THE ENERGY METHOD IN ADDITION CHAINS

THEOPHILUS AGAMA

Abstract. In this note, we introduce the energy method for constructing the length of addition chains leading to $2^n - 1$. This method is a generalization of the Brauer method. Using this method, we show that the conjecture is true for all addition chains with "low" energy.

1. Introduction

An addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* addition chain leading to n. In standard practice, we denote by $\iota(n)$ the length of an optimal addition chain that leads to n.

Example 1.1. The following is an example of an addition chain that leads to 15:

$$1, 2, 3, 5, 6, 8, 11, 14, 15$$

obtained from the sequence of additions

$$2 = 1+1, 3 = 2+1, 5 = 2+3, 6 = 3+3, 8 = 3+5, 11 = 5+6, 14 = 6+8,$$
$$15 = 14 + 1.$$

We remark that the same addition chain can also be obtained from the sequence of additions

$$2 = 1+1, 3 = 2+1, 5 = 2+3, 6 = 5+1, 8 = 6+2, 11 = 8+3, 14 = 11+3,$$
$$15 = 14 + 1.$$

The possibility to obtain an addition chain using distinct sequence of additions creates a subtle ambiguity. It suggests that knowing an addition chain leading to a fixed positive integer without specifying

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how the terms were obtained may be unsatisfactory, as it hides the procedure for obtaining the terms in the chain.

Alfred Brauer [1] is often credited with pioneering research on addition chains. A Brauer addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_{i-1} + s_j$ for $i > j \geq 0$. We denote the length of an optimal Brauer chain leading to n by $\iota^*(n)$. A number n for which the Brauer chain is optimal (i.e. $\iota^*(n) = \iota(n)$) is called a Brauer number. In fact, he proved the following results concerning the length of an optimal addition chain

Theorem 1.2. *[Braurer] We have*

$$\iota^*(2^{m+1} - 1) \leq m + \iota^*(m+1).$$

Brauer's original proof of Theorem 1.2 is as follows: Let $a_o = 1, 2, \ldots, m+1$ be the shortest Brauer chain leading to $m+1$ and of length $\iota^*(m+1)$. Next, we exponentiate the terms in the chain (with base 2) and take their unit left translates as follows

$$1, 2^2 - 1, \cdots, 2^{m+1} - 1.$$

We now artificially build an addition chain leading to $2^{m+1}-1$ using the above sequence as a building block. With each term 2^{a_v} $(v = 0, \ldots, m)$ in the sequence, we systematically double $(a_{v+1} - a_v)$ times and include the results of each step of double in the sequence. We obtain therefore a new sequence

$$1, 2, 2^2-1, 2(2^2-1), \ldots, 2^{a_v}-1, 2(2^{a_v}-1), \ldots, 2^{a_{v+1}-a_v}(2^{a_v}-1), \ldots, 2^{m+1}-1.$$

By construction, this sequence is now an addition chain leading to $2^{m+1} - 1$, since

$$2^{a_v} - 1 = 2^{a_v - a_{v-1}}(2^{a_{v-1}} - 1) + 2^{a_v - a_{v-1}}$$

and the exponents were terms in the Braurer chain leading to $m + 1$. We observe that the contribution of the number of terms adjoined to the resulting sequence by doubling to the length of the chain is

$$(a_1 - a_o) + (a_2 - a_1) + \cdots + (a_{k+1} - a_k) = m$$

and hence $\iota(2^{m+1}-1) \leq \iota^*(2^{m+1}-1) \leq m+\iota^*(m+1)$, since the Braurer chain is a special type of addition chain.

Example 1.3. To illustrate how the proof works, consider an example with $m = 5$. The shortest Brauer chain leading to 5 is $a_o = 1, a_1 = 2, a_2 = 3, a_4 = 5$ of length 3. We now mimic the proof in its exact form, exponentiating the terms in the chain leading to 5 with the base 2 and taking the unit left translates in the following way:

$$1 = 2^1 - 1, 3 = 2^2 - 1, 7 = 2^3 - 1, 31 = 2^5 - 1.$$

We observe that this sequence is not an addition chain, because 7 is not the sum of two previous terms in the sequence. According to Brauer's argument, we have to introduce new terms into the sequence, making it an addition chain. To do this, between the terms $2^{a_o} - 1 = 1$ and $2^{a_1} - 1 = 2$, we double the first term 1 for $(a_1 - a_o) = (2-1)$ times and include the result of each doubling in the sequence. Thus, we have to include $2^{2-1} \times 1 = 2$ in the sequence. Also, between the second term $2^{a_2} - 1 = 3$ and the third term $2^{a_3} - 1 = 7$, we double the second term $a_2 = 3$ for $(a_3 - a_2) = (3-2)$ number of times and include the result of each doubling in the chain. Therefore, we must include $2^{3-2} \times 3 = 6$ in the sequence. Finally, between the third term $2^{a_3} - 1 = 7$ and the last term $2^{a_4} - 1 = 31$, we double the third term 7 for $(a_4 - a_3) = (5-3) = 2$ number of times and include the result of each doubling in the sequence. Therefore, we must include $2 \times 7 = 14$ and $2^2 \times 7 = 28$ in the sequence. We now have the newly constructed sequence

$$1, 2, 3, 6, 7, 14, 28, 31$$

of length 7. On the other hand, $(5-1) + \iota^*(5) = 4 + 3 = 7$. We see that Theorem 1.2 holds in this case, since $\iota^*(2^5 - 1) = (5-1) + \iota^*(5) = 7$.

2. **The energy method**

In section, we extend Brauer's method to a general class of methods called the *energy method*. Roughly speaking, we show that Scholz's conjecture is true for a broad class of addition chains of which the Brauer chain is a subclass.

Definition 2.1 (The energy of an addition chain)**.** Let $n \geq 2$ be a fixed positive integer, and let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $1 \leq i \leq h$, where $\sigma, \tau : [1, i] \cap \mathbb{N} \longrightarrow [0, i-1] \cap \mathbb{Z}$. We put the energy of the chain to be

$$\mathbb{E}_n = \sum_{i=1}^{h} s_{\tau(i)}.$$

We say that the energy of the addition chain is *low* if $\mathbb{E}_n \leq n - 1$. On the other hand, we say that the addition chain has *excess* energy if $\mathbb{E}_n \geq n$.

In the discipline of general physics, the concept of energy is usually understood as an accumulated work *effort* or the work done by a *machine*. The concept of *additive energy* arises frequently in the area of study known as additive combinatorics [2]. In that context, the additive energy of a set A of positive integers, by convention $E(A)$, is formally defined as

$$E(A) := \#\{(a_1, a_2, a_3, a_4) \in A^4 \ : \ a_1 + a_2 = a_3 + a_4\}.$$

However, in the context of an addition chain, the energy of an addition chain leading to n, denoted $\mathbb{E}_n$, is understood to be total sum (accumulation) of the lower *weight* of pairs which generate each term in the addition chain. Intuitively, the sum

$$\mathbb{E}_n = \sum_{i=1}^{h} s_{\tau(i)}$$

may be viewed as the *accumulated* work effort or *computational* work required to build an addition chain leading to n.

Remark 2.2. It is clear that the energy of a Brauer chain leading to n is exactly $n-1$. In fact, there are addition chains with *excess energy*. We now formally launch the energy method as a generalization of Brauer's method.

Theorem 2.3 (The energy method)**.** *Let $n \geq 2$ be a fixed positive integer, and let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be the shortest addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $1 \leq i \leq h$. If the energy*

$$\mathbb{E}_n := \sum_{i=1}^{h} s_{\tau(i)} \leq n - 1$$

then $\iota(2^n - 1) \leq n - 1 + \iota(n)$, where $\iota(\cdot)$ denotes the length of the shortest addition chain leading to $\cdot$.

