Methods to Calculate Linear Momentum from Angular Momentum and to Link Their Conservation

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Abstract

In Newtonian physics, it commonly assumed that conservations of linear momentum and angular momentum are independent. This paper shows the gradient of the angular momentum field is populated by components of linear momentum: $\partial L_k/\partial x_j = \epsilon_{ijk} P_i$ where $i \notin (j,k)$, which allows calculation of linear momentum from a few angular momentum values displaced an arbitrary distance s:

$$\vec{P} = (\ L_z(\vec{O} + s\hat{y}) - L_z(\vec{O})\ ,\ L_x(\vec{O} + s\hat{z}) - L_x(\vec{O}))\ ,\ L_y(\vec{O} + s\hat{x}) - L_y(\vec{O})\ )/s\ .$$

Therefore conservation of linear momentum is a necessary condition for **Full** conservation of angular momentum (unchanging angular momentum at every (x, y, z), with unchanging gradient). These results are shown to be equally valid under left hand convention. Examples analyzed include a body orbiting in a central force, which has angular momentum changing at every observation point but one, contrasted with actual conservation of angular momentum (unchanging at every observation point, in every frame of reference). Translating angular momentum of the center of mass within and between frames of reference are discussed. A Python script is provided, to conveniently generate and evaluate random groups of particles.

1 Introduction

The pattern of Newtonian angular momentum around an undisturbed moving particle takes a distinctive tunnel shape. The pattern existed and exists, along the path, unchanged (conserved) over all time. If we could see the angular momentum visually, it would advertise the particle long before it arrived locally, and remain unchanged after the particle had passed. This static pattern does not indicate the changing particle position. The value of $\vec{L}$ at any one point means little. The overall pattern tells us much.

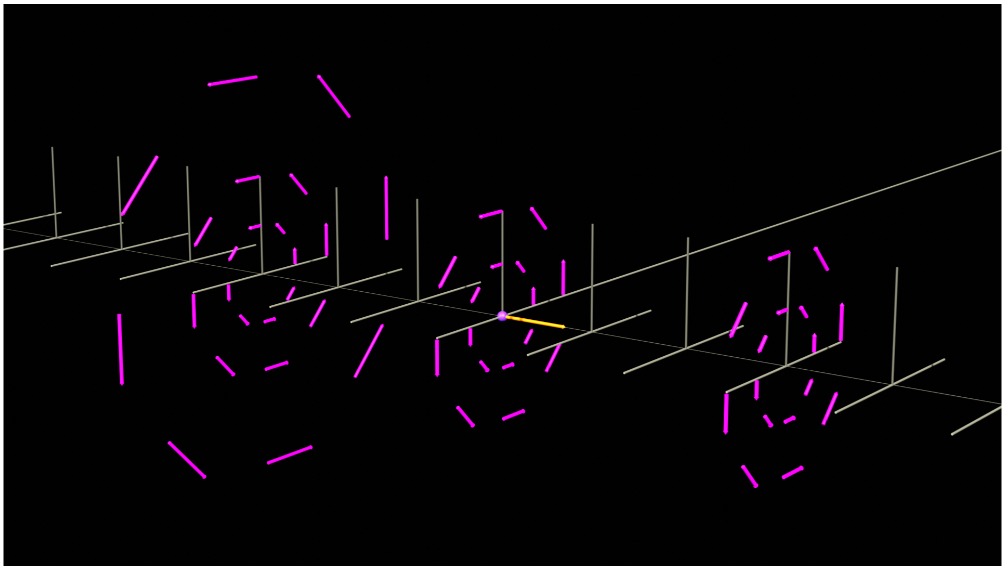

Figure 1: Angular momentum around a single particle

We can choose coordinates such that the particle is traveling along the x axis with position $= (r_x, 0, 0)$, momentum $(p_x, 0, 0)$. At observation point (x, y, z) the angular momentum is

$$\vec{L}(x,y,z) = ((r_x,0,0) - (x,y,z)) \times (p_x,0,0) \; ; \tag{1}$$

$$\vec{L}(x,y,z) = (r_x - x \;,\; -y \;,\; -z) \times (p_x,0,0) \; ; \tag{2}$$

$$\vec{L}(x,y,z) = (0, -zp_x, yp_x) \qquad r_x, x \text{ play no role} \; ; \tag{3}$$

$$\partial L_y/\partial z = -p_x \; ; \tag{4}$$

$$\partial L_z/\partial y = p_x \; ; \tag{5}$$

$$\partial L_k/\partial x_j = \epsilon_{ijk} p_i \qquad \text{where } i \notin (j,k) \text{, see Eq.(101)} \; . \tag{6}$$

Later we will see that (using $\vec{P}_{total}$), Eqs.(4), (5), (6), and (101) are always true, regardless of the the chosen observation point, or the number, positions, or momentums of particles, or even right or left hand convention.

The x coordinate of the observation point (in the direction of linear momentum) does not affect the value of $\vec{L}$ found there.

As we travel further from the path, angular momentum changes (Eqs. (3)-(6)), at a constant rate equal to the linear momentum of the particle. Given values of angular momentum at various positions, we can calculate the $\partial L_k/\partial x_j$ rates of change, and therefore the linear momentum, both magnitude and direction.

It is well-known that angular momentum around a system of particles is equal to angular momentum at the center of mass position, plus angular momentum of an imaginary particle at the position of the center of mass with momentum $\vec{P}_{total}$. So every system of bodies with non-zero linear momentum provides a single-particle-like tunnel pattern, possibly with a constant pseudovector $\vec{L}_{CoM}$ added to every pattern position. $\vec{L}_{CoM}$ does not affect the $\partial L_k/\partial x_j$ rate of change.

From values of angular momentum at various positions, we can calculate the rate of change, and therefore the linear momentum of the imaginary particle, which is the total linear momentum of the system (Eq. 20)). We can calculate **linear** momentum from **angular**.

2 Iterative Proof

This proof makes no specific reference to right or left hand convention, it is equally valid under either, assuming appropriate (left or right) orthogonal coordinates consistent with:

$$\hat{z} = \hat{x} \times \hat{y} \; ; \qquad \hat{x} = \hat{y} \times \hat{z} \; ; \qquad \hat{y} = \hat{z} \times \hat{x} \; . \tag{7}$$

For a single particle with position $\vec{r}_i$ and momentum $\vec{p}_i$, observed from point $O = (x, y, z)$:

$$\vec{L}_i(x, y, z) \equiv (\vec{r}_i - (x, y, z)) \times \vec{p}_i \; ; \tag{8}$$

$$\begin{aligned} \vec{L}_i(x, y, z) = (\quad & (r_{iy} - y)p_{iz} - (r_{iz} - z)p_{iy} \quad , \\ & (r_{iz} - z)p_{ix} - (r_{ix} - x)p_{iz} \quad , \\ & (r_{ix} - x)p_{iy} - (r_{iy} - y)p_{ix} \quad) \; ; \end{aligned} \tag{9}$$

$$L_{ix} = r_{iy}p_{iz} - yp_{iz} - r_{iz}p_{iy} + zp_{iy} \; ; \tag{10}$$

$$L_{iy} = r_{iz}p_{ix} - zp_{ix} - r_{ix}p_{iz} + xp_{iz} \; ; \tag{11}$$

$$L_{iz} = r_{ix}p_{iy} - xp_{iy} - r_{iy}p_{ix} + yp_{ix} \; . \tag{12}$$

Choosing an arbitrary distance s, if we evaluate

$$(L_{iz}(\vec{O} + s\hat{y}) - L_{iz}(\vec{O}) \, , \, L_{ix}(\vec{O} + s\hat{z}) - L_{ix}(\vec{O}) \, , \, L_{iy}(\vec{O} + s\hat{x}) - L_{iy}(\vec{O}))/s \tag{13}$$

$$\begin{aligned} = (\quad & (r_{ix}p_{iy} - xp_{iy} - r_{iy}p_{ix} + (y + s)p_{ix}) \\ & - (r_{ix}p_{iy} - xp_{iy} - r_{iy}p_{ix} + yp_{ix}) \; , \\ & (r_{iy}p_{iz} - yp_{iz} - r_{iz}p_{iy} + (z + s)p_{iy}) \\ & - (r_{iy}p_{iz} - yp_{iz} - r_{iz}p_{iy} + zp_{iy}) \; , \\ & (r_{iz}p_{ix} - zp_{ix} - r_{ix}p_{iz} + (x + s)p_{iz}) \\ & - (r_{iz}p_{ix} - zp_{ix} - r_{ix}p_{iz} + xp_{iz}) \quad)/s \; ; \end{aligned} \tag{14}$$

$$= (sp_{ix}, sp_{iy}, sp_{iz})/s \; ; \tag{15}$$

$$= (p_{ix}, p_{iy}, p_{iz}) \; ; \tag{16}$$

$$= \vec{p}_i \; ; \tag{17}$$

we get momentum. This equation embodies $(p_x, p_y, p_z) = (\partial L_z/\partial y, \partial L_x/\partial z, \partial L_y/\partial x)$ in a way that can be used with available angular momentum values.

