Four-elements tuples splitting

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Abstract. We consider a problem of splitting of the four-elements tuples of integer numbers into two two-elements tuples, providing a minimum of the difference between sums of the corresponding their elements. We give and prove the corresponding algorithm for its solution. Examples and generalizations for the tuples of arbitrary lengths and splitting are given.

1. Introduction

At the beginning we are going to consider splitting of 4-digits integers into two 2-digits sub-tuples, providing a minimum of the difference between sums of integers within each sub-tuple. Next, we consider more general algebraic systems. For general knowledge about considered further algebraic systems, see, e.g., [1 - 8].

Lets say, we have a 4-digits number: 6341.

It can be considered as a four-elements tuple (6, 3, 4, 1).

All possible non-ordered 2-digits subsets of the above tuple - (6, 3, 4, 1) are the following subsets:

(6, 3), (6, 4), (6, 1), (4, 3), (4,1), (3,1).

There are only the following three pairs of the above subsets, so that each pair doesn't have common elements(numbers):

(6, 3), (4,1), constitutes first splitting of the tuple (6, 3, 4, 1).
(6, 4), (3,1), constitutes second splitting of the tuple (6, 3, 4, 1).
(6, 1), (4, 3), constitutes third splitting of the tuple (6, 3, 4, 1).

For the first pair, the difference between the corresponding sums is:

$| (6 + 3) - (4 + 1) | = | 9 - 5 | = 4.$

For the second pair the difference between the corresponding sums is:

$| (6 + 4) - (3 + 1) | = | 10 - 4 | = 5.$

For the third pair the difference between the corresponding sums is:

$| (6 + 1) - (4 + 3) | = | 7 - 7 | = 0.$

Thus, third splitting(combination): (6, 1), (4, 3) gives a solution, providing the minimum: 0, of the differences between the corresponding sums.

2. Four-elements Tuples of Integer Numbers

Let us consider Four-elements non-ordered tuples of integer numbers:

$V := (i, j, m, n),\ i, j, m, n \in \mathbf{Z}_+ .$

What could be an algorithm that allows for any fixed tuple: $V_0 = (i_0, j_0, m_0, n_0)$, to split it into two two-elements sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides a minimum of the differences between the corresponding sums of E_1^0 and E_2^0 elements: $| (k_0 + l_0) - (s_0 + t_0) |$?

The corresponding graph illustration: $G = (V_0, E_1^0, E_2^0)$ can be given, considering a graph G, having vertices V_0 and edges E_1^0, E_2^0. Each vertex of G has degree 1.

Note that despite each sub-tuple: E_1^0 and E_2^0 is an "unique" subset of elements of the "parent" tuple V_0, they may include the same integers, as well as V_0, e.g., $V_0 = (3, 2, 2, 3)$, $E_1^0 = (3, 2)$, $E_2^0 = (2, 3)$.

3. Algorithm

Lemma. *Let* $x_{min} \leq x_2 \leq x_3 \leq x_{max}$,

$$x_{min}, x_2, x_3, x_{max} \in \boldsymbol{Z}_+.$$

Then:

$$(x_{max} + x_{min}) - (x_3 + x_2) \leq (x_{max} + x_2) - (x_3 + x_{min}),$$

and

$$(x_{max} + x_{min}) - (x_3 + x_2) \leq (x_{max} + x_3) - (x_2 + x_{min}).$$

Proof. Let us suppose that opposite:

$$(x_{max} + x_{min}) - (x_3 + x_2) > (x_{max} + x_2) - (x_3 + x_{min}).$$

Thus, it would mean that:

$$(x_{max} + x_{min}) - (x_3 + x_2) - (x_{max} + x_2) + (x_3 + x_{min}) > 0,$$

and therefore: $2(x_{min} - x_2) > 0$, which is wrong.

Similarly, let us suppose that opposite:

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and therefore: $2(x_{min} - x_3) > 0$, which is wrong. □

Theorem. *Let* $p_0 = max\ \{ i_0, j_0, m_0, n_0 \}$,
$q_0 = min\ \{ i_0, j_0, m_0, n_0 \}$.

Then:

$$E_1^0 = (p_0, q_0),\quad E_2^0 = (r_0, w_0),\quad r_0, w_0 \in V_0 \setminus p_0 \cup q_0 .$$

Proof. It follows from the above definitions and Lemma. □

For example, let $V_0 = (9, 3, 4, 1)$. Then, since $9 = \max \{9, 3, 4, 1)$ and $1 = \min \{9, 3, 4, 1\}$, splitting:

$E_1^0 = (9, 1)$, $E_2^0 = (4, 3)$, is a solution, where:

$| (9 + 1) - (4 + 3) | = | 10 - 7| = 3 = \min.$

4. Functions

Thus, to each integer tuple: (i_1, i_2, i_3, i_4), $i_1, i_2, i_3, i_4 \in \mathbf{Z}_+$, can be assigned the corresponding minimum and therefore we can define a function:

$f_{22} : \mathbf{Z}_+^4 \to \mathbf{Z}_+,$

$f_{22}(i_1, i_2, i_3, i_4) = | (p_0 + q_0) - (r_0 + w_0) |.$

Furthermore, there exists a pair of sub-tuples:

$E_1^0 = (p_0, q_0)$, $E_2^0 = (r_0, w_0)$, corresponding to that minimum, and we can define the following four functions as well:

$f_{22}p_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+$, $f_{22}p_0(i_1, i_2, i_3, i_4) = p_0$,

$f_{22}q_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+$, $f_{22}q_0(i_1, i_2, i_3, i_4) = q_0$,

$f_{22}r_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+$, $f_{22}r_0(i_1, i_2, i_3, i_4) = r_0$,

$f_{22}w_0 : \mathbf{Z}_+^4 \to \mathbf{Z}_+$, $f_{22}w_0(i_1, i_2, i_3, i_4) = w_0$.

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: $\mathbf{Z}[\mathbf{i}] := \{ a + b\mathbf{i} \mid a, b \in \mathbf{Z} \}$, where $\mathbf{i}^2 = -1$. Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice.

Let us define the following functions, corresponding to the above pair of "minimizing" sub-tuples:

$$f_{22C1}: \mathbf{Z}_+^4 \to \mathbf{Z[i]},\ f_{22C2}: \mathbf{Z}_+^4 \to \mathbf{Z[i]},$$

$$f_{22C3}: \mathbf{Z}_+^4 \to \mathbf{Z[i]},\ f_{22C4}: \mathbf{Z}_+^4 \to \mathbf{Z[i]},$$

$$f_{22C1}(i_1, i_2, i_3, i_4) = p_0, + \mathbf{i}\, q_0,\ \ f_{22C2}(i_1, i_2, i_3, i_4) = q_0, + \mathbf{i}\, p_0.$$

$$f_{22C3}(i_1, i_2, i_3, i_4) = r_0, + \mathbf{i}\, w_0,\ \ f_{22C4}(i_1, i_2, i_3, i_4) = w_0, + \mathbf{i}\, r_0.$$

Recall that quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers.

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$. This norm makes it possible to define the distance $d(p, q) = \|p - q\|$ which makes **H** into a metric space. Lipschits quaternions $\mathbf{L} := \{q:\ q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbf{Z}\}$.

Assigning to each integer tuple (i_1, i_2, i_3, i_4) the corresponding Lipschits quaternion: $i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}$, we can define the following functions:

$$f_{22L}: \mathbf{L} \to \mathbf{Z}_+,$$

$$f_{22L}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = |\,(p_0 + q_0) - (r_0 + w_0)\,|,$$

$$f_{22LC1}: \mathbf{L} \to \mathbf{Z[i]},\ f_{22LC2}: \mathbf{L} \to \mathbf{Z[i]},$$

$$f_{22LC3}: \mathbf{L} \to \mathbf{Z[i]},\ f_{22LC4}: \mathbf{L} \to \mathbf{Z[i]},$$

$$f_{22LC1}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = p_0, + \mathbf{i}\, q_0,$$

$$f_{22LC2}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = q_0, + \mathbf{i}\, p_0,$$

$$f_{22LC3}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = r_0, + \mathbf{i}\, w_0,$$

$f_{22LC4}(i_1 + i_2\mathbf{i} + i_3\mathbf{j} + i_4\mathbf{k}) = w_0, + \mathbf{i}\, r_0.$

Note that the following splitting scenarios can be defined as well: Split a tuple $V_0 = (i_0, j_0, m_0, n_0)$, into two two-elements sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides a minimum of the difference between the corresponding differences between E_1^0 and E_2^0 elements: $|\,|k_0 - l_0| - |s_0 - t_0|\,| \to \min$, or, so that it provides $|k_0 - l_0| + |s_0 - t_0| \to \min$, or, $(k_0 + l_0) + (s_0 + t_0) \to \min$.
The corresponding functions, similar to the above-defined can be defined as well.

5. Generalization

The above theory can be generalized as follows.
Let us consider n-elements non-ordered tuples of integer numbers:

$$V := (i_1, \ldots, i_n),\ i_1, \ldots, i_n \in \mathbf{Z}_+ .$$

Split any fixed tuple $V_0 = (i_1^0, \ldots, i_n^0)$, into two: m-elements and k-elements ($m + k \le n$, $m, k, n \in \mathbf{N}$) sub-tuples:
$E_1^0 = (u_1^0, \ldots, u_m^0)$, $E_2^0 = (v_1^0, \ldots, v_k^0)$, wherein:
$u_1^0, \ldots, u_m^0, v_1^0, \ldots, v_k^0 \in V_0$,
$E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 \subset V_0$, so that it provides a minimum of the difference between the corresponding sums (s_1 and s_2) of E_1^0 and E_2^0 elements:

$$|\, s_1 - s_2 \,| = |\, (u_1^0 + \ldots + u_m^0) - (v_1^0 + \ldots + v_k^0) \,| \to \min.$$

The corresponding functions: $f_{mk} : \mathbf{Z}_+^n \to \mathbf{Z}_+$, $f_{mk}(i_1, \ldots, i_n)$ can be considered.

