THE OPTIMIZED ELLIPSE PERIMETER FORMULA: THE ELLIPTIC BOUNDS THEOREM

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Abstract. This work discovers a unique, accurate, elementary formula for calculating the total perimeter of an ellipse demonstrably and heuristically consistently under the 1% maximum error range for all values and eccentricities, which surpasses Ramanujan's previous dominant formulas in terms of accuracy and simplicity. The formula also uniquely provides the definitive range of calculable deviation from any exact value, making it the optimal general approximation method. Several examples are provided which confirm the formula's precision. The theorem also provides a corollary which derives a definitive metric bound on all parametric elliptic geometry, with application to all of physics and astronomy.

Keywords: *elliptic perimeter, approximation, limit*

1. Formula

An elementary formula for the precise perimeter of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

has eluded grasp since the beginning of history; and ancient Greek knowledge of geometry. This work derives the correct elementary parametric formulation for ascertaining the precise value of elliptic perimeter, according to all objective standards of optimization, the results of which apply to all measure-theoretic and scientific methodologies.

The formula for the exact value is the elliptic integral; however it gives less parametric information concerning general elliptic geometry than the optimization formula provided. The elliptic integral is defined as

$$C = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta, \tag{2}$$

where e is the eccentricity of the ellipse, measured as

$$e = \sqrt{1 - \frac{b^2}{a^2}}, \tag{3}$$

and the integral is also expressed as a function of the eccentricity,

$$C = 4aE(e), \tag{4}$$

as

$$E(x) = \int_0^{\frac{\pi}{2}} \sqrt{1 - x^2 \sin^2 \theta}\, d\theta. \tag{5}$$

However, this formula only gives a value of an ellipse circumference, whereas much more empirical parametric data is available—which applies to the elliptic geometry of physics—by much more economical calculation method. This work derives such.

The formula is derived from taking the limit of the extreme limiting cases of ellipses for semi-major and semi-minor axes; circle, $2\pi a$ $(a = b)$, and maximum elongation limit, $4a$ $(b = 0)$, combining them through general term cancellation to a single limit formula.

It begins with the circle reduction: $\pi(a + b)$, and the reductive combinatory cancellation term, $\frac{(4-\pi)(a-b)^2}{a+b}$, to give the demonstrable reductive limit proportion:

The elliptic bounds theorem

$$C \sim \pi(a + b) + (4 - \pi)\frac{(a - b)^2}{a + b} \tag{6}$$

such that,

$$\lim_{b \to 0} \pi(a + b) + \frac{(4 - \pi)(a - b)^2}{a + b} = \pi(a + 0) + (4 - \pi)\frac{(a - 0)^2}{a + 0} \tag{7}$$
$$= a\pi + \frac{(4 - \pi)a^2}{a} = a\pi + 4a - a\pi = 4a,$$

recovering the unique unitary limit; with unique base root exponent limit cancellation ratio of a: 1 for $\frac{(a-b)^2}{a+b}$ as $\frac{a^2}{a}$, with a and b in the total limit approaching sufficient calculable static components with respect to the reduced denominator for the extreme cases of elongated ellipse and circle mutually, in the most reduced fundamental unique root limit power ratio of 2: 1; and

$$\lim_{b \to a} \pi(a + b) + \frac{(4 - \pi)(a - b)^2}{a + b} = \pi(2a) + (4 - \pi)\frac{(0)^2}{2a} = 2\pi a, \tag{8}$$

recovering the circle, in a unique dual limiting case. This proves the general uniqueness and calibrated optimization of limiting approach criteria of the formula for the elliptic parametric principle. Hence, this formula is analogous to recovering and deriving general term cancellation, similar to *completing the square* for derivation of the quadratic formula, and is the proper substitution methodology, as in other general partial differential equation solutions. The core principle is complementarity of root binary component cancellation for elementary, reductive, precise contributing part; this formula also then providing the minimal general parametric upper bound on the ultimate value as limit of all ranges. Thus, this is by definition the preeminent formula for estimating generalized circumferences accurately, without competitor in arbitrary ranges, which is then demonstrably confirmable as such to random values. Q.E.D.

