Decomposing the deformed Lie products in a four-dimensional space

Subtitle: First part - The initial theorem and its consequences

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This document continues the systematic exploration of diverse mathematical methods allowing the decomposition of deformed tensor products. Here, the discussion is focusing attention on deformed Lie products and on conditions generalizing, in a four-dimensional mathematical space, what has been called the initial theorem during the elaboration of the intrinsic method, the purpose of which was the decomposition of deformed cross products. This mathematical document sheds a particular light on the (2, 0) representations of the electromagnetic fields.

Contents

1 Introduction

1.1 Context and motivation

So far, the theory of the (E) question has:

- exposed discussions involving elements in $V_3 = \{\mathbb{C} \otimes E(3, \mathbb{R})$;
- introduced several intern operations acting on pairs of elements taken in $V_3 \times V_3$: the deformed tensor product, the alternated deformed tensor product and the deformed Lie product;
- presented two complementary methods yielding diverse divisions in the dual space of V_3 for the results of these operations: the intrinsic one for the deformed cross products [a] and the extrinsic one for the deformed tensor products [b].

The theory of relativity [01] is developed in a four-dimensional space. Therefore, it is meaningful to look for extrapolations of these methods in $V_4 = \{\mathbb{C} \otimes E(4, \mathbb{R})\}$. The extrinsic method can fortunately be applied in spaces having any finite dimension equal or greater than two. In opposition, until today, there is no intrinsic procedure decomposing the deformed Lie products involving pairs of elements in $V_4 \times V_4$.

1.2 Claim

Therefore, the long-term objective of this exploration is the construction of a method which would be in some way equivalent to the intrinsic one ... but for the decomposition of deformed tensor products (resp. of Lie products) acting in $V_4 \times V_4$.

1.3 The first obstacle

The intrinsic method developed in [a] began with the *initial theorem*. It concerns only deformed cross products. These products are deformed Lie products in a three-dimensional space. Two paths to get the theorem were proposed: (i) the poor man road (algebra) imposing inhuman calculations that will be even more difficult to perform in a four-dimensional space; and (ii) the more comfortable way of a logical analysis.

2 Generalizing the initial theorem

2.1 Preliminaries

In what follows:

$$\forall\, {}^{(4)}\mathbf{u},\, {}^{(4)}\mathbf{w} \in V_4 = \{\mathbb{C} \otimes E(4, \mathbb{R})\},\, \forall\, A \in \boxplus(4, \mathbb{C})$$

$$\exists\, ({}^{(4)}[P],\, {}^{(4)}\mathbf{z}) : |\otimes_A ({}^{(4)}\mathbf{u},\, {}^{(4)}\mathbf{w}) >= {}^{(4)}[P] \,.\, |{}^{(4)}\mathbf{w} > + |{}^{(4)}\mathbf{z} >$$

$$A^{\alpha}_{\chi\beta} \,.\, u^{\chi} = \Phi_{\alpha\beta}$$

A) Exactly as in a discussion concerning a three-dimensional space, one must calculate the difference:

$$\Lambda(\mathbf{u})$$
$$=$$
$$|{}_A\Phi({}^{(4)}\mathbf{u}) \, - \, [P]|$$
$$=$$
$$\begin{vmatrix} A^0_{\chi 0} . u^\chi - p_{00} & A^0_{\chi 1} . u^\chi - p_{01} & A^0_{\chi 2} . u^\chi - p_{02} & A^0_{\chi 3} . u^\chi - p_{03} \\ A^1_{\chi 0} . u^\chi - p_{10} & A^1_{\chi 1} . u^\chi - p_{11} & A^1_{\chi 2} . u^\chi - p_{12} & A^1_{\chi 3} . u^\chi - p_{13} \\ A^2_{\chi 0} . u^\chi - p_{20} & A^2_{\chi 1} . u^\chi - p_{21} & A^2_{\chi 2} . u^\chi - p_{22} & A^2_{\chi 3} . u^\chi - p_{23} \\ A^3_{\chi 0} . u^\chi - p_{30} & A^3_{\chi 1} . u^\chi - p_{31} & A^2_{\chi 3} . u^\chi - p_{32} & A^3_{\chi 3} . u^\chi - p_{33} \end{vmatrix}$$
$$=$$
$$c_{\alpha\beta\chi\delta} . u^\alpha . u^\beta . u^\chi . u^\delta$$
$$+ \, c_{\alpha\beta\chi} . u^\alpha . u^\beta . u^\chi$$
$$+ \, c_{\alpha\beta} . u^\alpha . u^\beta$$
$$+ \, c_\alpha . u^\alpha$$
$$+ \, (-1)^4 . |P|$$

A generalization of the initial theorem would be written: "*The previous expression is systematically a polynomial form of at most degree three when the discussion concerns deformed Lie products acting on elements in a four dimensional space.*"

B) Per definition, within the theory of the (E) question, any deformed Lie product is an alternated deformed tensor product built with an anti-symmetric cube; this is a simple extrapolation of [a; definition 1.7, p.5]. Hence, here one must calculate $\Lambda(\mathbf{u})$ when the deforming cube A is an element in $\boxplus^-(4, \mathbb{C})$:

$$\forall \, \alpha, \, \beta, \, \chi : \, A^\chi_{\alpha\beta} \, + \, A^\chi_{\beta\alpha} \, = \, 0$$

C) To be more efficient, it is useful to remark that all coefficients of degree four appearing in $\Lambda(\mathbf{u})$ are resulting from the calculation of the determinant of the simplest decomposition without residual part, precisely:

$$c_{\alpha\beta\chi\delta} . u^\alpha . u^\beta . u^\chi . u^\delta$$
$$=$$
$$|{}_A\Phi({}^{(4)}\mathbf{u})|$$
$$=$$
$$|A^\alpha_{\chi\beta} . u^\chi|$$
$$=$$
$$\begin{vmatrix} A^0_{\chi 0} . u^\chi & A^0_{\chi 1} . u^\chi & A^0_{\chi 2} . u^\chi & A^0_{\chi 3} . u^\chi \\ A^1_{\chi 0} . u^\chi & A^1_{\chi 1} . u^\chi & A^1_{\chi 2} . u^\chi & A^1_{\chi 3} . u^\chi \\ A^2_{\chi 0} . u^\chi & A^2_{\chi 1} . u^\chi & A^2_{\chi 2} . u^\chi & A^2_{\chi 3} . u^\chi \\ A^3_{\chi 0} . u^\chi & A^3_{\chi 1} . u^\chi & A^3_{\chi 2} . u^\chi & A^3_{\chi 3} . u^\chi \end{vmatrix}$$

This fact means that it is sufficient:

- (i) to calculate the previous determinant when the cube A is an element in $\boxplus^{-}(4,\, \mathbb{C})$;
- (ii) to discover the conditions annihilating it,

... to know when $\Lambda(\mathbf{u})$ is a polynomial form of degree at most equal to three.

2.2 Focus on anti-reduced cubes

Per definition, the knots of any completely anti-reduced cube are such that:

$$\forall\, \alpha,\, \beta,\, \chi : A^{\alpha}_{\chi\beta} \,+\, A^{\beta}_{\chi\alpha} \,=\, 0$$

This condition includes the particular one:

$$\forall\, \alpha,\, \chi : A^{\alpha}_{\chi\alpha} \,=\, 0$$

Proposition 2.1. *Each completely anti-reduced cube generates a set of anti-symmetric matrices. Each of them is representing the most trivial decomposition without residual part for some deformed tensor products.*

Proof. There is the logical suite:

$$\forall\, A \,\in\, \boxplus^{\downarrow}(4,\, \mathbb{C}) :$$

$$\Downarrow$$

$$\forall\, \alpha,\, \beta,\, \chi :\, A^{\alpha}_{\chi\beta} \,+\, A^{\beta}_{\chi\alpha} \,=\, 0$$

$$\Downarrow$$

$$\forall\, \alpha,\, \beta,\, \chi,\, \forall\, u^{\chi} :\, A^{\alpha}_{\chi\beta} \,.\, u^{\chi} \,+\, A^{\beta}_{\chi\alpha} \,.\, u^{\chi} \,=\, 0$$

$$\Downarrow$$

$$\forall\, \alpha,\, \beta,\, \chi,\, \forall\, u^{\chi} :\, \sum_{\chi} A^{\alpha}_{\chi\beta} \,.\, u^{\chi} \,+\, \sum_{\chi} A^{\beta}_{\chi\alpha} \,.\, u^{\chi} \,=\, 0$$

$$\Downarrow$$

$$\forall\, \alpha,\, \beta :\, \Phi_{\alpha\beta} \,+\, \Phi_{\beta\alpha} \,=\, 0$$

$$\Downarrow$$

$${}_{A}\Phi({}^{(4)}\mathbf{u}) \,+\, {}_{A}\Phi({}^{(4)}\mathbf{u}) \,=\, {}^{(4)}[0]$$