Since each of N particles obeys

$$\vec{p}_i = (L_{iz}(\vec{O} + s\hat{y}) - L_{iz}(\vec{O}) \, , \, L_{ix}(\vec{O} + s\hat{z}) - L_{ix}(\vec{O}) \, , \, L_{iy}(\vec{O} + s\hat{x}) - L_{iy}(\vec{O}))/s \; ; \tag{18}$$

$$\sum_i \vec{p}_i = \sum_i (L_{iz}(\vec{O} + s\hat{y}) - L_{iz}(\vec{O}) \, , \, L_{ix}(\vec{O} + s\hat{z}) - L_{ix}(\vec{O}) \, , \, L_{iy}(\vec{O} + s\hat{x}) - L_{iy}(\vec{O}))/s \; ; \tag{19}$$

$$\vec{P} = (\; L_z(\vec{O} + s\hat{y}) - L_z(\vec{O}) \; , \; L_x(\vec{O} + s\hat{z}) - L_x(\vec{O})) \; , \; L_y(\vec{O} + s\hat{x}) - L_y(\vec{O}) \;)/s \quad ; \tag{20}$$

$$P_i = \epsilon_{ijk}(L_k(O + s\hat{j}) - L_k(O))/s \qquad \text{where } \epsilon_{ijk} = 1 \; . \tag{21}$$

We can calculate linear momentum from spatially separated values of angular momentum.

Similarly,

$$
\begin{aligned}
&(L_{iy}(\vec{O}) - L_{iy}(\vec{O}+s\hat{z})\,,\, L_{iz}(\vec{O}) - L_{iz}(\vec{O}+s\hat{x})\,,\, L_{ix}(\vec{O}) - L_{ix}(\vec{O}+s\hat{y}))/s && (22)\\
&= (\quad (r_{iz}p_{ix} - zp_{ix} - r_{ix}p_{iz} + xp_{iz}) \\
&\quad - (r_{iz}p_{ix} - (z+s)p_{ix} - r_{ix}p_{iz} + xp_{iz}) \quad , \\
&\quad (r_{ix}p_{iy} - xp_{iy} - r_{iy}p_{ix} + yp_{ix}) \\
&\quad - (r_{ix}p_{iy} - (x+s)p_{iy} - r_{iy}p_{ix} + yp_{ix}) \quad , && (23)\\
&\quad (r_{iy}p_{iz} - yp_{iz} - r_{iz}p_{iy} + zp_{iy}) \\
&\quad - (r_{iy}p_{iz} - (y+s)p_{iz} - r_{iz}p_{iy} + zp_{iy}) \quad)/s \ ; \\
&=(sp_{ix}, sp_{iy}, sp_{iz})/s \ ; && (24)\\
&=(p_{ix}, p_{iy}, p_{iz}) \ ; && (25)\\
&=\vec{p}_i \ ; && (26)
\end{aligned}
$$

is also momentum. This equation embodies $(p_x, p_y, p_z) = (-\partial L_y/\partial z, -\partial L_z/\partial x, -\partial L_x/\partial y)$ in a useful fashion. Since each of N particles obeys

$$
\begin{aligned}
\vec{p}_i &= (L_{iy}(\vec{O}) - L_{iy}(\vec{O}+s\hat{z})\,,\, L_{iz}(\vec{O}) - L_{iz}(\vec{O}+s\hat{x})\,,\, L_{ix}(\vec{O}) - L_{ix}(\vec{O}+s\hat{y}))/s \ ; && (27)\\
\sum_i \vec{p}_i &= \sum_i (L_{iy}(\vec{O}) - L_{iy}(\vec{O}+s\hat{z})\,,\, L_{iz}(\vec{O}) - L_{iz}(\vec{O}+s\hat{x})\,,\, L_{ix}(\vec{O}) - L_{ix}(\vec{O}+s\hat{y}))/s \ ; && (28)\\
\vec{P} &= (\ L_y(\vec{O}) - L_y(\vec{O}+s\hat{z})\ ,\ L_z(\vec{O}) - L_z(\vec{O}+s\hat{x}))\ ,\ L_x(\vec{O}) - L_x(\vec{O}+s\hat{y})\ )/s \ ; && (29)\\
\vec{P} &= -(\ L_y(\vec{O}+s\hat{z}) - L_y(\vec{O})\ ,\ L_z(\vec{O}+s\hat{x}) - L_z(\vec{O}))\ ,\ L_x(\vec{O}+s\hat{y}) - L_x(\vec{O})\ )/s \ ; && (30)\\
P_i &= \epsilon_{ijk}(L_k(O+s\hat{j}) - L_k(O))/s \qquad \text{where } \epsilon_{ijk} \in (1,-1)\ . && (31)
\end{aligned}
$$

If we combine terms from Eq.(20) and Eq.(29), we only need $\vec{L}$ at three points: $\vec{O}, \vec{O}+s\hat{y}, \vec{O}+s\hat{z}$:

$$
\vec{P} = (\ L_z(\vec{O}+s\hat{y}) - L_z(\vec{O})\ ,\ L_x(\vec{O}+s\hat{z}) - L_x(\vec{O}))\ ,\ L_x(\vec{O}) - L_x(\vec{O}+s\hat{y})\ )/s \quad . \tag{32}
$$

This is the main finding of this paper: at time t, we can calculate total linear momentum solely from angular momentum values, without knowing the number of particles, their masses, positions, or momentums. If these angular momentum values are conserved over time, linear momentum is also conserved.

The demonstration script, shown later, can generate unlimited detailed examples on demand.

Linear momentum is associated with the gradients of angular momentum.

Any value of angular momentum at a single observation point can be associated with any value of linear momentum.

A series of angular momentum values across x, y, z defines a specific linear momentum.

Any single value of velocity can be associated with any value of acceleration.

A series of velocity values across time defines a specific acceleration.

3 Real Vector vs PseudoVector

We can doubly verify that this expression

$$(L_z(\vec{O}+s\hat{y})-L_z(\vec{O})\,,\,L_x(\vec{O}+s\hat{z})-L_x(\vec{O})\,,\,L_y(\vec{O}+s\hat{x})-L_y(\vec{O}))/s$$

is a vector, not a pseudovector. (s is a distance, typically in meters.)

Because it contains components of the pseudovector angular momentum $\vec{L}$, this expression may raise concern.

The result needs to be a real vector to actually equal the real vector $\vec{P}_{total}$.

Before we jump into the derivation, we should understand that

$$L_x(\vec{O}+s\hat{z})-L_x(\vec{O})=-(L_x(\vec{O}-s\hat{z})-L_x(\vec{O}))\ .$$

We know this because the components L_x, L_y, L_z are linear functions of $(r_{ix}, r_{iy}, r_{iz}, p_{ix}, p_{iy}, p_{iz}, O_x, O_y, O_z)$. Therefore, if we add $s\hat{z}$ to O_z and get a delta, then adding $-s\hat{z}$ to O_z must produce the equal but opposite delta.

The cross product right hand rule is only a convention. If we used a left hand rule instead, pseudovectors would point in the physically opposite direction, but physics would still work correctly, we would arrive at the same physical results.

If we change to left-hand cross-product, and left-hand coordinates, does the candidate expression transform like a vector or pseudovector?

Using left hand coordinates that reflect z:

$$x'=x\ ;\qquad y'=y\ ;\qquad z'=-z\ ;\tag{33}$$

$$x'\times_{left}y'=z'\ ;\qquad y'\times_{left}z'=x'\ ;\qquad z'\times_{left}x'=y'\ .\tag{34}$$

When switching to this left-hand system, real vectors like $\vec{P}'$ will physically be the same, but its z' coefficient will be negated:

$$P'_x=P_x\ ;\qquad P'_y=P_y\ ;\qquad P'_z=-P_z\ ;\tag{35}$$

$$P'_x\hat{x}'=P_x\hat{x}\ ;\qquad P'_y\hat{y}'=P_y\hat{y}\ ;\qquad P'_z\hat{z}'=P_z\hat{z}\ .\tag{36}$$

The left hand rule will physically negate pseudovectors like $\vec{L}'$, which means their x and y coefficients will negate, and their z coefficient will remain:

$$L'_x=-L_x\ ;\qquad L'_y=-L_y\ ;\qquad L'_z=L_z\ ;\tag{37}$$

$$L'_x\hat{x}'=-L_x\hat{x}\ ;\qquad L'_y\hat{y}'=-L_y\hat{y}\ ;\qquad L'_z\hat{z}'=-L_z\hat{z}\ .\tag{38}$$

Evaluation:

$$\begin{aligned}
&(\quad (L'_z(\vec{O}+s\hat{y}')-L'_z(\vec{O}))\hat{x}' &&+(L'_x(\vec{O}+s\hat{z}')-L'_x(\vec{O}))\hat{y}' &&+(L'_y(\vec{O}+s\hat{x}')-L'_y(\vec{O})\hat{z}') &&)/s\ ; &&(39)\\
=&(\quad (L_z(\vec{O}+s\hat{y}')-L_z(\vec{O}))\hat{x}' &&+(-L_x(\vec{O}+s\hat{z}')+L_x(\vec{O}))\hat{y}' &&+(-L_y(\vec{O}+s\hat{x}')+L_y(\vec{O})\hat{z}') &&)/s\ ; &&(40)\\
=&(\quad (L_z(\vec{O}+s\hat{y})-L_z(\vec{O}))\hat{x}' &&+(-L_x(\vec{O}-s\hat{z})+L_x(\vec{O}))\hat{y}' &&+(-L_y(\vec{O}+s\hat{x})+L_y(\vec{O})\hat{z}') &&)/s\ ; &&(41)\\
=&(\quad (L_z(\vec{O}+s\hat{y})-L_z(\vec{O}))\hat{x} &&+(-L_x(\vec{O}-s\hat{z})+L_x(\vec{O}))\hat{y} &&+(L_y(\vec{O}+s\hat{x})-L_y(\vec{O})\hat{z}) &&)/s\ ; &&(42)\\
=&(\quad L_z(\vec{O}+s\hat{y})-L_z(\vec{O}))\hat{x} &&+(L_x(\vec{O}+s\hat{z})-L_x(\vec{O}))\hat{y} &&+(L_y(\vec{O}+s\hat{x})+L_y(\vec{O})\hat{z}) &&)/s\ . &&(43)
\end{aligned}$$

This is the exact same physical vector, calculated right-hand or left hand. The expression is a vector, and it is eligible to be equated to other physical vectors like $\vec{P}$, when appropriate.

