

Quantum Evidence Theory

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Abstract

A quantum evidence theory is introduced for uncertainty modeling and reasoning in closed-world and open-world environments, termed QET and GQET, respectively. The framework integrates evidence theory with quantum probability to represent, fuse, and reason with heterogeneous multisource information. At the representation level, the (generalized) quantum basic probability amplitude ((G)QBPA) provides fine-grained uncertainty modeling for heterogeneous multisource information by encoding both the strength of evidential support strength and relative phases, thereby capturing specific support, partial and complete ignorance, and, in open-world environments, unknown states. At the fusion level, the (generalized) progressive quantum evidential combination rules ((G)PQECRs) perform coherent multisource fusion at the probability-amplitude level. The corresponding (generalized) quantum evidential combination rules ((G)QECRs) then yield real-valued fused support through the Born rule. At the decision level, the (generalized) quantum basic probability distribution ((G)QBPD), the (generalized) quantum Pignistic transformation, and the associated

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belief and plausibility functions support probability assessment and decision making.

In addition, selected components are further given Hilbert-space formulations: the (generalized) quantum evidence and progressively fused evidence admit linear- and density-operator representations, while the incoherent belief and plausibility functions are realized through basis-event measurements. By introducing superposition and interference into evidential reasoning, the framework models complex interactions among multiple items of evidence. With squared amplitude magnitudes identified with the corresponding classical masses, its belief and plausibility functions recover DSET in the closed world and GET in the open world in the incoherent limit. Its fusion rules capture inter-path interference effects; when the aggregate inter-path interference contribution vanishes for every nonempty output proposition, they recover Dempster’s rule and mGCR, respectively. Consequently, the quantum evidence theory provides a unified framework for uncertainty representation, dynamic multisource information fusion, and uncertainty-aware decision reasoning across closed- and open-world environments.

Keywords: Quantum evidence theory, Uncertainty representation, Uncertainty reasoning, Information fusion, Decision-making, Closed world, Open world

1. Introduction

Representing knowledge in uncertain environments and processing it for decision-making is a fundamental challenge in artificial intelligence (AI) systems. Dempster-Shafer evidence theory (DSET) provides a powerful mathe-

mathematical framework for uncertainty reasoning by generalizing Bayesian probability theory [1, 2]. Owing to its flexibility and capability of reasoning with evidence in the absence of prior information, DSET has been extensively studied and further extended from various perspectives.

Specifically, Yang and Xu presented the evidential reasoning approach for multiple attribute decision analysis [3]. Denœux reviewed approaches to decision-making with belief functions, including methods based on imprecise probabilities and Shafer’s constructive decision theory [4]. Smarandache and Dezert introduced DSmT by extending DSET to a higher-dimensional space for information fusion [5]. Deng further extended DSET to the open-world scenario and proposed the generalized evidence theory (GET) [6], which is capable of modeling not only partial or complete ignorance but also the uncertainty arising from the incompleteness of the frame of discernment. Xiao extended Dempster-Shafer evidence theory (DSET) to a high-dimensional complex plane and proposed the Complex Evidence Theory (CET) [7, 8]. Later on, Deng [9] extended DSET to random permutation set through considering the order of sets.

With the advancement of quantum information processing, increasing efforts have been made to bridge classical evidence theory with the quantum framework for addressing more complex problems. Recently, Xiao proposed the generalized quantum evidence theory, which enables uncertainty reasoning within both closed-world and open-world environments from a quantum-theoretic perspective [10, 11]. In this context, the theory is referred to as quantum evidence theory (QET) for closed-world scenario, and as generalized quantum evidence theory (GQET) for open-world scenario, in order to

distinguish between the two.

The main contributions of this work are summarized as follows. First, a unified quantum evidential representation framework is developed for both closed- and open-world environments through QBPA/GQBPA and their associated probability distributions. Second, quantum evidential combination mechanisms are established by integrating intra-path evidential conjunction with inter-path coherent superposition, thereby explicitly characterizing inter-path interference effects, including constructive and destructive interference, in multisource information fusion. Third, interference-aware belief, Plausibility, and Pignistic transformations are developed to support uncertainty quantification and probabilistic decision making. Finally, Hilbert-space representations and quantum measurement formulations are established, and the relationships between QET/GQET and their classical counterparts are systematically characterized.

2. Preliminaries

In this section, we review some basic concepts of Dempster-Shafer evidence theory, generalized evidence theory, and quantum probability theory.

2.1. DSET: Dempster-Shafer evidence theory

Definition 1 (Frame of discernment). Let Ω be a frame of discernment (FOD), consisting of a set of mutually exclusive and collectively exhaustive events:

$$\Omega = \{h_1, \dots, h_i, \dots, h_g, \dots, h_n\}, \quad (1)$$

where $i, g \in \{1, \dots, n\}$ with $i \neq g$, and h_i and h_g are two arbitrary nonempty events satisfying $h_i \cap h_g = \emptyset$.

Definition 2 (Power set of Ω). Let 2^Ω be the power set of Ω , denoted by:

$$2^\Omega = \{\emptyset, \{h_1\}, \{h_2\}, \dots, \{h_n\}, \{h_1, h_2\}, \dots, \{h_1, h_2, \dots, h_i\}, \dots, \Omega\}, \quad (2)$$

where $\emptyset$ is an empty set.

Definition 3 (Hypothesis or proposition). H_j is defined as a hypothesis or proposition when $H_j \subseteq \Omega$.

