalization is economically coherent. When more points are at stake, rational players will bet less money per point. If we assume that players are willing to risk a fixed amount of money per game, then the amount risked per point must be inversely proportional to the game's expected value. Using the ratio E/V as a measure of skill, the ranking is reversed: gammonless games emerge as the most skillful.

Table 2. Comparison of Money Formats

Format	E	V	**E/V**
Gammonless	0.0293	1.0000	**0.0293**
Cubeless	0.0329	1.2897	**0.0255**
Cubeful	0.0565	2.3850	**0.0237**

3. Match Play

We again consider 3 types of matches: *cubeful matches* with the Crawford rule, *cubeless matches* with gammons and backgammons counting and *gammonless matches* without the doubling cube. To compare the skill in these formats, two quantities are required for each: the probability that the stronger player wins an N-point match and its expected duration (assuming optimal play).

3.1. Defining Skill in Match Play

A natural way to define the skill S of a match relative to a gammonless game is as the ratio of the corresponding expectations of the stronger player:

$$S = \frac{2P - 1}{2W - 1} \tag{1}$$

where P and W denote the probabilities that the stronger player wins the match and the game, respectively. Under this definition, the larger the skill difference of the players is, the smaller the corresponding skill values will be.

An alternative approach derives skill from the Elo system. In chess, the rating difference between two players determines the expected score in a single game. In backgammon, however, the Elo difference corresponding to a 1-point match is scaled by a factor that depends on match length, reflecting the greater advantage of the stronger player in longer matches. This scaling factor may be interpreted as the skill associated with that particular match length. According to the Elo model, the probability that the stronger player wins an N-point match is given by

$$P(N) = \frac{1}{1 + 10^{-\frac{|\Delta R|}{C} S(N)}} \tag{2}$$

where R denotes a player’s rating and C is a constant. In backgammon, $S(N) = \sqrt{N}$ is typically used as the skill function [1]. However, it was adopted on theoretical rather than empirical grounds. Our approach instead uses rollout data to approximate the skill function directly. If we normalize the skill of a 1-point match to one unit, we can solve for the skill function:

$$S(N) = \frac{\log\left(\frac{1}{P(N)} - 1\right)}{\log\left(\frac{1}{W} - 1\right)} \tag{3}$$

where W denotes the probability that the stronger player wins a 1-point match. Under that definition, the skill values increase with the absolute value of the Elo difference. Although equations (1) and (3) might appear very different, they are actually equivalent in the limit as $W \rightarrow 1/2$. Because our interest lies in the opportunity for skill inherent in a match—that is, the extent to which it allows skill to influence the outcome regardless of the players' rating difference—we define the skill of an N-point match as this limiting value. Accordingly, by computing the *Match Winning Chances* (MWC) of closely matched players and substituting these values into equation (3), we obtain empirical estimates of the skill function across match lengths.

3.2. Expected Duration of Matches

The expected duration D[M,N] from an away score of -M/-N in a cubeless match can be computed recursively using the following relation:

$$D[M,N]=T+(1-G)\frac{D[M-1,N]+D[M,N-1]}{2}+(G-B)\frac{D[M-2,N]+D[M,N-2]}{2}+B\frac{D[M-3,N]+D[M,N-3]}{2} \quad (4)$$

where T denotes the average game duration, G the gammon rate and B the backgammon rate at the given score. Because the gammon rate is weakly dependent on the score, the outcome probabilities from Section 2.1 may be used as good approximations. To estimate average game duration, we used the time required for GNUbg 0ply to complete a rollout in a few special cases: DMP, which serves as our unit of measurement; -1/-2 Crawford for 1-away scores; and cubeless games for all remaining scores.

For cubeful matches, outcome probabilities and game duration are strongly score-dependent, requiring rollouts for every score. By contrast, gammonless matches require no assumptions; we simply set **G=B=0** by definition. Consequently, the resulting durations are mathematically exact in that case.