Two examples can illustrate why Eqs.(6) and (101 work equally in both right-hand rule and left-hand rule.

In this example, we know that $P_x = P'_x$, $L_z = L'_z$, and $y = y'$ are all unchanged by transforming to left-hand coordinates:

$$P_x = \partial L_z / \partial y \ ; \tag{44}$$

$$P'_x = \partial L'_z / \partial y' \ . \tag{45}$$

In this example, we know that $P_z = -P'_z$ and $L_y = -L'_y$, while x is unchanged by transforming to left-hand coordinates:

$$P_z = \partial L_y / \partial x \ ; \tag{46}$$

$$-P_z = -\partial L_y / \partial x \ ; \tag{47}$$

$$P'_z = \partial L'_y / \partial x' \ . \tag{48}$$

Regardless of chosen convention:

$$dL_k / dx_j = \epsilon_{ijk} P_i \qquad \text{where } i \notin (j,k) - Eq.(101). \tag{49}$$

4 Landau and Lifshitz

Total linear momentum is

$$\vec{P}_{total} \equiv \sum_{i=1}^{n} \vec{p}_i \,. \tag{50}$$

Angular momentum, which varies by the chosen observation point O, is

$$\vec{L}(\vec{O}) \equiv \sum_{i=1}^{n} (\vec{r}_i - \vec{O}) \times \vec{p}_i \,. \tag{51}$$

From these definitions, in their book *Mechanics*,[1] Landau and Lifshitz derived their Equation 9.4, proving that angular momentum $\vec{L}$, calculated in reference to observation points O_1 and O_2, differ by

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = \sum_{i=1}^{n} \Big(((\vec{r}_i - \vec{O}_2) \times \vec{p}_i) - ((\vec{r}_i - \vec{O}_1) \times \vec{p}_i) \Big) \tag{52}$$

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = \sum_{i=1}^{n} (\vec{O}_1 - \vec{O}_2) \times \vec{p}_i \tag{53}$$

$$\vec{L}(\vec{O}_2) = \vec{L}(\vec{O}_1) + (\vec{O}_1 - \vec{O}_2) \times \sum_{i=1}^{n} \vec{p}_i \tag{54}$$

$$\vec{L}(\vec{O}_2) = \vec{L}(\vec{O}_1) + (\vec{O}_1 - \vec{O}_2) \times \vec{P}_{total} \qquad \text{(LLM 9.4)}\,. \tag{55}$$

In classical mechanics, LLM 9.4 Eq.(55) is always true, for any group of particles, at any time t, in any chosen inertial frame of reference, as long as O_1, O_2 are unmoving in that frame, and $\vec{p}_1..\vec{p}_n$ are taken in reference to that frame, with or without external forces present. See Eqs. (133)-(138).

Thus, if you know angular momentum at a single observation point $\vec{L}(\vec{O}_1)$, and you know total linear momentum $\vec{P}_{total}$, this establishes angular momentum at every observation point. You now know the "angular momentum of the system". Knowing AM at a single observation point is feeling the elephant's trunk. Additionally knowing the total linear momentum, you see the entire elephant (in this frame of reference).

This specific linear momentum is necessary to produce this infinite pattern of angular momentum. Other systems, with a different linear momentum, can only match angular momentum values at a limited set of points.

Notice that the cross product term is unable to produce any component parallel to $\vec{P}_{total}$. If $\vec{L}$ has a component parallel to $\vec{P}_{total}$, it is equal at all observation points.

Notice that if $\vec{P}_{total} \neq 0$ and it is parallel to the displacement $(\vec{O}_1 - \vec{O}_2)$, then $(\vec{O}_1 - \vec{O}_2) \times \vec{P}_{total} = 0$, and $\vec{L}(\vec{O}_2) = \vec{L}(\vec{O}_1)$. So angular momentum has a constant unchanging value along any line parallel to $\vec{P}$. This means that every plane perpendicular to $\vec{P}_{total}$ repeats the angular momentum values of other perpendicular planes.

In each plane perpendicular to $\vec{P}_{total}$, every possible combination of angular momentum components perpendicular to $\vec{P}_{total}$ appear exactly once. If we knew the value of angular momentum at the center of mass, the single point in the plane with that same value would be a point on the projected path of the center of mass, parallel to the tunnel centerline. If $\vec{L}_{CoM}$ is zero, or parallel to $\vec{P}$, the centerline is the projected path.

By setting O_1 to the position of the center of mass, *Mechanics*[1] uses this equation to prove the well-known result that: total angular momentum at any point O_2 is equal to (angular momentum **at** the Center Of Mass (O_1)) plus angular momentum **of** (an imaginary particle with momentum $\vec{P}_{total}$ at) the Center of Mass:

$$\vec{L}(x,y,z) = \vec{L}(\vec{R}_{CoM}) + (\vec{R}_{CoM} - (x,y,z)) \times \vec{P}_{total} \,. \tag{56}$$

LLM 9.4 directly states this is true, not only for the center of mass, but for any O_1.

LLM 9.4 delivers another well-known result, that: if overall linear momentum $\vec{P} = 0$, then $(\vec{O}_1 - \vec{O}_2) \times \vec{P} = 0$, and $\vec{L}(\vec{O}_2) = \vec{L}(\vec{O}_1)$, angular momentum is equal at all (x, y, z) observation points.

5 Perpendicular Cross Division

Given any system of particles, if we know $\vec{L}(\vec{O}_1)$ and $\vec{L}(\vec{O}_2)$, then

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = (\vec{O}_1 - \vec{O}_2) \times \vec{P}_{total} \ . \tag{57}$$

If we want to calculate $\vec{P}$, we cannot simply divide $\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1)$ by $(\vec{O}_1 - \vec{O}_2)$ because the vector cross product ignores the component of $\vec{P}$ parallel to $(\vec{O}_1 - \vec{O}_2)$, and only uses the perpendicular component(s). The parallel component is discarded, and irretrievable.

We can obtain the component(s) of $\vec{P}$ perpendicular to $(\vec{O}_1 - \vec{O}_2)$ via perpendicular division:

Given $\vec{A}$ perpendicular to $\vec{B}_{perp}$, $\vec{A}$ parallel to $\vec{B}_{parr}$, and $\vec{C} = \vec{A} \times (\vec{B}_{perp} + \vec{B}_{parr})$, then:

$$0 = \vec{A} \times \vec{B}_{parr} \ ; \tag{58}$$

$$\vec{C} = \vec{A} \times \vec{B}_{perp} \ ; \tag{59}$$

$$\vec{B}_{perp} = (\vec{C} \times \vec{A})/(\vec{A} \cdot \vec{A}) \ ; \tag{60}$$

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = (\vec{O}_1 - \vec{O}_2) \times (\vec{P}_{perpendicular} + \vec{P}_{parallel}) \ ; \tag{61}$$

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = (\vec{O}_1 - \vec{O}_2) \times \vec{P}_{perpendicular} \ ; \tag{62}$$

$$\vec{P}_{perpendicular} = \Big((\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1)) \times (\vec{O}_1 - \vec{O}_2) \Big) \Big/ \Big((\vec{O}_1 - \vec{O}_2) \cdot (\vec{O}_1 - \vec{O}_2) \Big) \ . \tag{63}$$

In the case where $\vec{O}_2 - \vec{O}_1$ is parallel to an axis, for example $\vec{O}_2 - \vec{O}_1 = s\hat{x}$:

$$\vec{P}_{yz} = \Big((\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) \times (-s\hat{x}) \Big) /s^2 \ ; \tag{64}$$

$$\vec{P}_{yz} = \Big((L_x(\vec{O}_2) - L_x(\vec{O}_1) \ , \ L_y(\vec{O}_2) - L_y(\vec{O}_1) \ , \ L_z(\vec{O}_2) - L_z(\vec{O}_1)) \times (-1, 0, 0) \Big) /s \ ; \tag{65}$$

$$(0, P_y, P_z) = (\ 0 \ , \ L_z(\vec{O}_1) - L_z(\vec{O}_2) \ , \ L_y(\vec{O}_2) - L_y(\vec{O}_1) \ )/s \ ; \tag{66}$$

$$P_y = (L_z(\vec{O}_1) - L_z(\vec{O}_1 + s\hat{x}))/s \ ; \qquad (-\partial L_z/\partial x) \tag{67}$$

$$P_z = (L_y(\vec{O}_1 + s\hat{x}) - L_y(\vec{O}_1))/s \ . \qquad (\partial L_y/\partial x) \tag{68}$$