Hence, this formula may also be used to form a new, generalized, definitive expression for π:

$$\pi = \lim_{b\to a} \frac{\frac{C}{a+b} - h + \sqrt{\left(h - \frac{C}{a+b}\right)^2 - 16h}}{2}, \qquad h = \left(\frac{a-b}{a+b}\right)^2, \tag{9}$$

confirmable as the generalization of the direct, special case of the circle,

$$\frac{\frac{2\pi a}{2a} - 0 + \sqrt{\left(0 - \frac{2\pi a}{2a}\right)^2 - 16(0)}}{2} = \frac{\pi + \sqrt{(-\pi)^2}}{2} = \frac{2\pi}{2} = \pi\,. \tag{10}$$

Other common but less acute, accurate, useful (or directly interrelated to the structure of the ellipse) approximations have included Euler's arithmetic-geometric mean approximation [1]:

$$C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}, \tag{11}$$

and Ramanujan's completely random, non-rigorously founded first and second approximations, respectively,

$$C \approx \pi\left(3(a+b) - \sqrt{(a+3b)(3a+b)}\right), \tag{12}$$

$$C \approx \pi(a+b)\left(1 + \frac{3h}{10+\sqrt{4-3h}}\right), \tag{13}$$

which, by already established mathematical proof of the original *elliptic bounds theorem*, by definition cannot compare in terms of consistent accuracy.

2. Precision

This result consistently maintains an accuracy of an error percentage under 1% across all eccentricities, and gives the maximum upper bound within that percent range on an ellipse perimeter, and is particularly precise the closer the ellipse is to a circle, or the more elongated it is. For $a = 85$, $b = 43$, the result is 413.95, whereas the exact (corresponding decimal expansion) result is 413.02, deriving the optimal maximum upper limit within less than 1%. Increased by an order of magnitude to $a = 850$, $b = 430$, the result is 4139.53, and the exact result 4130.22, under 0.3% error; and for even $a = 850$, $b = 43$, it gives 3431.46 compared to 3416.85, maintaining an error percentage still well under 1% with arbitrary eccentricity, demonstrating immense superiority over Euler's geometric mean approximation formula—which in this case gives 3781.27, a 10% error. The closer the eccentricity is to 1, the more precise the elliptic bounds formula is in fact; for example, for $a = 1000000$, $b = 0.0000001$, the result is 4000000, compared to the answer, 4000000, still well under 1% error; the factor of error only decreasing as the sum eccentricity increasingly approaches the infinite limit—in this case, Ramanujan's second approximation, previously considered the most accurate, gives 3998391, completely off the mark, and gets even less close as the eccentricity increases. Therefore, that is ultimately comparatively ineffectual for the general case, without the same general consistency of optimized error margin as well.

Other examples:

Low eccentricity

a	b	C using (6)	Exact value of C	Relative error %
88	77	518.99	518.93	under 0.01%
96	85	569.20	569.15	under 0.0001%
1200	1000	6927.11	6925.79	under 0.02%
5022	4999	31481.94	31481.94	under 0.00002%

Moderate eccentricity

a	b	C using (6)	Exact value of C	Relative error %
459	221	2207.78	2202.22	under 0.3%
1000	500	4855.45	4844.22	under 0.3%
6800	4500	35901.85	35868.63	under 0.1%
10000	5000	48554.56	48442.24	under 0.3%

High eccentricity

a	b	C using (6)	Exact value of C	Relative error %
44	2	177.43	176.72	under 0.5%
100	15	415.21	412.61	under 0.631%
900	25	3616.47	3606.20	under 0.3%
10000	100	40060.03	40010.98	under 0.2%

Table 1: *Results for the perimeter proportion of a standard ellipse with arbitrary eccentricity, with a fixed semi-major axis* $a = 1$ *and varying semi-minor axis* b *from* 1 *to* 0.