□

Corollary 2.1. *Consequence of proposition 2.1.*

Because of proposition 2.1, each trivial decomposition without residual part is an anti-symmetric matrix that can be represented as follows:

$$
{}_A\Phi({}^{(4)}\mathbf{u})
$$

$$
=
$$

$$
[\Phi_{\alpha\beta}]
$$

$$
=
$$

$$
\begin{bmatrix} 0 & \Phi_{01} & \Phi_{02} & \Phi_{03} \\ -\Phi_{01} & 0 & \Phi_{12} & \Phi_{13} \\ -\Phi_{02} & -\Phi_{12} & 0 & \Phi_{23} \\ -\Phi_{03} & -\Phi_{13} & -\Phi_{23} & 0 \end{bmatrix}
$$

Proposition 2.2. *For any completely anti-reduced cube:*

$$
\forall\, A \,\in\, \boxplus^{\downarrow}(4,\, \mathbb{C}) :
$$

$$
\Lambda({}^{(4)}\mathbf{u})
$$

$$
=
$$

$$
|{}_A\Phi({}^{(4)}\mathbf{u}) \,-\, [P]|
$$

$$
=
$$

$$
(\Phi_{01}\,.\,\Phi_{23} \,-\, \Phi_{02}\,.\,\Phi_{13})^2 \,+\, (\Phi_{02}\,.\,\Phi_{13} \,-\, \Phi_{12}\,.\,\Phi_{03})^2 \,+\, (\Phi_{01}\,.\,\Phi_{23} \,+\, \Phi_{12}\,.\,\Phi_{03})^2
$$

$$
+\, c_{\alpha\beta\chi}\,.\,u^{\alpha}\,.\,u^{\beta}\,.\,u^{\chi}
$$

$$
+\, c_{\alpha\beta}\,.\,u^{\alpha}\,.\,u^{\beta}
$$

$$
+\, c_{\alpha}\,.\,u^{\alpha}
$$

$$
+\,(-1)^4\,.\,|P|
$$

Proof. In general:

$$
\Lambda({}^{(4)}\mathbf{u}) \,=\, \sum_{\beta\,=\,0}^{\beta\,=\,3} (-1)^{\beta}\,.\,(\Phi_{0\beta} \,-\, p_{0\beta})\,.\,\Delta_{0\beta} \,;\; \Phi_{0\beta} \,=\, A^{0}_{\chi\beta}\,.\,u^{\chi}
$$

But here one adds the constraint:

$$
\forall\,\alpha,\,\beta :\, \Phi_{\alpha\beta} \,+\, \Phi_{\beta\alpha} \,=\, 0
$$

As consequences:

- The first sub-determinant

$$\Delta_{00} = \begin{vmatrix} A^1_{\chi 1} . u^\chi - p_{11} & A^1_{\chi 2} . u^\chi - p_{12} & A^1_{\chi 3} . u^\chi - p_{13} \\ A^2_{\chi 1} . u^\chi - p_{21} & A^2_{\chi 2} . u^\chi - p_{22} & A^2_{\chi 3} . u^\chi - p_{23} \\ A^3_{\chi 1} . u^\chi - p_{31} & A^2_{\chi 3} . u^\chi - p_{32} & A^3_{\chi 3} . u^\chi - p_{33} \end{vmatrix}$$

... is of the same kind than the one intervening in the calculation of $\Lambda({}^{(3)}\mathbf{u})$. This one has already be done in [a] but with an argument ${}^{(3)}\mathbf{u}$ involving only three components. The result was a polynomial form of degree at most two depending on the three components u^1, u^2 and u^3. Here ${}^{(4)}\mathbf{u}$ has four components; hence, the calculation must be done again:

$$\begin{gathered} \Delta_{00} \\ = \\ -p_{11} . [p_{22} . p_{33} + (\Phi_{23} + p_{32}) . (\Phi_{23} - p_{23})] \\ -(\Phi_{12} - p_{12}) . [p_{33} . (\Phi_{12} + p_{21}) + (\Phi_{23} - p_{23}) . (\Phi_{13} + p_{31})] \\ +(\Phi_{13} - p_{13}) . [-(\Phi_{12} + p_{21}) . (\Phi_{23} - p_{32}) - p_{22} . (\Phi_{13} + p_{31})] \end{gathered}$$

The terms which are contributing to the degree three have been written in blue. Since $\Phi_{00} = 0$, the sub-determinant Δ_{00} must only be multiplied by -p_{00} which is not depending on the components of ${}^{(4)}\mathbf{u}$. Therefore, the part of $\Lambda({}^{(4)}\mathbf{u})$ depending on Δ_{00} is always a polynomial form of degree at most three depending on the four components of ${}^{(4)}\mathbf{u}$.

- Let now consider:

$$\begin{gathered} \Delta_{01} \\ = \\ -(\Phi_{01} + p_{10}). [p_{22}. p_{33} + (\Phi_{23} + p_{32}). (\Phi_{23} - p_{23})] \\ -(\Phi_{12} - p_{12}). [p_{33}. (\Phi_{02} + p_{20}) + (\Phi_{23} - p_{23}). (\Phi_{03} + p_{30})] \\ +(\Phi_{13} - p_{13}). [(\Phi_{02} + p_{20}). (\Phi_{23} + p_{32}) - p_{22}. (\Phi_{03} + p_{30})] \\ = \\ -\Phi_{01}. [p_{22}. p_{33} + (\Phi_{23} + p_{32}). (\Phi_{23} - p_{23})] \\ -p_{10}. [p_{22}. p_{33} + (\Phi_{23} + p_{32}). (\Phi_{23} - p_{23})] \\ -\Phi_{12}. [p_{33}. (\Phi_{02} + p_{20}) + (\Phi_{23} - p_{23}). (\Phi_{03} + p_{30})] \\ +p_{12}. [p_{33}. (\Phi_{02} + p_{20}) + (\Phi_{23} - p_{23}). (\Phi_{03} + p_{30})] \\ +\Phi_{13}. [(\Phi_{02} + p_{20}). (\Phi_{23} + p_{32}) - p_{22}. (\Phi_{03} + p_{30})] \\ -p_{13}. [(\Phi_{02} + p_{20}). (\Phi_{23} + p_{32}) - p_{22}. (\Phi_{03} + p_{30})] \\ = \\ -\Phi_{01} . p_{22} . p_{33} - \Phi_{01} . (\Phi_{23} + p_{32}). (\Phi_{23} - p_{23}) \\ -p_{10} . p_{22} . p_{33} . p_{10} . (\Phi_{23} + p_{32}) . (\Phi_{23} - p_{23}) \end{gathered}$$

$$- \Phi_{12} . p_{33} . (\Phi_{02} + p_{20}) - \Phi_{12} . (\Phi_{23} - p_{23}) . (\Phi_{03} + p_{30})$$
$$+ p_{12} . p_{33} . (\Phi_{02} + p_{20}) + p_{12} . (\Phi_{23} - p_{23}) . (\Phi_{03} + p_{30})]$$
$$+ \Phi_{13} . (\Phi_{02} + p_{20}) . (\Phi_{23} + p_{32}) - p_{22} . \Phi_{13} . (\Phi_{03} + p_{30})$$
$$- p_{13} . (\Phi_{02} + p_{20}) . (\Phi_{23} + p_{32}) - p_{13} . p_{22} . (\Phi_{03} + p_{30})$$

This sub-determinant must be multiplied by

$$-(\Phi_{01} - p_{01})$$

... which contains a term depending on the components of ${}^{(4)}\mathbf{u}$. Hence, there is a contribution to the set containing the terms of degree four:

$$-\Phi_{01} . \{-\Phi_{01} . \Phi_{23}^2 - \Phi_{12} . \Phi_{23} . \Phi_{03} + \Phi_{13} . \Phi_{02} . \Phi_{23}\}$$

- Let consider:

$$\Delta_{02}$$
$$=$$
$$-(\Phi_{01} + p_{10}) . [(\Phi_{12} + p_{21}) . p_{33} + (\Phi_{13} + p_{20}) . (\Phi_{23} - p_{23})]$$
$$+p_{11} . [p_{33} . (\Phi_{02} + p_{20}) + (\Phi_{23} - p_{23}) . (\Phi_{03} + p_{30})]$$
$$+(\Phi_{13} - p_{13}) . [(\Phi_{02} + p_{20}) . (\Phi_{13} + p_{20}) - (\Phi_{12} + p_{21}) . (\Phi_{03} + p_{30})]$$
$$=$$
$$-\Phi_{01} . [(\Phi_{12} + p_{21}) . p_{33} + (\Phi_{13} + p_{20}) . (\Phi_{23} - p_{23})]$$
$$-p_{10} . [(\Phi_{12} + p_{21}) . p_{33} + (\Phi_{13} + p_{20}) . (\Phi_{23} - p_{23})]$$
$$+p_{11} . [p_{33} . (\Phi_{02} + p_{20}) + (\Phi_{23} - p_{23}) . (\Phi_{03} + p_{30})]$$
$$+ \Phi_{13} . (\Phi_{02} + p_{20}) . (\Phi_{13} + p_{20}) - \Phi_{13} . (\Phi_{12} + p_{21}) . (\Phi_{03} + p_{30})$$
$$-p_{13} . [(\Phi_{02} + p_{20}) . (\Phi_{13} + p_{20}) - (\Phi_{12} + p_{21}) . (\Phi_{03} + p_{30})]$$

