A Feature of the Off-Shell Renormalization Schemes in Quantum Field Theory

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Abstract

It is shown that the off-shell renormalization schemes for subtraction of ultraviolet divergences in Quantum Field Theory produce zero for sums of perturbative corrections to physical quantities when all perturbation orders are taken into account. That is the off-shell renormalization schemes are in this sense unphysical. In this connection it is desirable to develope on-shell renormalization schemes for different quantum theories.

Keywords: Quantum Field Theory; off-shell renormalization schemes; on-shell renormalization schemes; Quantum Chromodynamics; Quantum Gravity

1 Introduction

Renormalization schemes for subtraction of ultraviolet divergences in Quantum Field Theory can be divided in two essentially different classes: the off-shell schemes and the on-shell schemes. The most known off-shell schemes are the Minimal Subtraction scheme - the MS-scheme [1], its commonly used modification the $\overline{MS}$-scheme [2], and the Momentum Subtraction scheme -

the MOM-scheme, see e.g. [3] and references therein. The essential point of the off-shell schemes is that one introduces an arbitrary renormalization parameter (usually denoted as μ) within these approaches during subtractions of ultraviolet divergences of Green functions. But physical quantities should be independent on this unphysical renormalization parameter μ. The coupling constants in the off-shell renormalization schemes are the so called running coupling constants depending on the free renormalization parameter μ of the schemes.

The on-shell renormalization schemes do not have free renormalization parameters and contain only physical parameters. The famous example of the application of the on-shell scheme is the anomalous magnetic moment of the electron. The agreement between the theory and the experiment in this case within at least ten decimal points convinces us that Quantum Field Theory and in particular the on-shell renormalization scheme is a valid approach. The coupling constants in the on-shell schemes do not depend on free parameters and are the fixed physical constants.

The purpose of the present paper is to show that the off-shell renormalization schemes produce zero for sums of perturbative corrections to physical quantities when all perturbation orders are taken into account. That is the off-shell renormalization schemes in Quantum Field Theory are in this sense unphysical. In this connection it seems to be desirable to develope on-shell renormalization schemes, in particular for Quantum Chromodynamics and Quantum Gravity.

2 Main Part

We will consider the famous R-ratio within perturbative Quantum Chromodynamics (QCD) in the MS-scheme as a typical illustrative example in view of its importance for phenomenological applications:

$$\tilde{R(s)} = \frac{\sigma_{total}(e^+e^- \to hadrons)}{\sigma_{tree}(e^+e^- \to \mu^+\mu^-)}. \tag{1}$$

I.e. $\tilde{R}$ is the total cross-section of the electron-positron annihilation into hadrons normalized by the tree level cross-section of the electron-positron annihilation into the muon-antimuon pairs. The momentum transfer squared q^2 of this reaction will be denoted as s: $s \equiv q^2$.

Ignoring masses one can write

$$\tilde{R(s)} = 3\sum_{i}^{n_f} q_i^2(1 + R(s)), \tag{2}$$

where q_i are quark electric charges in the units of the electron charge and n_f is the number of active quarks.

Here R has the following form within perturbative QCD:

$$R(s) = a(1 + \sum_{k=1}^{\infty} r_k a^k) = a(1 + r_1 a + r_2 a^2 + ...), \tag{3}$$

where $a \equiv a(\mu^2) = \frac{g^2}{16\pi^2}$ is the renormalized QCD running coupling constant depending on the arbitrary renormalization parameter μ of the MS-scheme. g is the strong coupling constant in the QCD Lagrangian.

The perturbative coefficients $r_k \equiv r_k(s, \mu^2)$ are calculable in perturbation theory and depend both on the momentum transfer squared s and μ^2.

The physical quantity R must be independent on the arbitrary parameter μ according to the basic principle of renormalization group invariance of physical quantities in Quantum Field Theory. This requirement generates the renormalization group equation:

$$\frac{dR(s)}{dln(\mu^2)} = \frac{\partial R}{\partial ln(\mu^2)} + \frac{\partial R}{\partial a}\frac{\partial a}{\partial ln(\mu^2)} = 0. \tag{4}$$

Usually one deals with the truncated perturbative series for R to compare R with experiments:

$$R_N = a(1 + \sum_{k=1}^{N-1} r_k a^k) = a + r_1 a^2 + ... + r_{N-1} a^N. \tag{5}$$

The truncated perturbative series R_N does depend on μ^2, but the derivative of the R_N in μ^2 is of the order of a^{N+1} only [4] :

$$\frac{dR_N}{dln(\mu^2)} = O(a^{N+1}). \tag{6}$$

This equation is the reminiscence of the μ-independence (4) of R.