Of course, we can repeat this process for $s\hat{y}$ and $s\hat{z}$, which is another path to deriving Eqs. (20),(29),(31)

So knowing $\vec{L}$ at two points delivers two components of linear momentum. Perpendicular cross division reveals components of linear momentum perpendicular to the displacement. We need a third point, not co-linear with $\vec{O}_1 - \vec{O}_2$, to find the remaining component, as in Eq.32

6 If $\vec{L}$ is unchanging at position O_1

Given a system of particles where angular momentum at a chosen point $\vec{L}(O_1)$ is unchanging over time, $\vec{L}(O_1, t_1) = \vec{L}(O_1, t_2)$. If $\vec{p}(t_1) \neq \vec{p}(t_2)$ linear momentum has changed, then

$$\vec{L}(O_2, t_1) = \vec{L}(O_1) + (\vec{O}_1 - \vec{O}_2) \times \vec{p}(t_1); \tag{69}$$

$$\vec{L}(O_2, t_2) = \vec{L}(O_1) + (\vec{O}_1 - \vec{O}_2) \times \vec{p}(t_2)\ ; \tag{70}$$

$$\vec{L}(O_2, t_2) - \vec{L}(O_2, t_1) = (\vec{O}_1 - \vec{O}_2) \times (\vec{p}(t_2) - \vec{p}(t_1))\ . \tag{71}$$

For some O_2 observation points, $(\vec{O}_1 - \vec{O}_2)$ will be parallel to $(\vec{p}(t_2) - \vec{p}(t_1))$, so the cross product will be zero, and $\vec{L}(O_2, t_2) = \vec{L}(O_2, t_1)$ angular momentum at $\vec{O}_2$ will be unchanged.

For most O_2, $(\vec{O}_1 - \vec{O}_2)$ will not be parallel to $(\vec{p}(t_2) - \vec{p}(t_1))$, so the cross product will be nonzero, and $\vec{L}(O_2, t_2) \neq \vec{L}(O_2, t_1)$.

So changing linear momentum demands changing angular momentum, not everywhere, but **almost everywhere**.

Conversely, if angular momentum is conserved **everywhere**, then linear momentum **must be conserved** also.

Of course, there are countless examples of angular momentum unchanging at some chosen reference point, while linear momentum is changing. In all of these examples, angular momentum is not conserved, changing at nearby observation points.

If $\vec{L}(O_1)$ is unchanging at O_1, and $\vec{p}(t_2) = \vec{p}(t_1)$ is also conserved, then

$$\vec{p}(t_2) - \vec{p}(t_1) = (0, 0, 0)\ ; \tag{72}$$

$$\vec{L}(O_2, t_2) - \vec{L}(O_2, t_1) = (\vec{O}_1 - \vec{O}_2) \times (0, 0, 0)\ ; \tag{73}$$

$$\vec{L}(O_2, t_2) = \vec{L}(O_2, t_1)\ ; \tag{74}$$

angular momentum is conserved at all O_2 (at every (x, y, z)).

Angular momentum unchanging at one observation point, plus conservation of linear momentum, is full conservation of angular momentum.

7 Linear Algebra Equations

For a system of particles at time t, the coefficient values of angular momentum L_x, L_y, L_z are a linear combination of the position coefficients r_{ix}, r_{iy}, r_{iz} of the particles, the momentum coefficients of the particles p_{ix}, p_{iy}, p_{iz}, and the position coefficients of the observation point x, y, z .

Each observation point $\vec{O}_j = (x_j, y_j, z_j)$ provides us with three linear equations:

$$L_{jx} = \sum_i r_{iy}p_{iz} - y_j p_{iz} - r_{iz}p_{iy} + z_j p_{iy} \ ; \tag{75}$$

$$L_{jy} = \sum_i r_{iz}p_{ix} - z_j p_{ix} - r_{ix}p_{iz} + x_j p_{iz} \ ; \tag{76}$$

$$L_{jz} = \sum_i r_{ix}p_{iy} - x_j p_{iy} - r_{iy}p_{ix} + y_j p_{ix} \ . \tag{77}$$

We have $6N$ unknowns: $r_{1x}, r_{1y}, r_{1z}, p_{1x}, p_{1y}, p_{1z}...r_{Nx}, r_{Ny}, r_{Nz}, p_{Nx}, p_{Ny}, p_{Nz}$. If we know the value of angular momentum at unlimited observation points, we can gather more equations than unknowns, which raises the possibility of solving for all the unknowns. But the equations are linearly co-dependent, so we are unable to solve for each particle position and momentum.

$$\vec{L}(x_2, y_2, z_2) = \vec{L}(x_1, y_1, z_1) + ((x_1, y_1, z_1) - (x_2, y_2, z_2)) \times \vec{P}_{total} \ . \qquad \text{Eq.(55)}$$

If two different systems have identical linear momentum, and identical angular momentum at one observation point (x_1, y_1, z_1), then they have identical angular momentum at every observation point (x_2, y_2, z_2). For any system, there are infinite other systems with identical AM values. Their angular momentum values will not distinguish between them. Their numerous linear equations will not solve to individual particle positions and momentums.

$$L_{jx} = \left(\sum r_{iy}p_{iz} - r_{iz}p_{iy}\right) - \left(\sum y_j p_{iz}\right) + \left(\sum z_j p_{iy}\right) \ ; \tag{78}$$

$$L_{jx} = L_x(0, 0, 0) - y_j P_z + z_j P_y \ . \tag{79}$$

Given angular momentum values in a *single* reference frame, we can only solve for total linear momentum (P_x, P_y, P_z). The center of mass remains elusive.

Given the total mass, position and velocity of the center of mass, and angular momentum at the center of mass in one frame, we can calculate angular momentum at any point, in any frame. We can also generate infinite other systems that share these same values, and therefore have identical angular momentum at every observation point, in every frame.

Not coincidentally, if we know angular momentum at enough points, in enough frames, we can calculate total mass, position and velocity of the center of mass, and angular momentum at the center of mass.

We can solve for $\vec{P}$ in two frames, obtaining total mass and CoM velocity, then find the frame where $\vec{P} = 0$. In this frame $\vec{L}$ is constant everywhere, which is also the value of $\vec{L}_{CoM}$ at the Center of Mass in every frame (Eq.115).

In every frame where the CoM is moving, the single line where $\vec{L}(x, y, z) = \vec{L}_{CoM}$ is the projected path of the CoM. In a frame where CoM is moving parallel to $\hat{x}$, the path provides the coordinates y and z. In a frame where CoM is moving parallel to $\hat{y}$, the path provides the coordinates x and z. Having the coordinates in the frame where CoM is unmoving, we can calculate its position in any frame.

Full knowledge of angular momentum fully describes the center of mass.

8 Camouflage

It is a curious mathematical quirk that the tunnel pattern can completely camouflage components of $\vec{L}_{CoM}$ perpendicular to $\vec{P}$. For example, two particles traveling in opposite directions along parallel paths produce an angular momentum pattern completely identical to an imaginary single particle, but the imaginary single particle travels parallel to the center of mass, such that it produces the necessary non-zero angular momentum at the center of mass position.

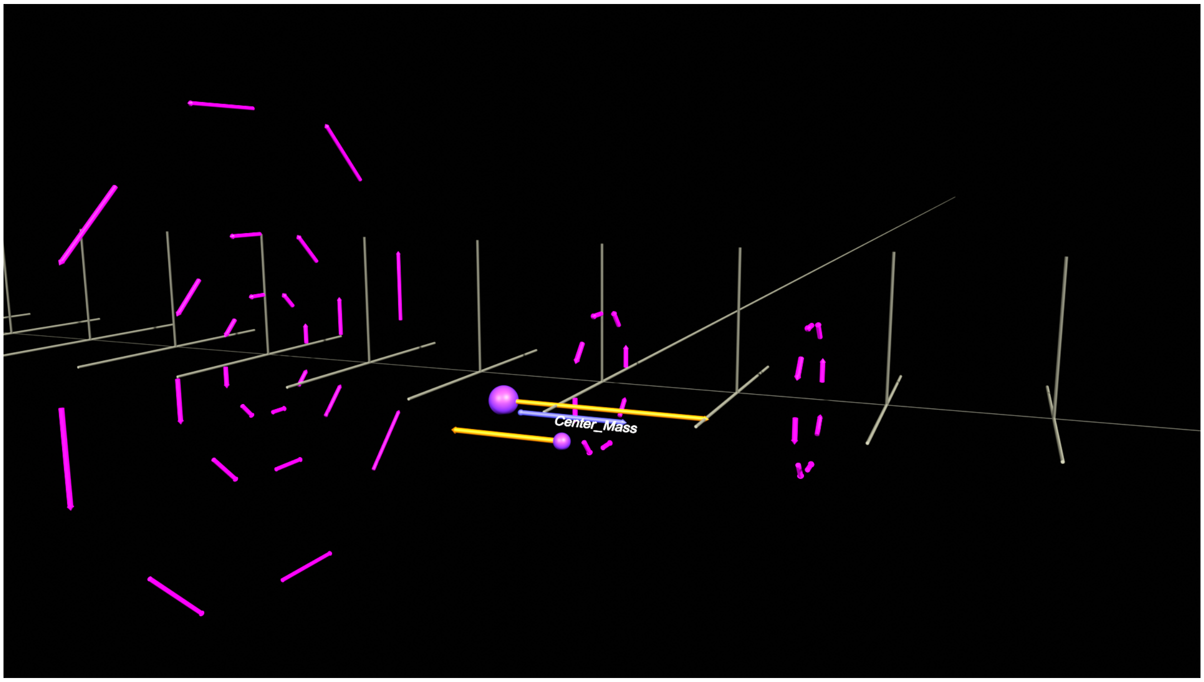

Figure 2: two particles anti-parallel

Given two particles:

$$\text{particle 1} \quad \text{mass } 3 \quad \text{position } (0,-1,0) \quad \text{momentum } (2,0,0) \tag{80}$$
$$\text{particle 2} \quad \text{mass } 1 \quad \text{position } (0,-2,0) \quad \text{momentum } (-1,0,0) \tag{81}$$

Total linear momentum is $(1,0,0)$.