Table 3. Duration of Gammonless Matches

Length	1	2	3	4	5
Duration	1	2.5	4.12	5.81	7.54

Table 4. Duration of Cubeless Matches

Length	1	2	3	4	5	6	7
Duration	1	1.85	2.99	4.13	5.32	6.52	7.74

Table 5. Duration of Cubeful Matches

Length	1	2	3	4	5	6	7	8	9	10	11	12	13
Duration	1	1	1.92	2.17	2.99	3.44	4.2	4.64	5.43	6.01	6.64	7.21	7.89

3.3. Skill in Match Play

The MWC of the stronger player from an away score of -M/-N in a gammonless match can be computed recursively using the following relation:

$$P[M,N]=W\cdot P[M-1,N]+(1-W)\cdot P[M,N-1] \tag{5}$$

where W denotes the win probability of the stronger player. Although this formula can be generalized to cubeless matches, the additional assumptions required introduce significant inaccuracies. Accordingly, the MWC of the stronger player in both cubeless and cubeful matches were computed by rolling out **38,880 games** for every score in a 7-point and 13-point match respectively; the resulting *Match Equity Tables* (METs) are provided in Appendix A. The empirical results differ substantially from the predictions of the theoretical model. All results are presented in the tables below. As with the expected duration, the skill values obtained for gammonless matches are mathematically exact.

Table 6. Skill in Gammonless Matches

Length	1	2	3	4	5
Skill	1	1.5	1.88	2.19	2.46
MWC	51.27	51.9	52.38	52.78	53.12

Table 7. Skill in Cubeless Matches

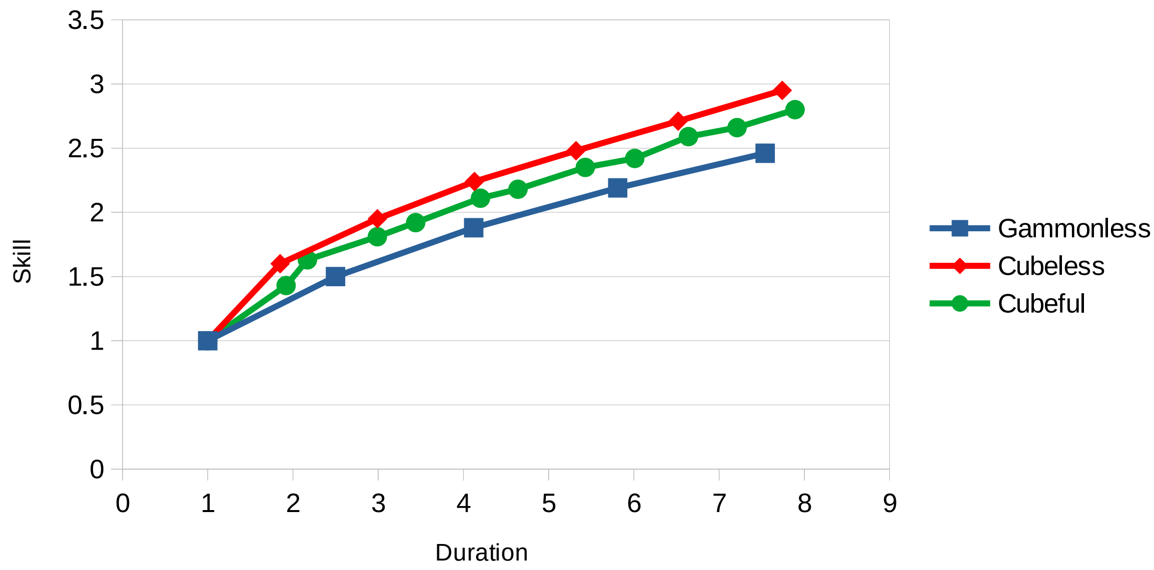
Length	1	2	3	4	5	6	7
Skill	1	1.6	1.95	2.24	2.48	2.71	2.95
MWC	51.27	52.03	52.48	52.84	53.14	53.44	53.74

Table 8. Skill in Cubeful Matches

Length	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sqrt{Length}$	1	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16	3.32	3.46	3.61
Skill	**1**	**1**	**1.43**	**1.63**	**1.81**	**1.92**	**2.11**	**2.18**	**2.35**	**2.42**	**2.59**	**2.66**	**2.8**
MWC	51.27	51.27	51.82	52.07	52.3	52.44	52.68	52.77	52.98	53.07	53.28	53.37	53.55

When comparing matches of equivalent length (or skill), cubeless matches dominate cubeful matches, which in turn dominate gammonless matches. We define one format as *dominating* another if matches of the former type are shorter in duration while containing more skill than those of the latter type. For example, gammonless matches contain nearly the same skill as cubeless matches but have a longer expected duration. Similarly, an N-point cubeless match contains approximately the same skill as a 2N-point cubeful match but its expected duration is comparable to that of a (2N−1)-point match. These relationships become most apparent when skill is plotted against expected duration.