The perturbative coefficients r_k have the well-known form:

$$r_k = \sum_{i=0}^{k} r_{k,i} ln^i(\mu^2/s), \tag{7}$$

where the powers of the logarithms are in accordance with the equation (6). Here $r_{k,i}$ are constants calculable in perturbative QCD.

The running coupling constant $a(\mu^2)$ obeys the standard renormalization group equation:

$$\frac{da(\mu^2)}{dln(\mu^2)} = \beta(a). \tag{8}$$

The renormalization group β-function has the well known form:

$$\beta(a) = -\sum_{n=0}^{\infty} \beta_n a^{n+2} = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 - ..., \tag{9}$$

where the constants β_n are calculable in perturbative QCD.

In particular, the famous result [5]-[7]

$$\beta_0 = 11 - \frac{2}{3} n_f \tag{10}$$

has led to the discovery of the asymptotic freedom in QCD.

The equation (8) can be solved by iterations in the region of the applicability of the method with the known result:

$$a(\mu^2) = \frac{1}{\beta_0 ln(\mu^2/\Lambda^2)} - \frac{\beta_1 lnln(\mu^2/\Lambda^2)}{\beta_0^3 ln^2(\mu^2/\Lambda^2)} + O\left(\frac{1}{ln^3(\mu^2/\Lambda^2)}\right), \tag{11}$$

where Λ is the known fundamental parameter of QCD with the dimension of the mass.

Substituting the coefficients r_k from the equation (7) and the expression for $a(\mu^2)$ from (11) into $R(s)$ in eq. (3) one gets the standard expression for $R(s)$ in terms of the logarithms $ln(\mu^2/s)$ and of the logarithms $ln(\mu^2/\Lambda^2)$:

$$R(s) = a + (r_{1,0} + r_{1,1} ln(\frac{\mu^2}{s}))a^2 + (r_{2,0} + r_{2,1} ln(\frac{\mu^2}{s}) + r_{2,2} ln^2(\frac{\mu^2}{s}))a^3 + ..., \tag{12}$$

where a is given in (11). Thus R in the above equarion (12) is the well defined function of μ.

One can note that in each perturbative order in the r.h.s. of (12) the minimal power of $ln(\mu^2)$ in the denominator is one more (due to the eqs. (11)) than the maximal power of $ln(\mu^2)$ in the numerator. Thus one can find that the limit of each perturbative order in (12) at $\mu \to \infty$ is zero. Hence the limit of the whole $R(s)$ at $\mu \to \infty$ is also zero within perturbation theory:

$$\lim_{\mu \to \infty} R(s) = 0. \tag{13}$$

Thus from one side $R(s)$ is independent on μ (see eq.(4)) and from another side the limit of $R(s)$ at $\mu \to \infty$ is zero (see eq.(13)). Thus one can conclude that $R(s)$ is identically zero for any μ within the region of applicability of perturbation theory. The inclusion of the non-sero masses will not change the result.

3 Conclusions

We have demonstrated above that the off-shell renormalization schemes for subtraction of ultraviolet divergences in Quantum Field Theory produce zero for sums of perturbative corrections to physical quantities when all perturbation orders are taken into account. Thus it seems to be desirable to develope on-shell renormalization schenes, in particular for Quantum Chromodynamics and also for Gravity which was quite recently quantized for the relativistic R^2-model [8],[9].

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References

[1] G. ‘t Hooft, Nucl. Phys. B61 (1973) 455.

[2] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D18 (1978) 3998.

[3] W. Celmaster and R.J. Gonsalves, Phys. Rev. Lett. 42 (1979) 1435.

[4] P.M. Stevenson, Phys.Rev. D23 (1981) 2916.

[5] G. 't Hooft, report at the Marseille Conference on Yang-Mills Fields, 1972.

[6] D.J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343.

[7] H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.

[8] S.A. Larin, Mod.Phys.Lett. A 33 (2018) 05, 1850028; e-Print: 1711.02975 [physics.gen-ph].

[9] S.A. Larin, Symmetry 11 (2019) 11, 1334.
S.A. Larin, Universe 7 (2021) 11, 435; e-Print: 2204.08007 [hep-th.