Proof. Let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be the shortest addition chain leading to $n \geq 2$, and denote the length of the chain by $\iota(n)$. Let us define $m_i = 2^{s_i} - 1$ for $i = 0, 1, \ldots, h$ and consider the sequence $2^{s_o} - 1 := 2^1 - 1, 2^{s_1} - 1 := 2^2 - 1, \ldots, 2^{s_h} - 1 = 2^n - 1$. We now build an addition chain leading to $2^n - 1$ using the above sequence as a building block. To do this, we double the term $2^{s_{\sigma(i+1)}} - 1$ for $(s_{i+1} - s_{\sigma(i+1)})$ $(0 \leq i \leq h-1)$ number of times and include the result of each doubling in the sequence, where $s_{i+1} = s_{\sigma(i+1)} + s_{\tau(i+1)}$ $(i+1 > \sigma(i+1) \geq \tau(i+1))$. We claim that this construction yields an addition chain that leads to $2^n - 1$. To see this, we observe that we can write

$$2^{s_{i+1}} - 1 = 2^{s_{i+1} - s_{\sigma(i+1)}} \left(2^{s_{\sigma(i+1)}} - 1\right) + 2^{s_{i+1} - s_{\sigma(i+1)}} - 1$$

for each $0 \leq i \leq h-1$. The total length of the addition chain constructed is therefore

$$\leq \sum_{i=0}^{h-1} s_{\tau(i+1)} + \iota(n).$$

Thus, we deduce

$$\iota(2^n - 1) \leq \sum_{i=0}^{h-1} s_{\tau(i+1)} + \iota(n) \leq n - 1 + \iota(n)$$

by using the upper bound for the energy of the shortest addition chain leading to n. $\square$

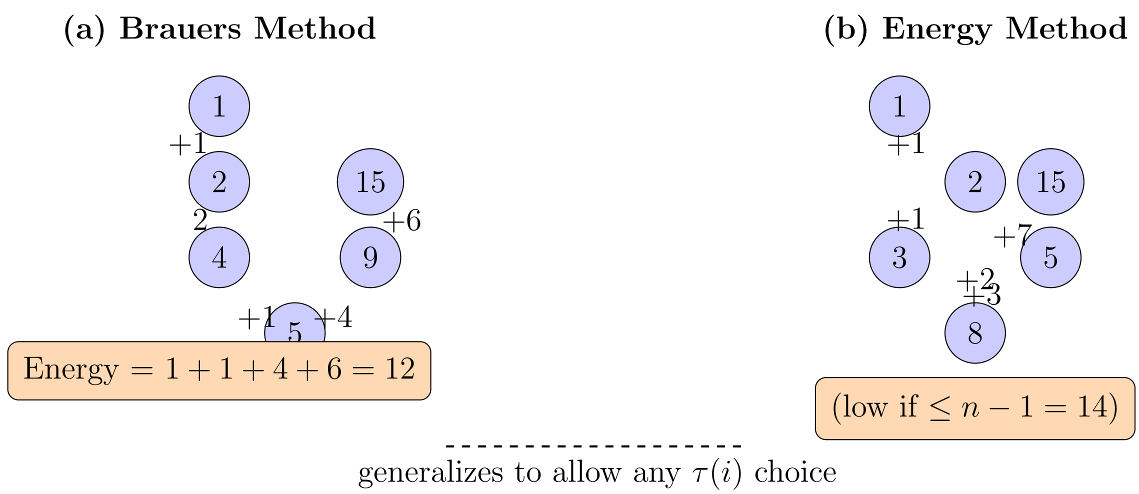

FIGURE 1. (a) Brauers method always uses the *immediately previous* term (energy $= n-1$). (b) The energy method picks the smaller summand $s_{\tau(i)}$ at each step, accumulating $\mathbb{E}_n = \sum s_{\tau(i)}$. Brauers chain is the special case where each $\tau(i) = i-1$, so $\mathbb{E}_n = n-1$, and the new method allows lower-energy choices in general.

It is important to observe that Theorem 2.3 generalizes Brauer's method. In particular, when we replace $\iota(n)$ with $\iota^*(n)$, then we can recover Brauer's result in the form $\iota^*(2^n - 1) \leq n - 1 + \iota^*(n)$. We may now reformulate Scholz's conjecture in the following way.

Conjecture 2.4 (Scholz). *Let $n \geq 2$ be a fixed positive integer, and let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be the shortest addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(i > \sigma(i) \geq \tau(i) \geq 0)$ for each $1 \leq i \leq h$. Then the energy*

$$\mathbb{E}_n := \sum_{i=1}^{h} s_{\tau(i)} \leq n - 1$$

References

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Departement de mathematiques et de statistique, Universite Laval, Quebec, Canada

E-mail address: thaga1@ulaval.ca/Theophilus@aims.edu.gh/emperordagama@yahoo.com