Figure 1. Skill versus Expected Duration



3.4. The Elo Rating System

For applications to the Elo system, it is desirable to obtain explicit formulas for the skill functions associated with the various formats. In the case of gammonless matches, exact expressions can be derived (see Appendices B and C) for both the skill measure and the expected duration:

$$S(N)=\binom{2N}{N}\frac{2N}{2^{2N}} \tag{6}$$

$$D(N)=2N-S(N) \tag{7}$$

The last equation reveals a deep connection between skill and expected duration. The more skillful the stronger player is, the shorter the match is likely to be, with the difference in points at the end of the match being a quantitative reflection of the winner's relative strength. Similar relationships between skill and expected duration appear to hold across all major match formats. Specifically, the sum of the duration and skill functions is well approximated by a linear function of the match length, while the skill function itself is closely fitted by a square-root function of the match length.

Table 9. Approximate Skill and Duration Formulas

Format	$S(N)$	$D(N)+S(N)$
Gammonless	$\sqrt{1.27N-0.27}$	$2N$
Cubeless	$\sqrt{1.31N-0.31}$	$1.45N+0.55$
Cubeful	$\sqrt{0.55N+0.45}$	$0.71N+1.29$

4. Conclusion

We examined a range of formats to determine which most favors the stronger player. The analysis indicates that the most skillful forms of play are gammonless money games and cubeless matches.

In money play, one possible explanation is that DMP strategy leads to more complex games, while the additional skill introduced by cube handling may be offset by the opportunity for skill lost when the game ends with a pass. In match play, the trailer—who is statistically more likely to be the weaker player—often faces simpler cube decisions. Moreover, when the stakes of a game are increased, both players move closer to the end of the match, thereby reducing the opportunity for skill. Finally, the comparatively poor performance of the gammonless format in match play is consistent with the absence of score-specific strategy—the goal is always to win the current game.

More broadly, these results illustrate that increasing strategic complexity does not necessarily increase the opportunity for skill, and that the measurement of skill itself warrants further examination.

5. References

1. Bastian, Kevin. “The FIBS Rating System” Backgammon Galore, 1998, https://www.bkgm.com/articles/McCool/ratings.html

A. Unequal-Skill Equity Tables

Table 10. Cubeless MET

	-1	-2	-3	-4	-5	-6	-7
-1	**51.27**	69.90	82.70	89.91	94.12	96.58	98.01
-2	33.47	**52.03**	67.59	78.38	85.81	90.81	94.10
-3	19.97	36.50	**52.48**	65.28	75.31	82.77	88.15
-4	12.32	25.35	39.77	**52.84**	64.05	73.30	80.55
-5	7.47	17.32	29.34	41.62	**53.14**	63.27	71.86
-6	4.57	11.71	21.27	32.09	43.04	**53.44**	62.77
-7	2.78	7.81	15.22	24.26	34.24	44.22	**53.74**

Table 11. Cubeful MET

	PC	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13
PC	**51.27**		52.62	69.90	71.24	82.70	83.63	89.91	90.54	94.12	94.54	96.58	96.83	98.01
-1		**51.27**	69.90	76.43	83.20	85.77	90.12	91.83	94.30	95.27	96.70	97.25	98.08	98.41
-2	49.80	33.47	**51.27**	61.76	69.10	76.50	81.87	85.92	89.11	91.57	93.53	94.97	96.16	97.00
-3	33.47	26.26	41.97	**51.82**	59.63	67.28	73.66	78.55	82.83	86.07	89.00	91.15	93.04	94.48
-4	32.12	19.65	34.69	44.28	**52.07**	60.01	66.72	72.35	77.02	81.18	84.59	87.42	89.79	91.71
-5	19.97	17.01	27.40	36.78	44.38	**52.30**	59.24	65.31	70.59	75.31	79.34	82.80	85.70	88.22
-6	18.98	12.05	21.37	30.39	37.60	45.61	**52.44**	58.88	64.42	69.65	74.17	78.09	81.54	84.49
-7	12.32	10.08	17.07	25.34	31.99	39.63	46.22	**52.68**	58.41	63.81	68.76	73.04	76.88	80.29
-8	11.59	7.29	13.52	20.93	27.05	34.27	40.65	47.08	**52.77**	58.42	63.42	68.11	72.28	76.01
-9	7.47	6.17	10.79	17.31	22.81	29.46	35.44	41.76	47.38	**52.98**	58.15	62.99	67.43	71.58
-10	6.99	4.44	8.45	14.12	19.09	25.11	30.85	36.85	42.33	47.94	**53.07**	58.09	62.63	66.95
-11	4.57	3.73	6.76	11.66	16.04	21.49	26.64	32.36	37.68	43.10	48.29	**53.28**	57.94	62.35
-12	4.26	2.70	5.31	9.45	13.34	18.24	22.95	28.28	33.35	38.73	43.66	48.72	**53.37**	57.96
-13	2.78	2.28	4.23	7.73	11.12	15.50	19.79	24.60	29.45	34.45	39.39	44.29	48.94	**53.55**