Particle two is twice as far from the x axis, but has half the momentum, in the opposite direction. At the axis, the two angular momentums add to zero.

At any observation point (x,y,z), total angular momentum is

$$\vec{L}(x,y,z) = (((0,-1,0)-(x,y,z)) \times (2,0,0)) + (((0,-2,0)-(x,y,z)) \times (-1,0,0)) \; ; \tag{82}$$
$$\vec{L}(x,y,z) = (0,0,2) - (0,2z,-2y) + (0,0,-2) - (0,-z,y) \; ; \tag{83}$$
$$\vec{L}(x,y,z) = (0,-z,y) \; ; \tag{84}$$

which is exactly the angular momentum produced by a particle traveling along the x axis with momentum (1,0,0).

It is also equal to

$$\vec{L}(x,y,z) = (0,0,-1.25) + (0,-z,y+1.25) \; ;$$

which is exactly the angular momentum at the center of mass, plus angular momentum produced by an imaginary particle at the center of mass, with this same momentum (1,0,0).

While the pattern of angular momentum reveals $\vec{P}$, the value of $\vec{L}_{CoM}$ is completely hidden among the forest of various $\vec{L}(x,y,z)$ values.

So any system that has same linear momentum as our imaginary x-axis particle, and same angular momentum at any single observation point, has the same identical pattern of angular momentum at every observation point. Using only AM values in this single frame, we cannot distinguish between these systems, we cannot deduce the path of the center of mass, or deduce the angular momentum at the CoM.

We can produce an infinite number of particle systems with this identical pattern of angular momentum. Each system must have linear momentum of $(1, 0, 0)$, and a center of mass offset from the pattern centerline to compensate for angular momentum at the center of mass.

Shifting the pattern sideways completely hides components of angular momentum at the center of mass that are perpendicular to linear momentum, but tunnel pattern has no component parallel to total linear momentum. If angular momentum at the center of mass has a component parallel to total linear momentum, every observation point has that same component. The pattern cannot hide it, it will be obvious along the centerline of the pattern, and elsewhere. The angular momentum pseudovectors will not lie in a plane perpendicular to $\vec{P}$.

These two particles are not moving parallel to each other:

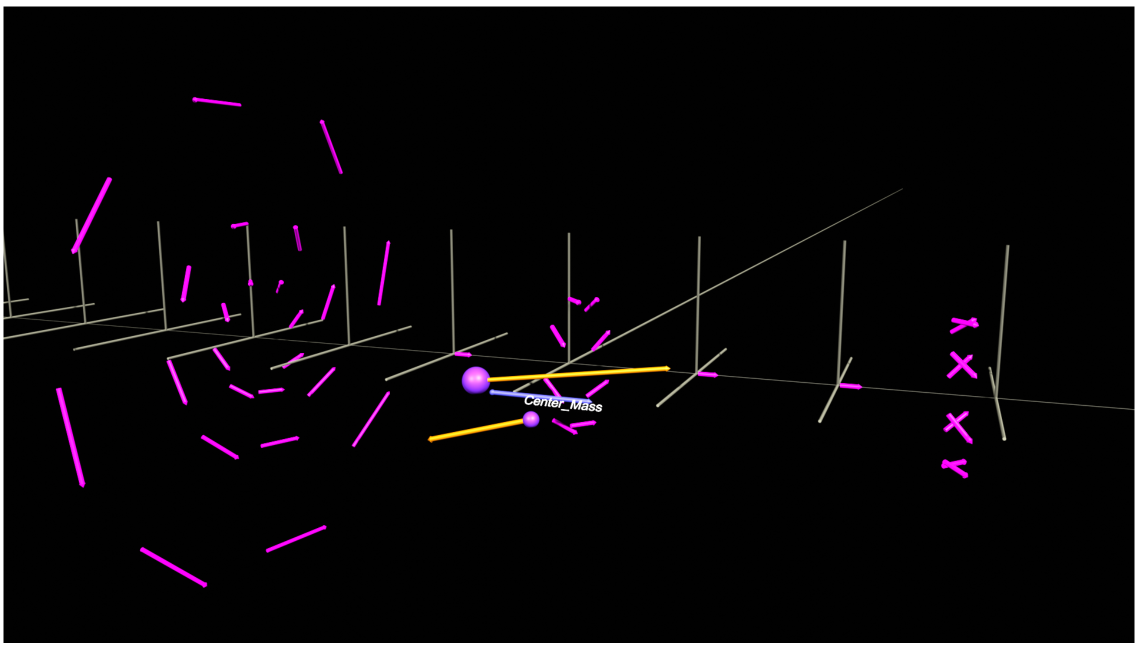

Figure 3: two particles with $\vec{L} \cdot \vec{P} \neq 0$

9 Torque Translation

Around any observation point $\vec{O}$, external forces provide net force and net torque.

The net force provides acceleration of the center of mass.

Over time, the net torque alters the angular momentum around $\vec{O}$.

$$\vec{\tau}((0,0,0)) = \sum_i \vec{r}_i \times \vec{f}_i \ ; \tag{85}$$

$$\vec{\tau}(\vec{O}) = \sum_i (\vec{r}_i - \vec{O}) \times \vec{f}_i \ ; \tag{86}$$

$$\vec{\tau}(\vec{O}_2) - \vec{\tau}(\vec{O}_1) = \sum_i \Big(((\vec{r}_i - \vec{O}_2) \times \vec{f}_i) - ((\vec{r}_i - \vec{O}_1) \times \vec{f}_i) \Big) \ ; \tag{87}$$

$$\vec{\tau}(\vec{O}_2) = \vec{\tau}(\vec{O}_1) + \sum_i (\vec{O}_1 - \vec{O}_2) \times \vec{f}_i \ ; \tag{88}$$

$$\vec{\tau}(\vec{O}_2) = \vec{\tau}(\vec{O}_1) + (\vec{O}_1 - \vec{O}_2) \times \sum_i \vec{f}_i \ ; \tag{89}$$

$$\vec{\tau}(\vec{O}_2) = \vec{\tau}(\vec{O}_1) + (\vec{O}_1 - \vec{O}_2) \times \vec{F}_{net} \ . \tag{90}$$

Given torque $\vec{\tau}$ with respect to one observation point, and net force $\vec{F}_{net}$, we can derive torque at any observation point.

Due to pairwise $\vec{f}_{ij} = -\vec{f}_{ji}$ cancellation, the sum of internal forces is zero, so $\vec{F}_{net}$ (sum of all forces) is also the sum of all external forces.

If the forces add to zero $\vec{F}_{net} = 0$, conserving linear momentum, $(\vec{O}_1 - \vec{O}_2) \times \vec{F}_{net} = 0$, so then $\vec{\tau}(\vec{O}_2) = \vec{\tau}(\vec{O}_1)$, torque is the same at all observation points. The angular momentum at all observation points may change, but the differences in angular momentum between any two points remain unchanged.

If $\vec{F}_{net} \neq 0$ then torque varies across x, y, z (but stays constant in the direction of $\vec{F}_{net}$):

$$\vec{\tau}((x,y,z)) = \vec{\tau}((0,0,0)) - (x,y,z) \times \vec{F}_{net} \ ; \tag{91}$$

$$\vec{\tau}((x,y,z)) = \vec{\tau}((0,0,0)) - (yF_z - zF_y, zF_x - xF_z, xF_y - yF_x) \ ; \tag{92}$$

$$\nabla\vec{\tau}(x,y,z) = \begin{bmatrix} \frac{\partial\tau_x}{\partial x} & \frac{\partial\tau_x}{\partial y} & \frac{\partial\tau_x}{\partial z} \\ \frac{\partial\tau_y}{\partial x} & \frac{\partial\tau_y}{\partial y} & \frac{\partial\tau_y}{\partial z} \\ \frac{\partial\tau_z}{\partial x} & \frac{\partial\tau_z}{\partial y} & \frac{\partial\tau_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -F_z & F_y \\ F_z & 0 & -F_x \\ -F_y & F_x & 0 \end{bmatrix} \ ; \tag{93}$$

$$\partial\tau_k / \partial j = \epsilon_{ijk} F_i \qquad \text{where } i \notin (j,k) \ . \tag{94}$$

At time t, the torque τ varies **linearly** across space, which helps explain why angular momentum varies linearly across space. If $\vec{F}_{net} \neq 0$, torque produces a tunnel pattern.

