FOUR INDEX EINSTEIN FIELD EQUATIONS

TOMASZ KOBIERZYCKI

Abstract. Einstein field equations are most tested theory of gravity. In this paper I will show a possible way to extend them into four index equation.

Date: June 12, 2025.

Contents

1. Useful mathematical equations

I will start this paper with key equations that will be used in it. First equation is just Einstein tensor, that first covariant derivative does vanish:

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{1.1}$$

Tensor itself is expressed as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{1.2}$$

From it follows contracted Bianachi [1] identity:

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla^{\mu} g_{\mu\nu} R \tag{1.3}$$

That can be written using contraction of contravariant derivative [2] with metric (that will be useful later) as:

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R \tag{1.4}$$

Now for Riemann tensor [3], its first contravariant derivative is equal to:

$$\nabla^{\alpha} R_{\alpha\mu\beta\nu} = \nabla_{\beta} R_{\mu\nu} - \nabla_{\nu} R_{\mu\beta} \tag{1.5}$$

Covariant derivative of energy-stress tensor is equal to zero with respect to first index so:

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{1.6}$$

Einstein field equations can be trace reversed by using first contraction [4][5] of equation that will lead to:

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = \kappa g^{\mu\nu} T_{\mu\nu} \tag{1.7}$$

$$R - \frac{D}{2} R = \kappa T \tag{1.8}$$

That leads in four dimensions to:

$$-R = \kappa T \tag{1.9}$$

That leads to trace reversed field equations in form of:

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{1.10}$$

First covariant derivative of Weyl tensor [6] is equal to Cotton tensor times constants so that:

$$\nabla^{\alpha} C_{\alpha\mu\beta\nu} = -\frac{n-3}{n-2} \left(\left(\nabla_{\nu} R_{\mu\beta} - \nabla_{\beta} R_{\mu\nu} \right) - \frac{1}{2(n-1)} \left(\nabla_{\nu} g_{\mu\beta} - \nabla_{\beta} g_{\mu\nu} \right) R \right) \tag{1.11}$$

All those equations are pretty basic and will be important in construction field equations of four order. But it's still good to write them down before moving forward.

2. Idea behind field equations of four index

Four index equation will be in form of:

$$G_{\alpha\mu\beta\nu} = \kappa K_{\alpha\mu\beta\nu} \tag{2.1}$$

Where covariant derivative of both tensors vanish so from it follows that:

$$\nabla^{\alpha} G_{\alpha\mu\beta\nu} = \kappa \nabla^{\alpha} K_{\alpha\mu\beta\nu} = 0 \tag{2.2}$$

Contraction of first equation will lead to Einstein tensor, so:

$$g^{\alpha\beta} G_{\alpha\mu\beta\nu} = G_{\mu\nu} \tag{2.3}$$

Contraction of energy momentum tensor part will lead to energy momentum tensor plus additional term, that term is non conservative part of energy stress tensor:

$$g^{\alpha\beta} K_{\alpha\mu\beta\nu} = T_{\mu\nu} - \left(\tilde{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{T} \right) \tag{2.4}$$

That itself has property that it's conserved as a whole so with it's trace part but not conserved by itself. It's conservation is equal to Ricci tensor:

$$\nabla^{\mu} \left(\tilde{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{T} \right) = 0 \tag{2.5}$$

$$\kappa \nabla^{\mu} \tilde{T}_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} \tag{2.6}$$

So whole two index equation is conserved:

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{2.7}$$

$$\nabla^{\mu} \left(T_{\mu\nu} - \left(\tilde{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{T} \right) \right) = 0 \tag{2.8}$$

Two index equation then can be written as:

$$G_{\mu\nu} = \kappa \left(T_{\mu\nu} - \left(\tilde{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{T} \right) \right) \tag{2.9}$$

Idea is to extend those equations into four order equations by using only five components, Weyl tensor, Ricci tensor, Stress energy tensor, non conservative stress energy tensor and metric tensor.

3. Field equation

Field equation can be in full form expressed as:

$$
\begin{aligned}
G_{\alpha\mu\beta\nu} = \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\mu\beta} + R_{\mu\nu}g_{\alpha\beta} - R_{\mu\beta}g_{\alpha\nu}\right) \\
-\frac{n}{2\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)R \\
K_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(T_{\alpha\beta}g_{\mu\nu} - T_{\alpha\nu}g_{\mu\beta} + T_{\mu\nu}g_{\alpha\beta} - T_{\mu\beta}g_{\alpha\nu}\right) \\
-\frac{1}{\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)T \\
-\frac{1}{n-2}\left(\tilde{T}_{\alpha\beta}g_{\mu\nu} - \tilde{T}_{\alpha\nu}g_{\mu\beta} + \tilde{T}_{\mu\nu}g_{\alpha\beta} - \tilde{T}_{\mu\beta}g_{\alpha\nu}\right) \\
-\frac{1}{2\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\mu\beta}\right)\tilde{T}
\end{aligned} \tag{3.1}
$$

It fulfills all needed assumptions from section before. There are three types of solutions to this equation, vacuum, matter and gravity dominated. Non conservative stress energy tensor is responsible for gravity field energy. Let's start with simplest so vacuum solutions, vacuum solutions are in form of:

$$G_{\mu\nu} = 0 \tag{3.2}$$

$$T_{\mu\nu} = \left(\tilde{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{T}\right) \tag{3.3}$$

It means Ricci tensor vanishes and im left with only Weyl tensor equal to $K_{\alpha\mu\beta\nu}$ tensor. Contraction of this equation will lead exactly to zero. Meaning behind is that energy contribution from matter field and gravity field cancel each other out. Second type of solutions is where matter field is stronger than gravity field energy tensor, then:

$$G_{\mu\nu} \approx \kappa T_{\mu\nu} \tag{3.4}$$

$$T_{\mu\nu} > \left(\tilde{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{T}\right) \tag{3.5}$$

Those types of matter model increasing energy of physical objects, like collapsing star or dust collapsing to from a star. last type of solutions are opposite of them so gravity is stronger than matter field:

$$G_{\mu\nu} \approx -\kappa\left(\tilde{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{T}\right) \tag{3.6}$$

$$T_{\mu\nu} < \left(\tilde{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{T}\right) \tag{3.7}$$

This type of solutions will lead to expansion of space like in expanding universe. Solutions can be combinations of all three but in general they are bound to this three scenario in general case.

References

[1] https://mathworld.wolfram.com/ContractedBianchiIdentities.html

[2] https://mathworld.wolfram.com/CovariantDerivative.html

[3] https://mathworld.wolfram.com/RiemannTensor.html

[4] https://mathworld.wolfram.com/RicciCurvatureTensor.html

[5] https://mathworld.wolfram.com/ScalarCurvature.html

[6] https://mathworld.wolfram.com/WeylTensor.html

Email address: kobierzyckitomasz@gmail.com