This sub-determinant must be multiplied by

$$(\Phi_{02} - p_{02})$$

... which contains a term depending on the components of ${}^{(4)}\mathbf{u}$. Hence, there is a contribution to the set containing the terms of degree four:

$$\Phi_{02} . \{-\Phi_{01} . \Phi_{13} . \Phi_{23} + \Phi_{13} . \Phi_{02} . \Phi_{13} - \Phi_{13} . \Phi_{12} . \Phi_{03}\}$$

- Let consider:

$$\Delta_{03}$$
$$=$$
$$-(\Phi_{01} + p_{10}) . [(\Phi_{12} + p_{21}) . (\Phi_{23} + p_{32}) - p_{22} . (\Phi_{13} + p_{20})]$$
$$+p_{11} . [(\Phi_{02} + p_{20}) . (\Phi_{23} + p_{32}) - p_{22} . (\Phi_{03} + p_{30})]$$
$$+(\Phi_{12} - p_{12}) . [(\Phi_{02} + p_{20}) . (\Phi_{13} + p_{20}) - (\Phi_{12} + p_{21}) . (\Phi_{03} + p_{30})]$$

...

$$=$$

$$-\Phi_{01}\,.\,(\Phi_{12}+p_{21})\,.\,(\Phi_{23}+p_{32})+p_{22}\,.\,\Phi_{01}\,.\,(\Phi_{13}+p_{20})$$
$$-p_{10}\,.\,[(\Phi_{12}+p_{21})\,.\,(\Phi_{23}+p_{32})-p_{22}.\,(\Phi_{13}+p_{20})]$$
$$+p_{11}\,.\,[(\Phi_{02}+p_{20}).\,(\Phi_{23}+p_{32})-p_{22}.\,(\Phi_{03}+p_{30})]$$
$$+\Phi_{12}\,.\,(\Phi_{02}+p_{20})\,.\,(\Phi_{13}+p_{20})\;-\;\Phi_{12}\,.\,(\Phi_{12}\;+\;p_{21})\,.\,(\Phi_{03}\;+\;p_{30})$$
$$-p_{12}\,.\,[(\Phi_{02}\;+\;p_{20})\,.\,(\Phi_{13}\;+\;p_{20})\;-\;(\Phi_{12}\;+\;p_{21})\,.\,(\Phi_{03}\;+\;p_{30})]$$

This sub-determinant must be multiplied by

$$(-1)^3\,.\,(\Phi_{03}\;-\;p_{03})$$

... which contains a term depending on the components of $^{(4)}\mathbf{u}$. Hence, there is a contribution to the set containing the terms of degree four:

$$-\Phi_{03}\;.\;\{-\Phi_{01}\,.\,\Phi_{12}\,.\,\Phi_{23}\;+\;\Phi_{12}\,.\,\Phi_{02}\,.\,\Phi_{13}\;-\;\Phi_{12}^2\,.\,\Phi_{03}\}$$

Let now add all contributions with degree four and state that:

$$-\Phi_{01}\,.\,\{-\Phi_{01}\,.\,\Phi_{23}^2\;-\;\Phi_{12}\,.\,\Phi_{23}\,.\,\Phi_{03}\;+\;\Phi_{13}\,.\,\Phi_{02}\,.\,\Phi_{23}\}$$
$$+\Phi_{02}\;.\;\{-\Phi_{01}.\;\Phi_{13}\,.\,\Phi_{23}\;+\;\Phi_{13}.\;\Phi_{02}.\;\Phi_{13}\;-\;\Phi_{13}.\;\Phi_{12}.\;\Phi_{03}\}$$
$$-\Phi_{03}\;.\;\{-\Phi_{01}\,.\,\Phi_{12}\,.\,\Phi_{23}\;+\;\Phi_{12}\,.\,\Phi_{02}\,.\,\Phi_{13}\;-\;\Phi_{12}^2\,.\,\Phi_{03}\}$$
$$=$$
$$\Phi_{01}^2\,.\,\Phi_{23}^2\;+\;\Phi_{01}\,.\,\Phi_{12}\,.\,\Phi_{23}\,.\,\Phi_{03}\;-\;\Phi_{01}\,.\,\Phi_{13}\,.\,\Phi_{02}\,.\,\Phi_{23}$$
$$-\Phi_{02}\,.\,\Phi_{01}\,.\,\Phi_{13}\,.\,\Phi_{23}\;+\;\Phi_{02}^2\,.\,\Phi_{13}^2\;-\;\Phi_{02}\,.\,\Phi_{13}\,.\,\Phi_{12}.\;\Phi_{03}$$
$$+\Phi_{03}\,.\,\Phi_{01}\,.\,\Phi_{12}\,.\,\Phi_{23}\;-\;\Phi_{03}\,.\,\Phi_{12}\,.\,\Phi_{02}\,.\,\Phi_{13}\;+\;\Phi_{12}^2\,.\,\Phi_{03}^2$$
$$=$$
$$\Phi_{01}^2\,.\,\Phi_{23}^2\;+\;\Phi_{02}^2\,.\,\Phi_{13}^2\;+\;\Phi_{12}^2\,.\,\Phi_{03}^2$$
$$-2\,.\,\{\Phi_{01}\,.\,\Phi_{13}\,.\,\Phi_{02}\,.\,\Phi_{23}\;+\;\Phi_{02}\,.\,\Phi_{13}\,.\,\Phi_{12}.\;\Phi_{03}\;-\;\Phi_{01}\,.\,\Phi_{12}\,.\,\Phi_{23}\,.\,\Phi_{03}\}$$
$$=$$
$$(\Phi_{01}\,.\,\Phi_{23}\;-\;\Phi_{02}\,.\,\Phi_{13})^2\;+\;(\Phi_{02}\,.\,\Phi_{13}\;-\;\Phi_{12}\,.\,\Phi_{03})^2\;+\;(\Phi_{01}\,.\,\Phi_{23}\;+\;\Phi_{12}\,.\,\Phi_{03})^2$$

□

Corollary 2.2. *Consequences of proposition 2.2.*

It is useful to note that:

1. This expression involves only three products:

$$a_1 = \Phi_{01} . \Phi_{23} \; ; \; a_2 = -\Phi_{02} . \Phi_{13} \; ; \; a_3 = \Phi_{03} . \Phi_{12}$$

At the end of this calculation:

$$c_{\alpha\beta\chi\delta} . u^\alpha . u^\beta . u^\chi . u^\delta$$

$$=$$

$$(a_1 + a_2)^2 + (-a_2 - a_3)^2 + (a_1 + a_3)^2$$

$$=$$

$$2 . (a_1^2 + a_2^2 + a_3^2 + a_1 . a_2 + a_2 . a_3 + a_3 . a_1)$$

$$=$$

$$||^{(3)}\mathbf{a}||^2 + (^{(3)}\mathbf{a}^\oplus)^2$$

With:

$$^{(3)}\mathbf{a} : (a_1, a_2, a_3)$$
$$||^{(3)}\mathbf{a}||^2 = a_1^2 + a_2^2 + a_3^2$$
$$^{(3)}\mathbf{a}^\oplus = a_1 + a_2 + a_3$$

The vanishing of this expression implies an interdependence between the a_is.

Example 2.1. *For the pedagogy*

The case involving the three complex roots of 1, $\{a_1, a_2, a_3\} = \{1, j, j^2\}$, illustrates one particular situation for which this (i) expression vanishes and (ii) the a_is are interdependent since $1 + j + j^2 = 0$.

2. Each anti-symmetric matrix contains at most only six different and non-vanishing entries which are usually regrouped in a vector and a pseudo-vector (synonym: axial vector):

$$^{(3)}\mathbf{X} : (\Phi_{01}, \Phi_{02}, \Phi_{03})$$

$$^{(3)}\mathbf{Y} : (-\Phi_{23}, \Phi_{13}, -\Phi_{12})$$

This choice allows a rewriting of this determinant as:

$$|_A\Phi(^{(4)}\mathbf{u})|$$

$$=$$

$$|\Phi_{\alpha\beta}|$$

$$=$$

$$\begin{vmatrix} 0 & X^1 & X^2 & X^3 \\ -X^1 & 0 & -Y^3 & Y^2 \\ -X^2 & Y^3 & 0 & -Y^1 \\ -X^3 & -Y^2 & Y^1 & 0 \end{vmatrix}$$

$$=$$

$$-X^1 \,.\, \begin{vmatrix} -X^1 & -Y^3 & Y^2 \\ -X^2 & 0 & -Y^1 \\ -X^3 & Y^1 & 0 \end{vmatrix} + X^2 \,.\, \begin{vmatrix} -X^1 & 0 & Y^2 \\ -X^2 & Y^3 & -Y^1 \\ -X^3 & -Y^2 & 0 \end{vmatrix} - X^3 \,.\, \begin{vmatrix} -X^1 & 0 & -Y^3 \\ -X^2 & Y^3 & 0 \\ -X^3 & -Y^2 & Y^1 \end{vmatrix}$$

$$=$$

$$\begin{gathered} -X^1 \,.\, \{-X^1 \,.\, (Y^1)^2 - Y^3 \,.\, X^3 \,.\, Y^1 - Y^2 \,.\, X^2 \,.\, Y^1\} \\ +X^2 \,.\, \{X^1 \,.\, Y^1 \,.\, Y^2 + X^2 \,.\, (Y^2)^2 + Y^2 \,.\, X^3 \,.\, Y^3\} \\ -X^3 \,.\, \{-X^1 \,.\, Y^1 \,.\, Y^3 - Y^3 \,.\, (X^2 \,.\, Y^2 + X^3 \,.\, Y^3)\} \end{gathered}$$

$$=$$

$$\begin{gathered} (X^1 \,.\, Y^1)^2 + (X^1 \,.\, Y^1) \,.\, (X^3 \,.\, Y^3) + (X^1 \,.\, Y^1) \,.\, (X^2 \,.\, Y^2) \\ +(X^2 \,.\, Y^2)^2 + (X^1 \,.\, Y^1) \,.\, (X^2 \,.\, Y^2) + (X^3 \,.\, Y^3) \,.\, (X^2 \,.\, Y^2) \\ +(X^3 \,.\, Y^3)^2 + (X^1 \,.\, Y^1) \,.\, (X^3 \,.\, Y^3) + (X^3 \,.\, Y^3) \,.\, (X^2 \,.\, Y^2) \end{gathered}$$

$$=$$

$$(<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y} >_{Id_3})^2$$

And it reveals that the Euclidean orthogonality between both vectors systematically induces the vanishing of this determinant.