For Table 1, it is observed that all relative error percentages are beneath 0.64%.

a	b	C using (6)	Exact value of C	Relative error %
1	0.999900	6.282871152206479	6.282871152	0.000000003286370
1	0.990000	6.251812516691246	6.251808848	0.000058682076436
1	0.900000	5.973543975222766	5.973160433	0.006421093608111
1	0.800000	5.673942495270743	5.672333578	0.028364292202121
1	0.700000	5.386152605912600	5.382368981	0.070296646810291
1	0.600000	5.112388980384690	5.105399773	0.136898336965742
1	0.500000	4.855456871453057	4.844224110	0.231879475391522
1	0.400000	4.618963032674049	4.602622519	0.355026153167994
1	0.300000	4.407623987929040	4.385910070	0.495083519326230
1	0.200000	4.227728435726529	4.202008908	0.612076944376833
1	0.100000	4.087851874032652	4.063974180	0.587545416753912
1	0.050000	4.036493838779008	4.019425619	0.424643254954792
1	0.040000	4.027937331321507	4.013143313	0.368639173028884
1	0.030000	4.019991377117258	4.007909450	0.301452097857597
1	0.020000	4.012673933614886	4.003839160	0.220657555457002
1	0.010000	4.006003669449100	4.001098330	0.122599822461768
1	0.005000	4.002917266738105	4.000309233	0.065195803279170
1	0.004000	4.002320201650834	4.000205049	0.052876105722701
1	0.003000	4.001729922076847	4.000120518	0.040233889694192
1	0.002000	4.001146448332054	4.000568070	0.014457405096835
1	0.001000	4.000569800813546	4.000155880	0.010347617091997
1	0.000100	4.000056671394296	4.000002020	0.001366284167433
1	0.000001	4.000000566374047	4.000000000	0.000014159351180

And so on, well under 1% error for all arbitrary eccentricities.

3. Deviation calculation

Specifically, the maximum error or deviation range is given by the maximum-to-minimum extrema general limiting ratio of maximum elongation to perfect circle, through all variable limit parameters,

$$\frac{4a}{2\pi a}=\frac{2}{\pi}, \tag{14}$$

which when converted to percentage value of such precise upper limit of deviation, becomes $\frac{2}{100\pi} \approx 0.00636619772\ldots$; or less than 0.64% precise deviation, statically and arbitrarily empirically verifiably; which, when thus subtracted as the maximum deviation fraction of the result from the result, can only give the minimum possible accurate perimeter range as the lower bound, through numeric calibration, as the total limit—deriving a minimum–maximum objective sufficient limit of general "*ellipse cohesiveness*" parameter, and a corollary, which may be called the *elliptic squeeze theorem*, where the original upper limit of C is technically $C_{\max}$:

The elliptic squeeze theorem

$$C_{\min} \leq C \leq C_{\max}, \tag{15}$$

where $C_{\min}$ is calculated as exactly

$$C_{\min} = C_{\max} - \frac{2}{100\pi} C_{\max} = C_{\max} - \frac{C_{\max}}{50\pi}, \tag{16}$$

the denominator factor of 100 necessary when correctly calibrated to base-10 decimal expansion to take into account the actual error percentage, or standard deviation.

Examples:

$a = 80, b = 4$

$$(C_{\min} = 320.86) \leq (C = 321.55) \leq (C_{\max} = 322.91) \tag{17}$$

$a = 2, b = 1$

$$(C_{\min} = 9.64) \leq (C = 9.68) \leq (C_{\max} = 9.71) \tag{18}$$

$a = 90, b = 45$

$$(C_{\min} = 434.20) \leq (C = 435.98) \leq (C_{\max} = 436.99) \tag{19}$$

$a = 9536, b = 322.15$

$$(C_{\min} = 38118.37) \leq (C = 38237.06) \leq (C_{\max} = 38362.60) \tag{20}$$

Table 2: *Results for the parametric bounds of a standard ellipse of arbitrary eccentricity, with a fixed semi-major axis* $a = 1$ *and varying semi-minor axis* b *from 1 to 0.*

For Table 2, it is observed that the exact perimeter is always bounded from below by $C_{\min}$, and from above by $C_{\max}$.