10 Jacobian of Angular Momentum

At time t, angular momentum at some chosen observation point (x, y, z) is

$$\vec{L}(x,y,z) \equiv \sum_{i=1}^{n} (\vec{r}_i - (x,y,z)) \times \vec{p}_i \ ; \tag{95}$$

$$\vec{L}(x,y,z) = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i - \sum_{i=1}^{n} (x,y,z) \times \vec{p}_i \ ; \tag{96}$$

$$\vec{L}(x,y,z) = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i - (x,y,z) \times \sum_{i=1}^{n} \vec{p}_i \ ; \tag{97}$$

$$\vec{L}(x,y,z) = \vec{L}(0,0,0) - (x,y,z) \times \vec{P}_{total} \ ; \tag{98}$$

$$\vec{L}(x,y,z) = \vec{L}(0,0,0) - (\ yP_z - zP_y,\ zP_x - xP_z,\ xP_y - yP_x\ ) \ ; \tag{99}$$

$$\nabla \vec{L}(x,y,z) = \begin{bmatrix} \frac{\partial L_x}{\partial x} & \frac{\partial L_x}{\partial y} & \frac{\partial L_x}{\partial z} \\ \frac{\partial L_y}{\partial x} & \frac{\partial L_y}{\partial y} & \frac{\partial L_y}{\partial z} \\ \frac{\partial L_z}{\partial x} & \frac{\partial L_z}{\partial y} & \frac{\partial L_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -P_z & P_y \\ P_z & 0 & -P_x \\ -P_y & P_x & 0 \end{bmatrix} \ ; \tag{100}$$

$$dL_k/dx_j = \epsilon_{ijk} P_i \qquad \text{where } i \notin (j,k). \tag{101}$$

At time t, Eq. (100) shows the jacobian has the same value at each and every (x, y, z). Components of Linear momentum are the constant partial derivatives of angular momentum. Significantly, each component of angular momentum varies linearly across (x, y, z).

The value of $\vec{L}(0, 0, 0)$ is arbitrary, so the value of angular momentum **at any single point** does not determine linear momentum. But at every x, y, z, the partial derivatives of $\vec{L}(x, y, z)$ are coefficients of $\vec{P}_{total}$, or zero , and can be used to determine linear momentum, as seen in the iterative proof Eq.(20).

Angular and linear momentum have a relationship something like velocity and constant acceleration, but across the three dimensions of space instead of the single dimension of time. Linear momentum linearly "accelerates" angular momentum over distance.

If $\vec{L}(x, y, z)$ is fully conserved (across all x, y, z) then the gradient $\nabla(\vec{L}(x, y, z))$ is also conserved, P_x, P_y, P_z are conserved, and $\vec{P}$ is conserved. Conservation of linear momentum is a necessary condition of full conservation of angular momentum.

"Conservation of Angular Momentum at all x, y, z" requires and enforces "Conservation of Linear Momentum".

This shows that cases where angular momentum is only unchanging at a single point are very different than those where angular momentum is fully conserved. There are countless examples of "angular momentum unchanging at some chosen point" where linear momentum is not conserved.

11 Rise over Run

Using the x component of Eq. (99),

$$L_x = L_x(0,0,0) - yP_z + zP_y \ . \tag{102}$$

At time t , $L_x(0,0,0)$, P_z , P_y are all constants. If we consider the line where $x = C_x$, $z = C_z$ are held constant, then along that line

$$L_x(C_x, y, C_z) = (L_x(0,0,0) + C_z P_y) - yP_z \tag{103}$$

is a straight line with slope $dL_x/dy = -P_z$ and intercept $L_x(0,0,0) + C_z P_y$.

($dL_x/dx = 0$, the value of L_x is constant with respect to x .)

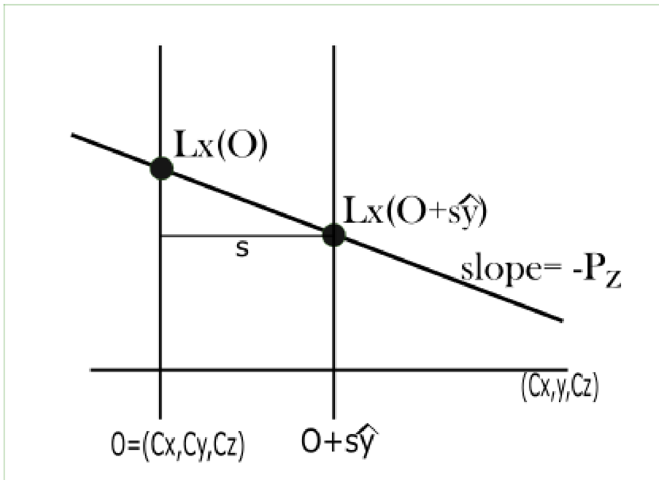

Figure 4: L_x plotted as a straight line over y with slope $-P_z$

To calculate the P_z (- slope) of the line, we calculate rise/run between two points on the line:

$$P_z = -(L_x(C_x, C_y + s, C_z) - L_x(C_x, C_y, C_z))/s \ ;$$

which is the P_z term of Eq. (29).

Changing the constant value $\vec{L}(0,0,0)$ will not affect the slope, but it will raise or lower the line, and change where $L_x = 0$. This shifts the position of the centerline of the angular momentum "tunnel" sideways (orthogonal to $\vec{P}$), but does not affect its direction.

12 Center of Mass

It is well known that angular momentum around a group of particles is equal to angular momentum at the center of mass position, plus angular momentum of an imaginary particle at the position of the center of mass with momentum $\vec{P}_{total}$. This is not a special property of the Center of Mass.

Laundau and Lifshitz *Mechanics*[1] Equation 9.4 (Eq.55) shows this applies to every point, not just the CoM.

We can illustrate this:

Given a system of bodies with center of mass at position $\vec{R}_{CoM1}$, we know that angular momentum at observation point $\vec{R}_{obv}$ is $\vec{L}(\vec{R}_{CoM1})$ plus $(\vec{R}_{CoM1} - \vec{R}_{obv}) \times \vec{P}_{total}$.

If we add a new unmoving body, at a new position, angular momentum will not change at any observation point, but the position of the Center of Mass will change. Depending on the position and mass of the new body, the Center of Mass $\vec{R}_{CoM2}$ can be anywhere we choose, without affecting linear or angular momentum (in this frame of reference).

We then know that angular momentum at observation point $\vec{R}_{obv}$, with or without the unmoving body, is $L(\vec{R}_{CoM2})$ plus $(\vec{R}_{CoM2} - \vec{R}_{obv}) \times \vec{P}_{total}$, and $\vec{R}_{CoM2}$ can be any point we want. Which agrees with *Mechanics*[1] Equation (9.4).

Although the CoM is not special in that way, it is special in another way:

$$M = \sum_i m_i \; ; \tag{104}$$
$$\vec{R}_{CoM} \equiv \sum_i \vec{r}_i m_i / M \; ; \tag{105}$$
$$\vec{V}_{CoM} = \sum_i \vec{v}_i m_i / M \; ; \tag{106}$$
$$\vec{L}_{CoM} = \sum_i (\vec{r}_i - \vec{R}_{CoM}) \times \vec{v}_i m_i \; . \tag{107}$$

In any other frame of reference, angular momentum at the center of mass $\vec{L}'_{CoM}$ has the same value :

$$\vec{v}'_i = v_i + V' \; ; \tag{108}$$
$$\vec{r}'_i - \vec{R}'_{CoM} = \vec{r}_i - \vec{R}_{CoM} \; ; \tag{109}$$
$$\vec{L}'_{CoM} = \sum_i (\vec{r}'_i - \vec{R}'_{CoM}) \times \vec{v}'_i m_i \; ; \tag{110}$$
$$= \sum_i (\vec{r}_i - \vec{R}_{CoM}) \times (\vec{v}_i + \vec{V}') m_i \; ; \tag{111}$$
$$= \sum_i (\vec{r}_i - \vec{R}_{CoM}) \times \vec{v}_i m_i + \sum_i (\vec{r}_i - \vec{R}_{CoM}) \times \vec{V}' m_i \; ; \tag{112}$$
$$= \vec{L}_{CoM} + \left(\left(\sum_i \vec{r}_i m_i \right) - \left(\sum_i \vec{R}_{CoM} m_i \right) \right) \times \vec{V}' \; ; \tag{113}$$
$$= \vec{L}_{CoM} + \left(M\vec{R}_{CoM} - M\vec{R}_{CoM} \right) \times \vec{V}' \; ; \tag{114}$$
$$\vec{L}'_{CoM} = \vec{L}_{CoM} \; . \tag{115}$$

The angular momentum at the CoM is equal in all frames of reference, (and equal along all projected paths of the CoM, because angular momentum does not vary in the direction of $\vec{P}_{total}$).

So if we find the frame of reference where $\vec{P}_{total} = 0$, which is also the frame where $\vec{L}$ is constant everywhere, that value of $\vec{L}$ is the value of $\vec{L}_{CoM}$ for every frame of reference.