Furthermore, since this determinant contains all terms of degree four, one can also write:

$$c_{\alpha\beta\chi\delta} \,.\, u^\alpha \,.\, u^\beta \,.\, u^\chi \,.\, u^\delta = (<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y} >_{Id_3})^2 = ||^{(3)}\mathbf{a}||^2 + (^{(3)}\mathbf{a}^\oplus)^2$$

Example 2.2. *The (2, 0) representations of the electromagnetic field tensor.*

It is well-known that [05; §23, p. 61, (23.5) left]:

$$[F(2,\,0)] = \begin{bmatrix} 0 & ^{(3)}\mathbf{E} \\ -^{(3)}\mathbf{E} & _{[J]}\Phi(^{(3)}\mathbf{H}) \end{bmatrix}$$

The corollary 2.1 suggests that the (2, 0) representation of the electromagnetic field is sometimes the simplest decomposition without residual part of some tensor product which has been deformed by an anti-reduced cube A.

$$[F(2,\,0)] = {_A\Phi}(^{(4)}\mathbf{u}),\ \textit{with}\ A \in \boxplus^{\downarrow}(4,\,\mathbb{C})$$

When it is the case:

$$^{(3)}\mathbf{E} \perp {^{(3)}\mathbf{H}} \Rightarrow |F(2,\,0)| = |{_A\Phi}(^{(4)}\mathbf{u})| = 0$$

And the polynomial:

$$\Lambda(\mathbf{u}) = |[F(2,\,0)] - [P]| = |{}_A\Phi({}^{(4)}\mathbf{u}) - [P]|$$

... is at most of degree equal to three. It depends on the four components of ${}^{(4)}\mathbf{u}$.

The eventuality which is suggested by this application of corollary 2.1 is only a pedagogical hypothesis. In a less theoretical approach, one can decompose $\otimes_{\Gamma(2)}({}^{(4)}\mathbf{u},\,{}^{(4)}\mathbf{u})$ appearing in the co-variant version of the Lorentz law [05; §90, p. 256, (90.7)] with the help of the extrinsic method [b]. In that case, $[\mathrm{F}(\uparrow,\downarrow)]$ is always the natural main part of the decomposition and one must write $[\mathrm{F}(2,\,0)] = {}^{(4)}[\mathrm{G}].{}_{\Gamma(2)}\Phi({}^{(4)}\mathbf{u})$ minus something where ${}^{(4)}\mathbf{u}$ represents the four-(dimensional)speed of the particle at hand and ${}^{(4)}[\mathrm{G}]$ is a metric tensor.

Furthermore, in this concrete approach: $\Lambda({}^{(4)}\mathbf{u}) = |{}_{\Gamma(2)}\Phi({}^{(4)}\mathbf{u})$ - $[\mathrm{F}(\uparrow,\downarrow)]|$ and the cube $\Gamma(2)$ containing the Christoffel's symbols of the second kind is an element in $\boxplus^{+}(4,\,\mathbb{R})$ [02, p. 49], not in $\boxplus^{-}(4,\,\mathbb{R})$. Hence, the generalization of the initial theorem cannot directly be applied to the co-variant version of the Lorentz law. At least, the so-called *gravitational term* $\otimes_{\Gamma(2)}({}^{(4)}\mathbf{u},\,{}^{(4)}\mathbf{u})$ must be transformed into a tensor product deformed by an anti-symmetric cube. For historical reasons [02, p. 48, (4.)], the cube $\Gamma(1)$ containing the Christoffel's symbols of the first kind cannot be a convenient choice because this one is also an element in $\boxplus^{+}(4,\,\mathbb{R})$.

This example will be developed further below in the subsection 3.4.

Proposition 2.3. *Anti-reduced cubes can also be anti-symmetric.*

Proof.

$$A^{\alpha}_{\chi\beta} = -A^{\alpha}_{\beta\chi} = A^{\chi}_{\beta\alpha} = -A^{\chi}_{\alpha\beta} = A^{\beta}_{\alpha\chi} = -A^{\beta}_{\chi\alpha} = A^{\alpha}_{\chi\beta}$$

□

Proposition 2.4. *Elements belonging to the intersection between anti-symmetric cubes and anti-reduced cubes are elements in V_4.*

Proof. The arguments are similar to the ones which have been involved in a three-dimensional space. A knot is different from another one if and only if none of the subscripts is repeated.

$$A \in \boxplus^{-}(4,\,\mathbb{C}) \cap \boxplus^{\downarrow}(4,\,\mathbb{C}) \rightarrow (A^{0}_{12},\,A^{0}_{13},\,A^{0}_{23},\,A^{1}_{23}) = (a,\,b,\,c,\,d) \equiv {}^{(4)}\mathbf{A}$$

□

Corollary 2.3. *Consequences of propositions 2.3 and 2.4.*

1. The results concerning the anti-reduced cubes can now be applied to cubes which are simultaneously anti-reduced and anti-symmetric. A first and necessary precaution is needed: the writing A must be replaced by ${}^{(4)}\mathbf{A}$.

2. Furthermore, in that case:

$$\Phi_{01} = \sum_{\chi} A^0_{\chi 1} \,.\, u^{\chi} = -a \,.\, u^2 - b \,.\, u^3 = X^1$$

$$\Phi_{02} = \sum_{\chi} A^0_{\chi 2} \,.\, u^{\chi} = a \,.\, u^1 - c \,.\, u^3 = X^2$$

$$\Phi_{03} = \sum_{\chi} A^0_{\chi 3} \,.\, u^{\chi} = b \,.\, u^1 + c \,.\, u^2 = X^3$$

$$\Phi_{12} = \sum_{\chi} A^1_{\chi 2} \,.\, u^{\chi} = -a \,.\, u^0 - d \,.\, u^3 = -Y^3$$

$$\Phi_{13} = \sum_{\chi} A^1_{\chi 3} \,.\, u^{\chi} = -b \,.\, u^0 + d \,.\, u^2 = Y^2$$

$$\Phi_{23} = \sum_{\chi} A^2_{\chi 3} \,.\, u^{\chi} = -c \,.\, u^0 - d \,.\, u^1 = -Y^1$$

   And:

$$a_1 = \Phi_{01} \,.\, \Phi_{23} = -X^1 \,.\, Y^1$$
$$a_2 = -\Phi_{02} \,.\, \Phi_{13} = -X^2 \,.\, Y^2$$
$$a_3 = \Phi_{03} \,.\, \Phi_{12} = -X^3 \,.\, Y^3$$

3. One can now calculate the terms of degree four for this specific family of cubes:

$$c_{\alpha\beta\chi\delta} \,.\, u^{\alpha} \,.\, u^{\beta} \,.\, u^{\chi} \,.\, u^{\delta}$$
$$=$$
$$(X^1 \,.\, Y^1 + X^2 \,.\, Y^2)^2 + (X^2 \,.\, Y^2 + X^3 \,.\, Y^3)^2 + (X^1 \,.\, Y^1 + X^3 \,.\, Y^3)^2$$
$$=$$
$$(<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3} - X^3 \,.\, Y^3)^2$$
$$+ (<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3} - X^1 \,.\, Y^1)^2$$
$$+ (<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3} - X^3 \,.\, Y^2)^2$$
$$=$$
$$(<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3})^2 + ||^{(3)}\mathbf{a}||^2$$

   - But in the circumstances at hand, it is relatively easy to prove that ${}^{(3)}\mathbf{X}$ and ${}^{(3)}\mathbf{Y}$ are systematically orthogonal:

$$<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3} = 0$$

Proof. Let calculate:

$$X^1 . Y^1$$

$$=$$

$$-(a . u^2 + b . u^3) . (c . u^0 + d . u^1)$$

$$=$$

$$-a . c . u^0 . u^2 - a . d . u^1 . u^2 - b . c . u^0 . u^3 - b . d . u^1 . u^3$$

$$X^2 . Y^2$$

$$=$$

$$(a . u^1 - c . u^3) . (-b . u^0 + d . u^2)$$

$$=$$

$$-a . b . u^0 . u^1 + a . d . u^1 . u^2 + b . c . u^0 . u^3 - c . d . u^2 . u^3$$

$$X^3 . Y^3$$

$$=$$

$$(b . u^1 + c . u^2) . (a . u^0 + d . u^3)$$

$$=$$

$$a . b . u^0 . u^1 + b . d . u^1 . u^3 + a . c . u^0 . u^2 + c . d . u^2 . u^3$$

Let add these terms and state with the help of the colors that:

$$X^1 . Y^1 + X^2 . Y^2 + X^3 . Y^3 = 0$$

□

The first consequence is that all terms with degree four systematically vanish when the cube is reduced to a vector **A**.