a	b	$C_{\min}$	Exact value of C	$C_{\max}$
1	0.999900	6.242873152179153	6.282871152	6.282871152206479
1	0.990000	6.212012242078639	6.251808848	6.251812516691246
1	0.900000	5.935515213165425	5.973160433	5.973543975222766
1	0.800000	5.637821055473083	5.672333578	5.673942495270743
1	0.700000	5.351863293453468	5.382368981	5.386152605912600
1	0.600000	5.079842501295220	5.105399773	5.112388980384690
1	0.500000	4.824546072970607	4.844224110	4.855456871453057
1	0.400000	4.589557800729697	4.602622519	4.618963032674049
1	0.300000	4.379564182130267	4.385910070	4.407623987929040
1	0.200000	4.200813880582687	4.202008908	4.227728435726529
1	0.100000	4.061827800737461	4.063974180	4.087851874032652
1	0.050000	4.010796720890942	4.019425619	4.036493838779008
1	0.040000	4.002294685851739	4.013143313	4.027937331321507
1	0.030000	3.994399317163057	4.007909450	4.019991377117258
1	0.020000	3.987128457952854	4.003839160	4.012673933614886
1	0.010000	3.980500658007617	4.001098330	4.006003669449100
1	0.005000	3.977433903946535	4.000309233	4.002917266738105
1	0.004000	3.976840639893663	4.000205049	4.002320201650834
1	0.003000	3.976254118156156	4.000120518	4.001729922076847
1	0.002000	3.975674358920589	4.000568070	4.001146448332054
1	0.001000	3.975101382454200	4.000155880	4.000569800813546
1	0.000100	3.974591519718292	4.000002020	4.000056671394296
1	0.000001	3.974535771873695	4.000000000	4.000000566374047

According to this precise elliptic range principle, all spatial orbits, as approached general ellipsoid limits, must fall within the range of this governing elliptic squeeze theorem, otherwise they will sufficiently deviate from geometric ellipsoid phenomena generally at a tangent, according to all limits of measure. With Earth's semi-major axis of orbit at measure 1 AU, and its semi-minor axis of orbit at 0.99986 AU, the stable orbital circumference of the Earth thus cannot exceed the exact calculated parametric range of 6.24274 AU and 6.28274 AU (0.04 AU difference).

For Earth's moon, with semi-major axis of 0.00257 AU and semi-minor axis of 0.00256 AU, the stable orbital circumference parametric range is precisely 0.01601 AU and 0.01611 AU (or 0.0001 AU difference), any other range predictable as anomalous.

For Mercury, with semi-major axis of 0.38709 AU, and semi-minor axis of 0.37881 AU, the formula gives a precise stable orbital circumference parametric range of 2.39090 AU and

2.40622 *AU* (0.01532 *AU* difference), any other, all things considered, indicating a spacetime geometric anomaly, which could be chiefly due to gravity of other orbital bodies. And thus this formula can predict any anomalous geometric orbits as approached measurable limit—in all limits, as precise general calculable elliptic limit; namely, classical, general relativistic, and quantum—which happen to for whatever reason exceed the limiting elliptic squeeze theorem's range.

4. Conclusions

This precise elliptic measurable range principle extends to all of physics as well, indicating the range of a predictably stable orbit, for example. Any deviation from this precise range thus indicates an unstable orbit according to precise numerical analysis, and then the elliptic squeeze theorem's range in astronomy may be referred to as "the anomalous orbit indicator (AOI) formula", a correct cornerstone of the field (as well as for classical and quantum physics in relation to precise measurable deviating influences on elliptic orbital trajectories), which can track anomalous influences on all orbital bodies, including satellites.

The rarity of eons of orbital formation and evolution being so stable as to allow life and intelligent life to develop explains its empirical rarity in cosmology and biology as well, in application; hence, life on Earth is an anthropic limit as such. The ellipse is the generalized symmetry principle, and as such, this optimal unique precision general formula in all its basic applications may be referred to centrally as *the fundamental theorem of numerical analysis*.

REFERENCES

[1] Daepp, Ulrich, Gorkin, Pamela, Shaffer, Andrew, and Voss, Karl, Finding Ellipses (Carus Mathematical Monographs), American Mathematical Society, 2018.