13 Parabolic Projectile

When a projectile travels in a parabolic curved arc due to a constant external force ($-mg$) of gravity, we do not usually think of it as "conserving" angular momentum.

If we choose an observation point that the projectile passes over, at that instant, the force will be co-linear with the observation-projectile displacement, so instantaneous torque will be zero.

If we choose a moving inertial frame, in the same horizontal direction and speed as the projectile, then it looks like the projectile is moving and accelerating only in the z direction. If we choose an observation point in that frame directly above or below the projectile, angular momentum at that point is zero, and unchanging.

This does not mean that projectiles "conserve" angular momentum. When angular momentum is conserved, it is unchanging at every observation point. We are choosing to ignore observation points with changing angular momentum.

Still, Eq. (63) holds true, under any and all conditions. In our chosen frame, suppose the projectile at time t_1 is at $z = z_1$ rising with momentum $(0, 0, 3)$kg·m/s. If we choose $\vec{O}_1$ to be at ground level below the projectile, and we choose a second observation point such that $\vec{O}_2 - \vec{O}_1 = (2, 0, 0)$m , we can get $\vec{P}$ perpendicular to $\vec{O}_2 - \vec{O}_1$ from their angular momentum values:

$$\vec{O}_2 - \vec{O}_1 = (0, 0, 2); \tag{116}$$

$$\vec{L}_1 = (0, 0, z_1) \times (0, 0, 3) = (0, 0, 0); \tag{117}$$

$$\vec{L}_2 = (-2, 0, z_1) \times (0, 0, 3) = (0, 6, 0); \tag{118}$$

$$\vec{L}_2 - \vec{L}_1 = (0, 6, 0); \tag{119}$$

$$\vec{P}_{perpendicular} = \Big((\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1)) \times (\vec{O}_1 - \vec{O}_2)\Big) \,/\, \Big((\vec{O}_1 - \vec{O}_2) \cdot (\vec{O}_1 - \vec{O}_2)\Big) \qquad Eq.(63);$$

$$\vec{P}_{yz} = ((0, 6, 0) \times (-2, 0, 0)) \,/ (2 \cdot 2) \ ; \tag{120}$$

$$(0, P_y, P_z) = (0, 0, 12)/(2 \cdot 2); \tag{121}$$

$$(0, P_y, P_z) = (0, 0, 3) \ . \tag{122}$$

Components P_y and P_z calculated from angular momentum values match the momentum given. We would need to use a different displacement to obtain P_x.

If later, at time t_2, the projectile is at $z = z_2$ with momentum (0,0,-7).

$$\vec{L}_1 = (0, 0, z_2) \times (0, 0, -7) = 0; \tag{123}$$

$$\vec{L}_2 = (-2, 0, z_2) \times (0, 0, -7) = (0, -14, 0); \tag{124}$$

$$\vec{P}_{yz} = ((0, -14, 0) \times (-2, 0, 0)) \,/ (2 \cdot 2) \ ; \tag{125}$$

$$(0, P_y, P_z) = (0, 0, -28)/(2 \cdot 2); \tag{126}$$

$$(0, P_y, P_z) = (0, 0, -7) \ . \tag{127}$$

Any time we have angular momentum values at two points, we can calculate the component(s) of momentum perpendicular to their displacement. If the difference between their AM values changes between t_1 and t_2, momentum has changed. If the difference between their AM values remains the same, those components of momentum remain unchanged. If all AM values across (x, y, z) remain the same, linear momentum is entirely unchanged.

14 Central Force

If particle i is orbiting in an external gravitational field centralized around origin $\vec{O}_1 = (0,0,0)$, if we only observe at the origin, it appears as if there is no torque, and that angular momentum is unchanging, but linear momentum is changing. There is zero torque around (0,0,0).

This is like viewing a leaky pump from one specific position, and deciding it looks good.

There is non-zero torque around all points not linearly aligned with the $(\vec{r}_i - \vec{O})$.

Example: if the particle is at position $\vec{r}_1 = (10,10,10)$ meters, and the central force is currently $\vec{f}_1 = (-3,-3,-3)$ newtons, then the chosen alternative observation point $\vec{O}_2 = (10,0,0)$ will view that force as a torque of

$$\begin{aligned}
\vec{\tau}(\vec{O}_2) &= (\vec{r}_1 - \vec{O}_2) \times \vec{f}_1 \ ; \\
\vec{\tau}(\vec{O}_2) &= ((10,10,10) - (10,0,0)) \times (-3,-3,-3)\mathrm{n\,m} \ ; \\
\vec{\tau}(\vec{O}_2) &= (0,-30,30)\mathrm{n\,m} \, .
\end{aligned}$$

Angular momentum around $\vec{O}_2 = (10,0,0)$ is not conserved, it is changing. If all observation points voted, the proposition "angular momentum is being conserved" would lose, infinity-to-1. Unbalanced central forces do not obey the law "Conservation of Angular Momentum at all x, y, z" like isolated systems do. We choose specific observation points to filter out and ignore existing inconvenient non-conservation.

And still, at any time t, Eqs. (55) and (63) both hold true. Even though $\vec{P}$ and $\vec{L}(10,0,0)$ are both changing over time, they are linked. Since we chose an $\vec{O}_2$ with displacement $\vec{O}_2 - \vec{O}_1 = (10,0,0)$:

$$\vec{P}_{yz} = (0, Py, Pz) = \Big((\vec{L}(10,0,0) - \vec{L}(0,0,0)) \times (-10,0,0) \Big) / (10 \cdot 10) \ ; \tag{128}$$

$$\vec{P}_{yz} = (0, Py, Pz) = (0 \ , \ L_z(0,0,0) - L_z(10,0,0) \ , \ L_y(10,0,0) - L_y(0,0,0))\,/10 \ ; \tag{129}$$

is always true, for any group of particles, under any forces, in an inertial frame.

If, instead, for n particles, $\vec{L}(0,0,0)$ and $\vec{L}(10,0,0)$ were both unchanging (neither see torque, which rules out the unbalanced central force, but would allow internal forces, and certain law-abiding external forces), then P_y and P_z would therefore be unchanging also. And if $\vec{L}(0,7,0)$ was unchanging as well, by Eq. (63), P_x would also be unchanging:

$$\vec{P}_{xz} = (Px, 0, Pz) = \Big((\vec{L}(0,7,0) - \vec{L}(0,0,0)) \times (0,-7,0) \Big) / (7 \cdot 7) \ ; \tag{130}$$

$$\vec{P}_{xz} = (Px, 0, Pz) = (L_z(0,7,0) - L_z(0,0,0) \ , \ 0 \ , \ L_x(0,0,0) - L_x(0,7,0))\,/7 \ . \tag{131}$$

So conservation of angular momentum at any three non-linear points, like $(0,0,0)$, $(10,0,0)$,$(0,7,0)$, completely guarantees conservation of total linear momentum $\vec{P} = (P_x, P_y, P_z)$.

15 Angular Momentum Conserved

Given a group of bodies moving with various momentums, they may possibly completely conserve angular momentum.

It does not matter if the bodies are converging, orbiting, or separating. It does not matter if they interact using internal body-body forces, or if there are external forces present.

It does matter that internal forces directly oppose, as in $\vec{f}_{ij} = -\vec{f}_{ji}$ and $\vec{f}_{ij} \times (\vec{r}_i - \vec{r}_j) = 0$, which cancels internal torques pairwise.

It does matter that external torque is $\sum(\vec{r}_i - \vec{O}) \times \vec{f}_{i\_Ext} = 0$ zero at some chosen observation point.

It does matter that external forces balance $\sum \vec{f}_{i\_Ext} = 0$, so torque is equal at all observation points.

(Therefore torque is zero at every observation point.)

The angular momentum at any (every) point $\vec{O}$ is

$$\vec{L}(\vec{O}) = \vec{L}(\vec{R}_{CoM}) + (\vec{R}_{CoM} - \vec{O}) \times (\vec{P}_{total}) \ . \tag{132}$$

The unchanging angular momentum at the center of mass, along with the unchanging linear momentum, guarantees that the angular momentum is unchanging everywhere, forming the classic tunnel pattern. This also carries over to all other frames of reference. Every frame of reference has an unchanging pattern of angular momentum.

This is what conservation of angular momentum looks like.

In each frame of reference, the unchanging pattern of angular momentum there expresses the frame-relative unchanging total linear momentum. The complete angular momentum pattern cannot remain unchanged without the partnership of conserved linear momentum.

The constraints $\vec{f}_{ij} = -\vec{f}_{ji}$ and $\sum \vec{f}_{i\_Ext} = 0$ enact the foundation of linear momentum conservation. Upon that foundation, the constraints $\vec{f}_{ij} \times (\vec{r}_i - \vec{r}_j) = 0$ and $\sum(\vec{r}_i - \vec{O}) \times \vec{f}_{i\_Ext} = 0$ are able to construct conservation of angular momentum.

16 Demonstration Script

A computer script can quickly choose random particles and perform the math, to easily demonstrate the relationship between Linear and Angular momentum.

The script "angularLinearDemo01" can be run at this www.glowscript.org link (user davidbgraham).

Glowscript.org allows users to safely run scripts shared by others, which is useful to educators.