- Since one must write in general:

$$c_{\alpha\beta\chi\delta} . u^\alpha . u^\beta . u^\chi . u^\delta = (<^{(3)} \mathbf{X}, {}^{(3)}\mathbf{Y} >_{Id_3})^2 = ||^{(3)}\mathbf{a}||^2 + ({}^{(3)}\mathbf{a}^\oplus)^2$$

  And since, specifically here:

$$c_{\alpha\beta\chi\delta} . u^\alpha . u^\beta . u^\chi . u^\delta = 0 = ||^{(3)}\mathbf{a}||^2$$

  It is obvious that:

$$(<^{(3)} \mathbf{X}, {}^{(3)}\mathbf{Y} >_{Id_3})^2 = ({}^{(3)}\mathbf{a}^\oplus)^2 = 0$$

2.3 The initial theorem in V_4

Let A be a given anti-symmetric cube in $\boxplus^-(4,\mathbb{C})$. Per definition, it allows the construction of deformed Lie products $[\mathbf{u},...]_A$ and these products may eventually be decomposed as [P].|**...**> + |**z**>.

The polynomial $\Lambda(\mathbf{u}) = |_A\Phi(\mathbf{u})$ - [P]| measuring the difference between the simplest decomposition without residual part $_A\Phi(\mathbf{u})$ and the main part of a generic non-trivial decomposition [P] is systematically a polynomial form of at most degree three when the anti-symmetric cube A is also anti-reduced and therefore equivalent to some vector $^{(4)}\mathbf{A}$.

Corollary 2.4. *of the initial theorem in V_4.*

When a cube A is equivalent to an element $^{(4)}\mathbf{A}$ in V_4:

- The simplest decomposition without residual part has the formalism:

$$_{\mathbf{A}}\Phi(^{(4)}\mathbf{u}) = \begin{bmatrix} 0 & <^{(3)}\mathbf{X}| \\ -|^{(3)}\mathbf{X}> & _{[J]}\Phi(^{(3)}\mathbf{Y}) \end{bmatrix}$$

  Its determinant vanishes.

- The vectors $^{(3)}\mathbf{X}$ and $^{(3)}\mathbf{Y}$ are orthogonal.

$$|_{\mathbf{A}}\Phi(^{(4)}\mathbf{u})| = 0 \iff <^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3} = 0 \iff {}^{(3)}\mathbf{X} \perp {}^{(3)}\mathbf{Y}$$

- The vectors $^{(3)}\mathbf{X}$ and $^{(3)}\mathbf{Y}$ are associated with a specific type of isotropic vectors, $^{(3)}\mathbf{a}$ in V_3 [03], the sum of the components of which vanishes.

- If $^{(4)}\mathbf{A} = {}^{(4)}\mathbf{u}$:

$$\begin{aligned} (a,\, b,\, c,\, d) &= (u^0,\, u^1,\, u^2,\, u^3) \\ \Phi_{01} &= -(a\,.\,c + b\,.\,d) = X^1 \\ \Phi_{02} &= a\,.\,b - c\,.\,d = X^2 \\ \Phi_{03} &= (b^2 + c^2) = X^3 \\ \Phi_{12} &= -(a^2 + d^2) = -Y^3 \\ \Phi_{13} &= -b\,.\,a + d\,.\,c = Y^2 \\ \Phi_{23} &= -(c\,.\,a + d\,.\,b) = -Y^1 \end{aligned}$$

  Hence:

$$\begin{aligned} \Phi_{01} &= -(a\,.\,c + b\,.\,d) = X^1 = -Y^1 = \Phi_{23} \\ \Phi_{02} &= a\,.\,b - c\,.\,d = X^2 = -Y^2 = -\Phi_{13} \end{aligned}$$

  The simplest decomposition without residual part writes:

$$_{\mathbf{A}}\Phi(^{(4)}\mathbf{A}) = {}_{\mathbf{u}}\Phi(^{(4)}\mathbf{u}) = \begin{vmatrix} 0 & X^1 & X^2 & X^3 \\ -X^1 & 0 & -Y^3 & -X^2 \\ -X^2 & Y^3 & 0 & X^1 \\ -X^3 & X^2 & -X^1 & 0 \end{vmatrix}$$

Corollary 2.5. *The case of projectiles with only one non-vanishing component.*

Inside the family of $[{}^{(4)}\mathbf{u}, {}^{(4)}\ldots]_{\mathbf{A}}$ deformed Lie products let consider the ones which are such that $u^0 \neq 0$, $u^1 = u^2 = u^3 = 0$. For all Lie products in this sub-family:

$$\Lambda({}^{(4)}\mathbf{u}) = c_{000} \,.\, (u^0)^3 + c_{00} \,.\, (u^0)^2 + c_0 \,.\, u^0 + |P|$$

Any decomposition without residual part is characterized by; see [a]:

$$\Lambda({}^{(4)}\mathbf{u}) = 0$$

Therefore, each solution of the polynomial which is associated with the given sub-family of deformed Lie products at hand characterizes one decomposition without residual part:

$$\Lambda(u^0, 0, 0, 0) = 0 \iff \exists\,[{}_nP] \,:\, |[{}_nu^0 \,.\, \mathbf{e}_0, {}^{(4)}\ldots]_{\mathbf{A}} >= {}^{(4)}[{}_nP] \,.\, |{}^{(4)}\ldots >$$

The three roots (n = 1, 2, 3) of this polynomial can be calculated with the Tartaglia-Cardan method [04] as soon as the coefficients are precisely known.

3 Complement concerning the cubes which are only anti-reduced

The initial theorem does not concern the tensor products which are exclusively deformed by anti-reduced cubes because they are not deformed Lie products.

3.1 The terms of degree four

Nevertheless, this kind of tensor products can be decomposed too. In this document, it has been proved that -for these products- the simplest decomposition without residual part has the formalism:

$${}_A\Phi({}^{(4)}\mathbf{u}) = \begin{bmatrix} 0 & <^{(3)}\mathbf{X}| \\ -|^{(3)}\mathbf{X}> & {}_{[J]}\Phi({}^{(3)}\mathbf{Y}) \end{bmatrix}$$

And that:

$$c_{\alpha\beta\chi\delta} \,.\, u^\alpha \,.\, u^\beta \,.\, u^\chi \,.\, u^\delta = |{}_A\Phi({}^{(4)}\mathbf{u})| = (<^{(3)}\mathbf{X}, {}^{(3)}\mathbf{Y}>_{Id_3})^2 = ||{}^{(3)}\mathbf{a}||^2 + ({}^{(3)}\mathbf{a}^{\oplus})^2$$

Here, the determinant does not systematically vanish and the polynomial $\Lambda(\mathbf{u})$ may contain terms of degree four. But what does the vector ${}^{(3)}\mathbf{a}$ describe? What do its components represent? Looking for answers, let build the Pythagorean table:

$$T_2(\otimes)({}^{(3)}\mathbf{X}, {}^{(3)}\mathbf{Y})$$
$$=$$
$$\begin{bmatrix} -\Phi_{01}\,.\,\Phi_{23} & -\Phi_{02}\,.\,\Phi_{23} & -\Phi_{03}\,.\,\Phi_{23} \\ \Phi_{01}\,.\,\Phi_{13} & \Phi_{02}\,.\,\Phi_{13} & \Phi_{03}\,.\,\Phi_{13} \\ -\Phi_{01}\,.\,\Phi_{12} & -\Phi_{02}\,.\,\Phi_{12} & -\Phi_{03}\,.\,\Phi_{12} \end{bmatrix}$$
$$= \begin{bmatrix} -a^1 & -\Phi_{02}\,.\,\Phi_{23} & -\Phi_{03}\,.\,\Phi_{23} \\ \Phi_{01}\,.\,\Phi_{13} & -a^2 & \Phi_{03}\,.\,\Phi_{13} \\ -\Phi_{01}\,.\,\Phi_{12} & -\Phi_{02}\,.\,\Phi_{12} & -a^3 \end{bmatrix}$$

Remark 3.1. *The trace of the Pythagorean table $T_2(\otimes)(\boldsymbol{X}, \boldsymbol{Y})$*

The trace of the Pythagorean table $\mathrm{T}_2(\otimes)(\mathbf{X}, \mathbf{Y})$ has two representations:

- as Euclidean scalar products of the arguments in the pair $(\mathbf{X}, \mathbf{Y})$:

$$Trace\{T_2(\otimes)({}^{(3)}\mathbf{X},\ {}^{(3)}\mathbf{Y})\}\ = <\mathbf{X},\ \mathbf{Y}>_{Id_3}$$

