The four Second Order axiomatizations of the Categorical theory

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Abstract:
There are four possible axiomatizations of the Zeroth Order Categorical theory, which are the four possible forms in which the implication operator can be constructed (they are implicit). Analogously, there are four possible axiomatizations of the Second Order Categorical theory (the explicit form of the Zeroth Order Categorical theory). They are presented here. Only one is known to date with rigor (Peano axioms + Hereditarily Finite Sets axioms).

1) First form (axioms):

- $\forall a(\forall b(a = b \Leftrightarrow \forall c(c \in a \Leftrightarrow c \in b)))$
- $\forall a(0 \neq S(a))$
- $\forall a(\forall b(a \neq b \Leftrightarrow S(a) \neq S(b)))$
- $\exists a(\forall b(b \notin a))$
- $\forall a(\forall b(\exists c(\forall d(d \in c \Leftrightarrow (d \in a \nLeftrightarrow d = b)))))$
- $\forall \Phi(\forall a(\forall b(S(b) = a \rightarrow \Phi(b)) \rightarrow \Phi(a)) \Leftrightarrow \forall d(\forall c(d \in c \rightarrow \Phi(d)) \rightarrow \Phi(c)))$

2) Second form:

- $\forall a(\forall b(a = b \Leftrightarrow \forall c(c \in a \Leftrightarrow c \in b)))$
- $\forall a(0 \neq S(a))$
- $\forall a(\forall b(a \neq b \Leftrightarrow S(a) \neq S(b)))$
- $\exists a(\forall b(b \notin a))$
- $\forall a(\forall b(\exists c(\forall d(d \in c \Leftrightarrow (d \in a \nLeftrightarrow d = b)))))$
- $\forall a(\forall b(S(b) = a \rightarrow b) \rightarrow a) \Leftrightarrow \forall d(\forall c(d \in c \rightarrow d) \rightarrow c)$

3) Third form:

- $\forall a(\text{Addition}(a, 0) = a \land \forall b(\text{Addition}(a, S(b)) = S(\text{Addition}(a, b))))$
- $\forall a(\text{Multiplication}(a, 0) = 0 \land \forall b(\text{Multiplication}(a, S(b)) = \text{Addition}(a, \text{Multiplication}(a, b))))$
- $\forall a((a \neq 0 \rightarrow \text{Exponentiation}(a, 0) = 1) \land \forall b(\text{Exponentiation}(a, S(b)) = \text{Multiplication}(a, \text{Exponentiation}(a, b))))$
- $\forall a(\forall b(\forall c(c \in \text{Union}(a, b) \Leftrightarrow c \in a \lor c \in b)))$
- $\forall a(\forall b(\forall c(c \in \text{Intersection}(a, b) \Leftrightarrow c \in a \land c \in b)))$
- $\forall a(\forall b(b \in \text{Complementation}(a) \Leftrightarrow b \notin a))$

4) Fourth form:

- $\forall a(a = P(S(a)))$
- $\forall a(a = S(P(a)))$
- $\forall a(\forall b(\forall c(\text{NumbersOperation}(a, S(b), S(c)) = \text{NumbersOperation}(a, b, \text{NumbersOperation}(a, S(b), c)))))$
- $\forall a(a \notin \text{EmptySet})$
- $\forall a(a \in \text{FullSet})$
- $\forall a(\forall b(\forall c(\exists d(\forall e(e \in d \Leftrightarrow (e \in a \rightarrow e \in b)) \land \forall f(f \in \text{SetsOperation}(a, b, c) \Leftrightarrow (f \in d \nLeftrightarrow f = c))))))$