Conjecture. *If* $|m - k| < |r - s|$,
$m + k \le n,\ r + s \le n,$
$m, k, r, s, n \in \mathbf{N},$
then $f_{mk}(i_1, \ldots, i_n) \le f_{rs}(i_1, \ldots, i_n)$.

More general case may consider splitting of V_0 into more than two: m ($m \geq 2$) sub-tuples, providing a minimum, e.g., of the mean absolute deviation of the corresponding sums: $s_1, \dots, s_m$ around of central point: $\mathbf{D}(s_1, \dots, s_m) \to \min$.

The corresponding functions: $f_\mathbf{r}: \mathbf{Z}_+^n \to \mathbf{Z}_+$, $f_\mathbf{r}(\mathbf{i})$, $\mathbf{r} = (r_1, \dots, r_m)$, $\mathbf{i} = (i_1, \dots, i_n)$, $r_1 + \dots + r_m \leq n$ can be considered as well.

6. Normed Spaces and N-tuples Splittings

The above theory can be further generalized for the tuples, consisted of elements of some Normed Space.

Recall that Normed Space is a Vector Space $\mathbf{X}$ over a subfield $\mathbf{F}$ of the complex numbers $\mathbf{C}$, where the corresponding real-valued function(the norm), $\| x \| : \mathbf{X} \to \mathbf{R}$, is defined.

A Vector(Linear) Space is a set whose elements(vectors) can be added together and multiplied by numbers(scalars), e.g., Banach and Hilbert Spaces.

Let us consider n-elements non-ordered tuples of elements of some Normed Space $\mathbf{X}$: $V := (a_1, \dots, a_n)$, $a_1, \dots, a_n \in \mathbf{X}$.

Split any fixed tuple $V_0 = (a_1^0, \dots, a_n^0)$, into two: m-elements and k-elements ($m + k \leq n$) sub-tuples:

$E_1^0 = (u_1^0, \dots, u_m^0)$, $E_2^0 = (v_1^0, \dots, v_k^0)$, wherein:

$u_1^0, \dots, u_m^0, v_1^0, \dots, v_k^0 \in V_0$,

$E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 \subset V_0$, so that it provides a minimum of the difference between the corresponding norm sums of E_1^0 and E_2^0 elements:

$$| (\|u_1^0\| + \dots + \|u_m^0\|) - (\|v_1^0\| + \dots + \|v_k^0\|) | \to \min.$$

The following criteria can be suggested as well:

$$\| (u_1^0 + \dots + u_m^0) - (v_1^0 + \dots + v_k^0) \| \to \min.$$

The corresponding affinity of functions: $f_{mk} : \mathbf{X}^n \to \mathbf{R}$, $f_{mk}(a_1, \dots, a_n)$ can be considered.

Similar scenarios can be considered for the Normed Abelian groups.

7. Splitting, based on Two Algebraic Operations

Let us consider the following scenarios:

Split a tuple $V_0 = (i_0, j_0, m_0, n_0)$, $i_0, j_0, m_0, n_0 \in \mathbf{Z}_+$ into two two-elements sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides:
$| k_0 * l_0 - s_0 * t_0 | \to \min$, or, $k_0 * l_0 + s_0 * t_0 \to \min$, or,
$(k_0 + l_0) * (s_0 + t_0) \to \min$, or, $| k_0 - l_0 | * | s_0 - t_0 | \to \min$.

To generalize this approach, let $\mathbf{X}$ be a normed commutative ring. The following scenarios then can be suggested:

Split a tuple $V_0 = (i_0, j_0, m_0, n_0)$, $i_0, j_0, m_0, n_0 \in \mathbf{X}$, into two two-elements sub-tuples: $E_1^0 = (k_0, l_0)$, $E_2^0 = (s_0, t_0)$, wherein: $k_0, l_0, s_0, t_0 \in V_0$, $E_1^0 \cap E_2^0 = \varnothing$, $E_1^0 \cup E_2^0 = V_0$, so that it provides:
$\| k_0 * l_0 - s_0 * t_0 \| \to \min$, or, $\| k_0 * l_0 + s_0 * t_0 \| \to \min$, or,
$\| (k_0 + l_0) * (s_0 + t_0) \| \to \min$, or, $\| (k_0 - l_0) * (s_0 - t_0) \| \to \min$.

Similar to the above-given section 6, the corresponding scenarios, comprising splitting of n-tuples into two (or even more) m- and k-sub-tuples can be considered.

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