- as minus one times the sum of the components of $\mathbf{a}$:

$$Trace\{T_2(\otimes)({}^{(3)}\mathbf{X},\ {}^{(3)}\mathbf{Y})\}\ =\ -(\mathbf{a})^{\oplus}$$

Therefore, for strictly and completely anti-reduced cubes:

$$\forall\, A\ \in\ \boxplus^{\downarrow}(4,\ \mathbb{C}):$$

1. The Euclidean scalar product between the arguments of the pair $(\mathbf{X}, \mathbf{Y})$ is equal to minus the sum of the components of $\mathbf{a}$:

$$<\mathbf{X},\ \mathbf{Y}>_{Id_3}\ + (\mathbf{a})^{\oplus}\ =\ 0$$

2. The vector $\mathbf{a}$ which can be associated with the pair $(\mathbf{X}, \mathbf{Y})$ is an isotropic vector in V_3 [03; see the definitions at the bottom of page 3 and p.41]:

$$||{}^{(3)}\mathbf{a}||^2\ =\ 0$$

3. The sum of all terms of degree four in the polynomial $\Lambda(\mathbf{u})$ is equal to the square of Euclidean scalar product between the arguments of the pair $(\mathbf{X}, \mathbf{Y})$ or, equivalently, to the square of the sum of the components of the isotropic vector which is associated with this pair:

$$c_{\alpha\beta\chi\delta}\,.\,u^{\alpha}\,.\,u^{\beta}\,.\,u^{\chi}\,.\,u^{\delta}\ =\ |{}_{A}\Phi({}^{(4)}\mathbf{u})|\ =\ (<^{(3)}\ \mathbf{X},\ {}^{(3)}\mathbf{Y}>_{Id_3})^2\ =\ ({}^{(3)}\mathbf{a}^{\oplus})^2$$

3.2 The problem

Following the approach explained in [03; §55] one can propose a representation for ${}^{(3)}\mathbf{a}$:

$${}^{(3)}\mathbf{a}\ \rightarrow \begin{bmatrix} a^3 & a^1\ -\ i\,.\,a^2 \\ a^1\ +\ i\,.\,a^2 & -a^3 \end{bmatrix}$$

But here, one must add two constraints:

$$(a^1)^2\ +\ (a^2)^2\ +\ (a^3)^2\ =\ 0$$

$$a^1\ +\ a^2\ +\ a^3 +\ <^{(3)}\ \mathbf{X},\ {}^{(3)}\mathbf{Y}>_{Id_3} = 0$$

The first one implies that ${}^{(3)}\mathbf{a}$ is either the null vector or an element in $\mathrm{V}_3 = \mathbb{C} \otimes \mathrm{E}(3, \mathbb{R})$. In that case, the second constraint allows a discussion for which ${}^{(3)}\mathbf{X}$ and ${}^{(3)}\mathbf{Y}$ are elements in V_3 too (they can have components in $\mathbb{C}$). Inspired by the approach developed in [07], one may also introduce a vector $\mathbf{w}_2$, the existence of which will be justified a little bit later in this document, and write for convenience:

$$<^{(3)}\mathbf{X},\, ^{(3)}\mathbf{Y}>_{Id_3} = -(w_2^0)^2,\ \forall\, i = 1,\, 2,\, 3:\ a^i = (w_2^i)^2 \iff \mathbf{a}^{\oplus} = ||\mathbf{w}_2||^2$$

This choice transforms the two constraints respectively into:

$$\sum_i (w_2^i)^4 = 0\, and\ <^{(4)}\mathbf{w}_2,\, ^{(4)}\mathbf{w}_2>_{[\hat{\eta}]} = 0$$

... where $[\hat{\eta}]$ is the element in M(4, $\mathbb{R}$) representing the Minkowski's geometry. The first advantages of this choice are to allow (i) incorporating this mathematical discussion into a geometrical context usually associated with the empty regions of the universe; (ii) the involving of Lorentz's transformations. Its disadvantage is that neither the components of $^{(3)}\mathbf{a}$ nor the ones of $^{(4)}\mathbf{w}_2$ can be related to Stokes parameters. At this stage, one has no interpretation for these vectors.

With the hope to progress, let reconsider the first proposition which has been made for representing an isotropic vector with the two components of a spinor in [03; §52]:

$$a^1 = (\eta^0)^2 + (\eta^1)^2$$
$$a^2 = i\,.\,\{(\eta^0)^2 - (\eta^1)^2\}$$
$$a^3 = 2i\,.\,\eta^0\,.\,\eta^1$$

It's easy to check the coherence of the proposition (no surprise):

$$||\mathbf{s}||^2$$
$$=$$
$$(a^1)^2 + (a^2)^2 + (a^3)^2$$
$$=$$
$$\{(\eta^0)^4 + (\eta^1)^4 + 2\,.\,(\eta^0)^2\,.\,(\eta^1)^2\} - \{(\eta^0)^4 + (\eta^1)^4 - 2\,.\,(\eta^0)^2\,.\,(\eta^1)^2\} - 4\,.\,(\eta^0)^2\,.\,(\eta^1)^2$$
$$=$$
$$0$$

But is this historical proposition the unique possible procedure associating the isotropic vector $^{(3)}\mathbf{a}$ with a spinor having the pair $(\eta^0\,,\eta^1)$ as components?

Proposition 3.1. *A given isotropic vector can be associated in many different manners with a given pair of scalars.*

Proof. Let consider three polynomials of degree two depending on two components denoted $\eta^0\,,\eta^1$ such that:

$$a^1(\eta^0\,,\,\eta^1) = a^1_{00}\,.\,(\eta^0)^2 + a^1_{01}\,.\,\eta^0\,.\,\eta^1 + a^1_{11}\,.\,(\eta^1)^2$$
$$a^2(\eta^0\,,\,\eta^1) = a^2_{00}\,.\,(\eta^0)^2 + a^2_{01}\,.\,\eta^0\,.\,\eta^1 + a^2_{11}\,.\,(\eta^1)^2$$

$$a^3(\eta^0\,,\,\eta^1)\;=\;a^3_{00}\,.\,(\eta^0)^2\;+\;a^3_{01}\,.\,\eta^0\,.\,\eta^1\;+\;a^3_{11}\,.\,(\eta^1)^2$$

Let calculate the Euclidean norm of ${}^{(3)}\mathbf{a}$ with the help of previous definitions. Let then annihilate the norm. All terms can be reorganized in five subsets; each of them is yielding a condition:

1. $$(\eta^0)^4\,terms\;:\;\{(a^1_{00})^2\;+\;(a^2_{00})^2\;+\;(a^3_{00})^2\}\;=\;0$$

2. $$(\eta^0)^3\,.\,(\eta^1)\,terms\;:\;a^1_{00}\,.\,a^1_{01}\;+\;a^2_{00}\,.\,a^2_{01}\;+\;a^3_{00}\,.\,a^3_{01}\;=\;0$$

3. $$(\eta^0)^2\,.\,(\eta^1)^2\,terms\;:\;a^1_{00}\,.\,a^1_{11}+a^2_{00}\,.\,a^2_{11}+a^3_{00}\,.\,a^3_{11}+\{(a^1_{01})^2+(a^2_{01})^2+(a^3_{01})^2\}\;=\;0$$

4. $$(\eta^0)\,.\,(\eta^1)^3\,terms\;:\;a^1_{01}\,.\,a^1_{11}\;+\;a^2_{01}\,.\,a^2_{11}\;+\;a^3_{01}\,.\,a^3_{11}\;=\;0$$

5. $$(\eta^1)^4\,terms\;:\;\{(a^1_{11})^2\;+\;(a^2_{11})^2\;+\;(a^3_{11})^2\}\;=\;0$$

The definitions of the components a^i (i = 1, 2, 3) introduce a matrix:

$$[W]\;=\;\begin{bmatrix} a^1_{00} & a^1_{01} & a^1_{11} \\ a^2_{00} & a^2_{01} & a^2_{11} \\ a^3_{00} & a^3_{01} & a^3_{11} \end{bmatrix}$$

It can be decoded/interpreted as a set of three vectors which are disposed in the following manner:

$$[W]\;=\;[|\mathbf{w}_1>,|\mathbf{w}_2>,|\mathbf{w}_3>]$$

With this interpretation, the five conditions can be reformulated as:

1. $$(\eta^0)^4\,terms\;:\;||\mathbf{w}_1||^2\;=\;0$$

2. $$(\eta^0)^3\,.\,(\eta^1)\,terms\;:<\mathbf{w}_1\,,\,\mathbf{w}_2>_{Id_3}=\;0$$

3. $$(\eta^0)^2\,.\,(\eta^1)^2\,terms\;:<\mathbf{w}_1\,,\,\mathbf{w}_3>_{Id_3}\;+||\mathbf{w}_2||^2\;=\;0$$

4. $$(\eta^0)\,.\,(\eta^1)^3\,terms\;:<\mathbf{w}_2\,,\,\mathbf{w}_3>_{Id_3}=\;0$$

5. $$(\eta^1)^4\,terms\;:\;||\mathbf{w}_3||^2\;=\;0$$

The first and the third vectors in [W] are isotropic vectors. The Euclidean norm of the second vector in [W] is minus the Euclidean scalar product of the first and of the third vectors. The second vector is orthogonal to the first and to the third vectors. Each triad $({}^{(3)}\mathbf{w}_1,\ {}^{(3)}\mathbf{w}_2,\ {}^{(3)}\mathbf{w}_3)$ composing the matrix [W] contains a pair of isotropic vectors and any third one which is orthogonal to these two isotropic vectors.