The code of the script is:

```
Web VPython 3.2
cm = ','
print_options(width=1400, height=650, readonly=False, digits=5)

bodyCt =input('Enter number of particles to randomly generate\n ( integer )\n ( 3 is plenty ):')

s=input('Enter an offset distance\n ( floating point )\n ( 0 will cause divide errors )\n ( simplest is 1 ):')
s=float(s)

print( 'Please note that VPython uses <> to delimit vectors !' )

print()
print( 'randomly choose', bodyCt, 'body positions and momentums:' )
rvec = [vector( 20*random() - 10, 20*random() - 10, 20*random() - 10 ) for i in range(bodyCt)] # get random positions
pvec = [vector( 20*random() - 10, 20*random() - 10, 20*random() - 10 ) for i in range(bodyCt)] # get random momentums

ptotal = vector(0,0,0)
for i in range(bodyCt):
    print ( 'r_'+str(i),'=',rvec[i], ' p_'+str(i),'=',pvec[i] )
    ptotal = ptotal + pvec[i] # sum(p_i)

LO = vector(0,0,0) # A.M. observed at O
LOsx = vector(0,0,0) # A.M. observed at (O + (s,0,0)
LOsy = vector(0,0,0) # A.M. observed at (O + (0,s,0)
LOsz = vector(0,0,0) # A.M. observed at (O + (0,0,s)

ovec = vector( 20*random() - 10, 20*random() - 10, 20*random() - 10 ) # get random observation point
Osx = ovec + vector(s,0,0) # get offset observation point
Osy = ovec + vector(0,s,0) # get offset observation point
Osz = ovec + vector(0,0,s) # get offset observation point

for i in range(bodyCt):
    LO = LO + (rvec[i]-ovec).cross(pvec[i]) # AM = sum(r_i X p_i)
    LOsx = LOsx + (rvec[i]-Osx).cross(pvec[i]) # AM = sum(r_i X p_i)
    LOsy = LOsy + (rvec[i]-Osy).cross(pvec[i]) # AM = sum(r_i X p_i)
    LOsz = LOsz + (rvec[i]-Osz).cross(pvec[i]) # AM = sum(r_i X p_i)

print()
print( 's =',s, ' # offset distance between observation points' )
print( 'O = ',ovec, ' # random observation point' )
print( 'define Osx as O + (s,0,0) = ',Osx, ' # observation point offset by s in x direction' )
print( 'define Osy as O + (0,s,0) = ',Osy, ' # observation point offset by s in y direction' )
print( 'define Osz as O + (0,0,s) = ',Osz, ' # observation point offset by s in z direction' )

print()
print('Angular Momentum at observation points equals sum(( r_i - r_observationPt ) X p_i ):')
print( 'define LO as L(O) = ', LO )
print( 'define LOsx as L(Osx) = ', LOsx )
print( 'define LOsy as L(Osy) = ', LOsy )
print( 'define LOsz as L(Osz) = ', LOsz )

print()
print( 'Differential Angular Momentum:')
print( '(LOsx-LO)/s = <(LOsx_x-LO_x)/s , (LOsx_y-LO_y)/s , (LOsx_z-LO_z)/s >= ', (LOsx-LO)/s, ' # should equal <0, P_z, -P_y >')
print( '(LOsy-LO)/s = <(LOsy_x-LO_x)/s , (LOsy_y-LO_y)/s , (LOsy_z-LO_z)/s >= ', (LOsy-LO)/s, ' # should equal <-P_z, 0, P_x >')
print( '(LOsz-LO)/s = <(LOsz_x-LO_x)/s , (LOsz_y-LO_y)/s , (LOsz_z-LO_z)/s >= ', (LOsz-LO)/s, ' # should equal <P_y, -P_x, 0 >')

print()
print( 'Linear Momentum:' )
print( 'P_total = <P_x, P_y, P_z >= calculated as sum(p_i) = ', ptotal )
print( '<(LOsy_z-LO_z)/s, (LOsz_x-LO_x)/s, (LOsx_y-LO_y)/s >= <', ((LOsy-LO)/s).dot(vector(0,0,1)), cm, ((LOsz-LO)/s).dot(vector(1,0,0)),
cm, ((LOsx-LO)/s).dot(vector(0,1,0)), ">" )
print( '<(LO_y-LOsz_y)/s, (LO_z-LOsx_z)/s, (LO_x-LOsy_x)/s >= <', ((LO-LOsz)/s).dot(vector(0,1,0)), cm, ((LO-LOsx)/s).dot(vector(0,0,1)),
cm, ((LO-LOsy)/s).dot(vector(1,0,0)), ">" )

print()
print( 'For this random group of particles, \n\ndid differential angular momentum components produce accurate linear momentum?' )
```

Typically, the rate of change of angular momentum matches total linear momentum, as in these sample results from 17 randomly generated particles:

P_total = <P_x, P_y, P_z >= calculated as sum(p_i) = <-15.271, 31.293, 12.191 >

<(LOsy_z-LO_z)/s, (LOsz_x-LO_x)/s, (LOsx_y-LO_y)/s >= <-15.271 , 31.293 , 12.191 >

<(LO_y-LOsz_y)/s, (LO_z-LOsx_z)/s, (LO_x-LOsy_x)/s >= <-15.271 , 31.293 , 12.191 >

17 Momentum Field

Equation (55) and its descendants still hold true even if there is an integrated momentum field, regardless of forces present :

$$\vec{L}(\vec{O}) \equiv \sum (\vec{r}_i - \vec{O}) \times \vec{p}_i + \int (\vec{r} - \vec{O}) \times d\vec{p} \; ; \tag{133}$$

$$\begin{aligned} \vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) &= \sum (\vec{r}_i - \vec{O}_2) \times \vec{p}_i - \sum (\vec{r}_i - \vec{O}_1) \times \vec{p}_i \\ &\quad + \int (\vec{r} - \vec{O}_2) \times d\vec{p} - \int (\vec{r} - \vec{O}_1) \times d\vec{p} \; ; \end{aligned} \tag{134}$$

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = \sum (\vec{O}_1 - \vec{O}_2) \times \vec{p}_i + \int (\vec{O}_1 - \vec{O}_2) \times d\vec{p} \; ; \tag{135}$$

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = (\vec{O}_1 - \vec{O}_2) \times \left(\sum \vec{p}_i + \int d\vec{p} \right) \; ; \tag{136}$$

$$\vec{L}(\vec{O}_2) - \vec{L}(\vec{O}_1) = (\vec{O}_1 - \vec{O}_2) \times \vec{P} \; ; \tag{137}$$

$$\vec{L}(\vec{O}_2) = \vec{L}(\vec{O}_1) + (\vec{O}_1 - \vec{O}_2) \times \vec{P} \; . \tag{138}$$

18 Concluding

Linear momentum determines the rate of variation of angular momentum across x, y, z, and the rate of variation of angular momentum across x, y, z can be used to calculate total linear momentum.

Using Eq. (20), Eq. (29), or perpendicular cross division Eq. (63), linear momentum can be calculated from differences of angular momentum, without needing the number of particles, their mass, positions, velocities, or the position of Center of Mass.

Angular momentum values in various frames of reference are determined by, and also divulge, total mass, total momentum, position, velocity, and angular momentum of/at the center of mass.

The MKS units for angular momentum are $Kg \cdot m^2 \cdot /s$, units for linear momentum $Kg \cdot m \cdot /s$, which is appropriate as linear momentum acts as the derivative with respect to displacement.

The mechanisms which enforce full conservation of angular momentum

$$\vec{f}_{ij} = -\vec{f}_{ji} \qquad \text{(equal and opposite);} \tag{139}$$

$$(\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij} = 0 \qquad \text{(internal forces are co-linear);} \tag{140}$$

$$\sum (\vec{r}_i - \vec{O}) \times \vec{f}_{i\_Ext} = 0 \qquad \text{(zero torque at some } \vec{O}\text{);} \tag{141}$$

$$\sum \vec{f}_{i\_Ext} = 0 \qquad \text{(zero torque gradient);} \tag{142}$$

also enforce conservation of linear momentum.

Angular momentum changing at infinite observation points, but unchanging at some specific points, is not conservation of angular momentum.

Conservation of linear momentum, plus angular momentum unchanging at one point guarantees, and is, full conservation of angular momentum.

Full conservation of angular momentum implies conservation of linear momentum.

19 Acknowledgments

I am not affiliated with a university, which makes it difficult for me to search the literature, to determine how much of this is new. I included canonical knowledge, to show how it all fits together.

I have not seen equations that can calculate linear from angular momentum published elsewhere.

If it was known that conservation of linear momentum is necessary condition for full conservation of angular momentum, I would think that one of the several mechanics textbooks I consulted (Taylor's *Classical Mechanics*[2] for example) would have said so.

I posted earlier versions of these proofs in the physics StackExchange online forum. Educated persons indicated that they believe that these conservations are independent.

I created a youTube video on this subject, under the channel name "David Gahrsch", which did not achieve wide viewership.

If the math was not speaking so clearly, I would have given up.

Circa 1981, at University of Pittsburgh, I had a brief conversation with a physics professor, Dr. Alec Stewart, about this. I had no equations at that time, only ideas.

Thanks to Dr. Eric Swanson at University of Pittsburgh for advice and encouragement.

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20 Bibliography

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[2] John R.Taylor, *Classical Mechanics*, (University Science Books 2005)