B. The Gammonless Skill Formula

Since an N-point gammonless match is equivalent to a best-of-(2N-1) match, the probability that the stronger player wins can be calculated using the binomial distribution:

$$P(N)=\sum_{K=N}^{2N-1}\binom{2N-1}{K}W^{K}(1-W)^{2N-1-K}$$

Applying L’Hôpital’s rule to the definition of skill from Section 3.1 yields

$$S(N)=\lim_{W\to 1/2}\frac{\log\left(\frac{1}{P(N)}-1\right)}{\log\left(\frac{1}{W}-1\right)}=\lim_{W\to 1/2}\frac{(1-W)W}{[1-P(N)]P(N)}\sum_{K=N}^{2N-1}\binom{2N-1}{K}W^{K-1}(1-W)^{2N-2-K}[K-(2N-1)W]$$

Since $\lim_{W\to 1/2}P(N)=1/2$ we have $\sum_{K=N}^{2N-1}\binom{2N-1}{K}=2^{2N-2}$ and therefore

$$S(N)=2^{3-2N}\sum_{K=N}^{2N-1}\binom{2N-1}{K}K-\frac{2N-1}{2^{2N-2}}\sum_{K=N}^{2N-1}\binom{2N-1}{K}=(2N-1)\cdot 2^{3-2N}\sum_{K=N}^{2N-1}\binom{2N-2}{K-1}-(2N-1)$$

Let $A=\sum_{K=N}^{2N-1}\binom{2N-2}{K-1}=\sum_{K=N-1}^{2N-2}\binom{2N-2}{K}=\sum_{K=0}^{N-1}\binom{2N-2}{K}\Rightarrow$

$$2A=\sum_{K=0}^{N-1}\binom{2N-2}{K}+\sum_{K=N-1}^{2N-2}\binom{2N-2}{K}=\binom{2N-2}{N-1}+\sum_{K=0}^{2N-2}\binom{2N-2}{K}=\binom{2N}{N}\frac{N}{2\cdot(2N-1)}+2^{2N-2}\Rightarrow$$

$$S(N)=\binom{2N}{N}\frac{2N}{2^{2N}}\quad\blacksquare$$

C. The Gammonless Duration Formula

Since an N-point gammonless match must end after at least N games and at most 2N-1 games, the expected duration can be expressed as:

$$D(N)=\sum_{K=N}^{2N-1}K\cdot P(K)$$

where P(K) is the probability that the match lasts exactly K games. For the match to end in exactly K games, the winner of game K must win N-1 of the first K-1 games. There are C(K-1,N-1) combinations in which this can occur. Assuming that both players are equally likely to win a single game, each combination has probability 2^{1-K} and therefore

$$D(N)=\sum_{K=N}^{2N-1}K\cdot\binom{K-1}{N-1}\cdot 2^{1-K}=N\sum_{K=N}^{2N-1}\binom{K}{N}\cdot 2^{1-K}=N\sum_{K=0}^{N-1}\binom{K+N}{N}\cdot 2^{1-K-N}=N\cdot 2^{1-2N}\sum_{K=0}^{N-1}\binom{N+K}{N}\cdot 2^{N-K}$$

To simplify the calculation, consider a best-of-(2N+1) match, which is equivalent to an (N+1)-point match. Let K be the number of games won by the loser. The decisive game is then preceded by exactly N games won by the winner and K games won by the loser. These N+K games may occur in any order. Now imagine that all 2N+1 games are played, even if the winner has already been determined. In that case, there remain N−K irrelevant games which may be allocated arbitrarily. By symmetry—assuming both players are equally likely to win—the total number of possible sequences of 2N+1 games is equal to twice the number of sequences in which the match is decided in the (N+K+1)-st game. Hence

$$2\sum_{K=0}^{N}\binom{N+K}{K}\cdot 2^{N-K}=2^{2N+1}\Rightarrow\sum_{K=0}^{N-1}\binom{N+K}{N}\cdot 2^{N-K}=2^{2N}-\binom{2N}{N}\Rightarrow$$

$$D(N)=2N-S(N)\quad\blacksquare$$