Let now analyze the calculations and state that a triad generating the vector ${}^{(3)}\mathbf{a}$ with the help of the above definitions in respecting the five previous conditions is:

- (i) insuring the isotropic character of this vector ${}^{(3)}\mathbf{a}$,
- ... and (ii) compatible with the existence of a pair $(\eta^0\,,\eta^1)$ which can be understood as the components of a spinor associated with ${}^{(3)}\mathbf{a}$.

With these conditions, ${}^{(3)}\mathbf{a}$ can be associated with any pair $(\eta^0\,,\eta^1)$ and the proposition is true. □

3.3 An interpretation for the isotropic vector associated with the pair $(\mathbf{X}, \mathbf{Y})$

Unfortunately, this fact is telling an embarrassing question: "Are the above definitions compatible with the concept of spinor proposed in [03; §52]? Or do they introduce new mathematical objects mimicking the spinors?

Considering [03; §53], one gets a new information: each definition proposed previously for the components of ${}^{(3)}\mathbf{a}$ can also be interpreted as the square of the modified component of some classical (synonym: Cartan's) spinor resulting from a rotation of the vector with which the initial version of that spinor was associated.

Here, this interpretation is seemingly useless since spinors associated with isotropic vectors in V_3 have only two components whilst the isotropic vector ${}^{(3)}\mathbf{a}$ has three components and is not necessarily representing a spinor in some space greater than V_3.

However, among the five conditions, the third one attracts attention because it roughly resembles to one of both constraints linking $<{}^{(3)}\mathbf{X}, {}^{(3)}\mathbf{Y}>_{Id}$ and ${}^{(3)}\mathbf{a}^{\oplus}$. The third condition and this constraint coincide when:

$$\mathbf{w}_1 = \mathbf{X}$$
$$\mathbf{w}_2 : ||\mathbf{w}_2||^2 = \mathbf{a}^{\oplus}$$
$$\mathbf{w}_3 = \mathbf{Y}$$

In that case, the five conditions write:

1. $$(\eta^0)^4\, terms : ||\mathbf{X}||^2 = 0$$

2.
$$(\eta^0)^3 \,.\, (\eta^1)\, terms \,: < \mathbf{X} \,,\, \mathbf{w}_2 >_{Id_3} = 0$$

3.
$$(\eta^0)^2 \,.\, (\eta^1)^2\, terms \,: < \mathbf{X} \,,\, \mathbf{Y} >_{Id_3} + \mathbf{a}^{\oplus} \,=\, 0$$

4.
$$(\eta^0) \,.\, (\eta^1)^3\, terms \,: < \mathbf{w}_2 \,,\, \mathbf{Y} >_{Id_3} = 0$$

5.
$$(\eta^1)^4\, terms \,:\, ||\mathbf{Y}||^2 \,=\, 0$$

Hence each triad $({}^{(3)}\mathbf{X},\, {}^{(3)}\mathbf{w}_2,\, {}^{(3)}\mathbf{Y})$ of which the arguments respect the five previous conditions and the supplementary condition $||\mathbf{w}_2||^2 = \mathbf{a}^{\oplus}$ generates an isotropic vector ${}^{(3)}\mathbf{a}$.

3.4 The (2, 0) representations of the electromagnetic field tensor

Each (2, 0) representation of the electromagnetic field tensor (as already mentioned in example 2.2) is a concrete realization of the simplest decomposition without residual part for some deformed tensor product $\otimes_A({}^{(4)}\mathbf{u},\, {}^{(4)}\mathbf{...})$ when A is an anti-reduced cube. In that case, obviously:

$$({}^{(3)}\mathbf{X},\, {}^{(3)}\mathbf{Y}) \,=\, ({}^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H})$$

The new information here concerns the existence of an isotropic vector ${}^{(3)}\mathbf{a}$ of which the physical meaning is unclear although it is seemingly systematically associated with the pair $({}^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H})$.

Recalling basic knowledge, one can note that the determinant of the (2, 0) representation of any electromagnetic field tensor is an invariant quantity for each given field [05; §25]. Therefore, the unidentified vector ${}^{(3)}\mathbf{a}$ associated with this field respects at least the condition:

$$<^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H} >_{Id_3} = -{}^{(3)}\mathbf{a}^{\oplus}({}^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H}) \,=\, Invariant_1({}^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H})$$

A lecture on partially polarized light is instructive [05; §50, pp. 121-124] when one is looking for an interpretation for the vector ${}^{(3)}\mathbf{a}$ within a context concerning electromagnetic fields. For example, one may ask if the components of this vector coincide with Stokes parameters [05; §50, p. 122, (50.12)]?

If they would, they would be real scalars and constrained to respect:

$$\forall\, i \,=\, 1,\, 2,\, 3 : \, a^i \,\in\, [-1,\, +1]$$

As consequence of their reality ($a^i \in \mathbb{R}$) and of the isotropic character of ${}^{(3)}\mathbf{a}$, this vector would necessary be null. The simultaneous achievement of all these criteria would strongly restrict the domain of validity of this first and spontaneous interpretation. Precisely, it would only be useful for the description of a natural and not polarized light characterized with:

$$a^1 = a^2 = a^3 = <^{(3)} \mathbf{X},\, ^{(3)}\mathbf{Y} >_{Id_3} = 0$$

A cube which would have directly been condensed in a four-vector would have brought the same result.

Here, one is facing a fairly classical situation in mathematical physics: the mathematics proposes more possibilities than physical reality can offer. The mathematical discussion admits the existence of a vector $^{(3)}\mathbf{a}$ with components in $\mathbb{C}$. This fact opens a debate that can only reach a conclusion if there are experiences of polarization needing Stokes-like parameters in $\mathbb{C}$ to be correctly interpreted or if the interpretation for $^{(3)}\mathbf{a}$ as Stokes parameters is incomplete - an idea which has been very recently suggested in [06].

Among others alternative interpretations, but not new (2003), one must also cite the unusual Lorentzian interpretation of the Stokes parameters [07], including the interpretation of the polarization P (here related to $||\mathbf{a}||^2$) as equivalent to (citation) "*a relative velocity between the Stokes vectors respectively associated with natural and partially polarized lights*" (end of citation). Within this unusual context, a vector $^{(3)}\mathbf{a}$ in $\mathbb{C} \otimes \mathrm{E}(3,\, \mathbb{R})$ would be associated with very specific situations.

The previous discussion gives a visage to these very specific situations:

1. $$(\eta^0)^4 \, terms \, : \, ||^{(3)}\mathbf{E}||^2 = 0$$

2. $$(\eta^0)^3 \,.\, (\eta^1) \, terms \, : <^{(3)} \mathbf{E} \,,\, ^{(3)}\mathbf{w}_2 >_{Id_3} = 0$$

3. $$(\eta^0)^2 \,.\, (\eta^1)^2 \, terms \, : <^{(3)} \mathbf{E} \,,\, ^{(3)}\mathbf{H} >_{Id_3} + ^{(3)}\mathbf{a}^{\oplus} = 0$$

4. $$(\eta^0) \,.\, (\eta^1)^3 \, terms \, : <^{(3)} \mathbf{w}_2 \,,\, ^{(3)}\mathbf{H} >_{Id_3} = 0$$

5. $$(\eta^1)^4 \, terms \, : \, ||^{(3)}\mathbf{H}||^2 = 0$$

6. $$<^{(3)} \mathbf{E},\, ^{(3)}\mathbf{H} >_{Id_3} = -(w_2^0)^2,\, \forall\, i = 1,\, 2,\, 3 : \, a^i = (w_2^i)^2 \iff \, ^{(3)}\mathbf{a}^{\oplus} = ||^{(3)}\mathbf{w}_2||^2$$

7. $$\sum_i (w_2^i)^4 = 0 \, and \, <^{(4)} \mathbf{w}_2,\, ^{(4)}\mathbf{w}_2 >_{[\hat{\eta}]} = 0$$

All these relations are supporting only one interpretation concerning the vector ${}^{(4)}\mathbf{w}_2$: it can only be the four-speed associated with an isotropic plane wave in some empty region without curvature.

$$ {}^{(4)}\mathbf{w}_2 \,=\, {}^{(4)}\mathbf{u} $$

For these waves, the first invariant of the EM field is the sum of the components of the spatial speed.

$$ <^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H} >_{Id_3} = -{}^{(3)}\mathbf{u}^{\oplus}({}^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H}) \,=\, Invariant_1({}^{(3)}\mathbf{E},\, {}^{(3)}\mathbf{H}) $$

4 The difficulty with the cubes which are exclusively anti-symmetric

Per definition, the knots of any completely anti-symmetric cube are such that:

$$ \forall\, \alpha,\, \beta,\, \chi : A^{\alpha}_{\chi\beta} \,+\, A^{\alpha}_{\beta\chi} \,=\, 0 $$

This condition includes the particular one:

$$ \forall\, \alpha,\, \beta : A^{\alpha}_{\beta\beta} \,=\, 0 $$

A completely anti-symmetric cube doesn't automatically generate an anti-symmetric decomposition without residual part but, instead of that, gigantic calculations:

$$ c_{\alpha\beta\chi\delta} \,.\, u^{\alpha} \,.\, u^{\beta} \,.\, u^{\chi} \,.\, u^{\delta} $$

$$ = $$

$$ |_A\Phi({}^{(4)}\mathbf{u})| $$

$$ = $$

$$ |A^{\alpha}_{\chi\beta} \,.\, u^{\chi}| $$

$$ = $$

$$ \begin{vmatrix} A^0_{\chi 0} \,.\, u^{\chi} & A^0_{\chi 1} \,.\, u^{\chi} & A^0_{\chi 2} \,.\, u^{\chi} & A^0_{\chi 3} \,.\, u^{\chi} \\ A^1_{\chi 0} \,.\, u^{\chi} & A^1_{\chi 1} \,.\, u^{\chi} & A^1_{\chi 2} \,.\, u^{\chi} & A^1_{\chi 3} \,.\, u^{\chi} \\ A^2_{\chi 0} \,.\, u^{\chi} & A^2_{\chi 1} \,.\, u^{\chi} & A^2_{\chi 2} \,.\, u^{\chi} & A^2_{\chi 3} \,.\, u^{\chi} \\ A^3_{\chi 0} \,.\, u^{\chi} & A^3_{\chi 1} \,.\, u^{\chi} & A^3_{\chi 2} \,.\, u^{\chi} & A^3_{\chi 3} \,.\, u^{\chi} \end{vmatrix} $$

$$ = $$

$$ \begin{vmatrix} -A^0_{01} . u^1 - A^0_{02} . u^2 - A^0_{03} . u^3 & A^0_{01} . u^0 - A^0_{12} . u^2 - A^0_{13} . u^3 & A^0_{02} . u^0 + A^0_{12} . u^1 - A^0_{23} . u^3 & A^0_{03} . u^0 + A^0_{13} . u^1 + A^0_{23} . u^2 \\ -A^1_{01} . u^1 - A^1_{02} . u^2 - A^1_{03} . u^3 & A^1_{01} . u^0 - A^1_{12} . u^2 - A^1_{13} . u^3 & A^1_{02} . u^0 + A^1_{12} . u^1 - A^1_{23} . u^3 & A^1_{03} . u^0 + A^1_{13} . u^1 + A^1_{23} . u^2 \\ -A^2_{01} . u^1 - A^2_{02} . u^2 - A^2_{03} . u^3 & A^2_{01} . u^0 - A^2_{12} . u^2 - A^2_{13} . u^3 & A^2_{02} . u^0 + A^2_{12} . u^1 - A^2_{23} . u^3 & A^2_{03} . u^0 + A^2_{13} . u^1 + A^2_{23} . u^2 \\ -A^3_{01} . u^1 - A^3_{02} . u^2 - A^3_{03} . u^3 & A^3_{01} . u^0 - A^3_{12} . u^2 - A^3_{13} . u^3 & A^3_{02} . u^0 + A^3_{12} . u^1 - A^3_{23} . u^3 & A^3_{03} . u^0 + A^3_{13} . u^1 + A^3_{23} . u^2 \end{vmatrix} $$

Here again, the four sub-determinants must be calculated.

$$ \Delta_{00} \,=\, \begin{vmatrix} A^1_{01} . u^0 - A^1_{12} . u^2 - A^1_{13} . u^3 & A^1_{02} . u^0 + A^1_{12} . u^1 - A^1_{23} . u^3 & A^1_{03} . u^0 + A^1_{13} . u^1 + A^1_{23} . u^2 \\ A^2_{01} . u^0 - A^2_{12} . u^2 - A^2_{13} . u^3 & A^2_{02} . u^0 + A^2_{12} . u^1 - A^2_{23} . u^3 & A^2_{03} . u^0 + A^2_{13} . u^1 + A^2_{23} . u^2 \\ A^3_{01} . u^0 - A^3_{12} . u^2 - A^3_{13} . u^3 & A^3_{02} . u^0 + A^3_{12} . u^1 - A^3_{23} . u^3 & A^3_{03} . u^0 + A^3_{13} . u^1 + A^3_{23} . u^2 \end{vmatrix} $$

$$ \Delta_{01} \,=\, \begin{vmatrix} -A^1_{01} . u^1 - A^1_{02} . u^2 - A^1_{03} . u^3 & A^1_{02} . u^0 + A^1_{12} . u^1 - A^1_{23} . u^3 & A^1_{03} . u^0 + A^1_{13} . u^1 + A^1_{23} . u^2 \\ -A^2_{01} . u^1 - A^2_{02} . u^2 - A^2_{03} . u^3 & A^2_{02} . u^0 + A^2_{12} . u^1 - A^2_{23} . u^3 & A^2_{03} . u^0 + A^2_{13} . u^1 + A^2_{23} . u^2 \\ -A^3_{01} . u^1 - A^3_{02} . u^2 - A^3_{03} . u^3 & A^3_{02} . u^0 + A^3_{12} . u^1 - A^3_{23} . u^3 & A^3_{03} . u^0 + A^3_{13} . u^1 + A^3_{23} . u^2 \end{vmatrix} $$

$$\Delta_{02} = \begin{vmatrix} -A^1_{01} . u^1 - A^1_{02} . u^2 - A^1_{03} . u^3 & A^1_{01} . u^0 - A^1_{12} . u^2 - A^1_{13} . u^3 & A^1_{03} . u^0 + A^1_{13} . u^1 + A^1_{23} . u^2 \\ -A^2_{01} . u^1 - A^2_{02} . u^2 - A^2_{03} . u^3 & A^2_{01} . u^0 - A^2_{12} . u^2 - A^2_{13} . u^3 & A^2_{03} . u^0 + A^2_{13} . u^1 + A^2_{23} . u^2 \\ -A^3_{01} . u^1 - A^3_{02} . u^2 - A^3_{03} . u^3 & A^3_{01} . u^0 - A^3_{12} . u^2 - A^3_{13} . u^3 & A^3_{03} . u^0 + A^3_{13} . u^1 + A^3_{23} . u^2 \end{vmatrix}$$

$$\Delta_{03} = \begin{vmatrix} -A^1_{01} . u^1 - A^1_{02} . u^2 - A^1_{03} . u^3 & A^1_{01} . u^0 - A^1_{12} . u^2 - A^1_{13} . u^3 & A^1_{02} . u^0 + A^1_{12} . u^1 - A^1_{23} . u^3 \\ -A^2_{01} . u^1 - A^2_{02} . u^2 - A^2_{03} . u^3 & A^2_{01} . u^0 - A^2_{12} . u^2 - A^2_{13} . u^3 & A^2_{02} . u^0 + A^2_{12} . u^1 - A^2_{23} . u^3 \\ -A^3_{01} . u^1 - A^3_{02} . u^2 - A^3_{03} . u^3 & A^3_{01} . u^0 - A^3_{12} . u^2 - A^3_{13} . u^3 & A^3_{02} . u^0 + A^3_{12} . u^1 - A^3_{23} . u^3 \end{vmatrix}$$

It's inhuman work and therefore, it will not be made! In stopping the calculation at this stage, one must keep in mind that some interesting situations canceling the determinant may have been omitted.

5 Conclusion of the first part.

The claim of this exploration was the discovery of conditions generalizing the so-called *initial theorem* in V_4. This theorem concerns deformed Lie products and these products are tensor products deformed by anti-symmetric cubes.

Since the calculation by hand of the discriminant of the system associated with a decomposition in a four-dimensional environment is an inhuman task, attention has first been focused on tensor products deformed by anti-reduced cubes. Because anti-reduced cubes can also simultaneously be anti-symmetric without being systematically null, this strategy gave a set of very simple conditions which generalize the initial theorem in V_4.

When the initial theorem holds true in V_4, then the simplest decomposition without residual part is equivalent to a pair of orthogonal vectors and this pair is automatically associated with an isotropic vector in V_3; the sum of the components of this isotropic vector vanishes.

The end of the document has looked for a correct interpretation of the isotropic vector (i) in a context studying the (2, 0) representations of the electromagnetic fields and (ii) in focusing attention on exclusively anti-reduced cubes. In that case, the initial theorem is not valid, the simplest decomposition without residual part is equivalent to a pair $(\mathbf{E}, \mathbf{H})$ representing the EM field and this pair is no more automatically a pair of orthogonal vectors. However, this pair remains automatically associated with an isotropic vector in V_3; the sum of the components of this isotropic vector does no more systematically vanish.

This quest confirmed the existence of a mathematical and natural link between the propagation of a plane wave in vacuum and the theory of spinors [03]. The progression came to the conclusion that, within this mathematical approach, *the spatial part* of the argument of the simplest decomposition without residual part of $\otimes_A({}^{(4)}\mathbf{u}, {}^{(4)}\mathbf{...})$ - more precisely the spatial part of the four-speed associated to the EM field at hand- can be the isotropic vector associated with the pair $(\mathbf{E}, \mathbf{H})$, provided it is a pair of isotropic vectors.

This strange conclusion will be studied and deepen elsewhere.

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