Definition 4 (Mass function in DSET). In FOD Ω , a mass function m in DSET is defined as a mapping:

$$m : 2^\Omega \rightarrow [0, 1], \quad (3)$$

satisfying

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{H_j \subseteq \Omega} m(H_j) = 1, \quad (4)$$

where m is also called a basic probability assignment (BPA) or a basic belief assignment (BBA).

Definition 5 (Focal element in DSET). Let m be a BPA. $\forall H_j \subseteq \Omega$, if $m(H_j) > 0$, H_j is called a focal element in DSET.

Definition 6 (Belief function). Let H_j and H_k be two propositions such that $H_j, H_k \subseteq \Omega$. A belief function Bel for H_j , mapping from 2^Ω to $[0, 1]$, is defined by

$$\text{Bel}(H_j) = \sum_{\emptyset \neq H_k \subseteq H_j} m(H_k). \quad (5)$$

Definition 7 (Plausibility function). Let H_j and H_k be two propositions such that $H_j, H_k \subseteq \Omega$. A Plausibility function Pl for H_j , mapping from 2^Ω to $[0, 1]$, is defined by

$$\text{Pl}(H_j) = \sum_{\emptyset \neq H_k, H_k \cap H_j \neq \emptyset} m(H_k) = 1 - \text{Bel}(\bar{H}_j), \quad (6)$$

where $\bar{H}_j = \Omega \setminus H_j$.

Clearly, $\forall H_j \subseteq \Omega$, $\text{Bel}(H_j) \leq \text{Pl}(H_j)$ where $\text{Bel}(H_j)$ and $\text{Pl}(H_j)$ are the lower and upper limit functions to support H_j , respectively.

Definition 8 (Dempster's rule of combination). Let m_1 and m_2 be two independent BPAs in FOD Ω with propositions $H_k, H_h \subseteq \Omega$, respectively. Dempster's rule of combination (DRC), represented in the form $m_1 \oplus m_2$, is defined by

$$m_1 \oplus m_2(H_j) = \begin{cases} \frac{1}{1-K} \sum_{H_k \cap H_h = H_j} m_1(H_k)m_2(H_h), & H_j \neq \emptyset, \\ 0, & H_j = \emptyset, \end{cases} \quad (7)$$

with

$$K = \sum_{H_k \cap H_h = \emptyset} m_1(H_k)m_2(H_h), \quad (8)$$

where K is the conflict coefficient between m_1 and m_2 .

Note that Eq. (7) is feasible under the condition $K < 1$.

2.2. GET: Generalized evidence theory

Definition 9 (Mass function in GET). In FOD Ω , a mass function m_G in GET is defined as a mapping:

$$m_G : 2^\Omega \rightarrow [0, 1], \quad (9)$$

satisfying

$$\sum_{H_j \in 2^\Omega} m_G(H_j) = 1, \quad (10)$$

where m_G is also called a generalized BPA (GBPA).

Definition 10 (Focal element in GET). Let m_G be a GBPA. $\forall H_j \in 2^\Omega$, if $m_G(H_j) > 0$, H_j is called a focal element in GET.

Definition 11 (Generalized belief function in GET). Let H_j and H_k be two propositions such that $H_j, H_k \in 2^\Omega$. A generalized belief function GBel for H_j in GET, mapping from 2^Ω to $[0, 1]$, is defined by

$$\text{GBel}(H_j) = \sum_{\emptyset \neq H_k \subseteq H_j} m_G(H_k), \quad (11)$$

$$\text{GBel}(\emptyset) = m_G(\emptyset). \quad (12)$$

Definition 12 (Generalized Plausibility function in GET). Let H_j and H_k be two propositions such that $H_j, H_k \in 2^\Omega$. A generalized Plausibility function GPI for H_j in GET, mapping from 2^Ω to $[0, 1]$, is defined by

$$\text{GPI}(H_j) = \sum_{\emptyset \neq H_k, H_k \cap H_j \neq \emptyset} m_G(H_k), \quad (13)$$

$$\text{GPI}(\emptyset) = m_G(\emptyset). \quad (14)$$

Definition 13 (Generalized combination rule in GET). Let m_{G_1} and m_{G_2} be two independent GBPAs in FOD Ω with propositions $H_k, H_h \in 2^\Omega$, respectively. The generalized combination rule (GCR), represented in the form $m_{G_1} \oplus m_{G_2}$, is defined, for $K_G < 1$, by

$$m_{G_1} \oplus m_{G_2}(H_j) = \begin{cases} \frac{1 - m_{G_1}(\emptyset)m_{G_2}(\emptyset)}{1 - K_G} \\ \quad \times \sum_{H_k \cap H_h = H_j} m_{G_1}(H_k)m_{G_2}(H_h), & H_j \neq \emptyset, \\ m_{G_1}(\emptyset)m_{G_2}(\emptyset), & H_j = \emptyset. \end{cases}$$

with

$$K_G = \sum_{H_k \cap H_h = \emptyset} m_{G_1}(H_k)m_{G_2}(H_h), \quad (15)$$

For the boundary case $K_G = 1$, the combined mass is assigned entirely to the empty proposition, i.e., $m_G(\emptyset) = 1$.

Definition 14 (Modified generalized combination rule in GET). Let m_{G_1} and m_{G_2} be two independent GBPAs on FOD Ω . The modified generalized combination rule (mGCR) [12] is defined, for $K_G^m < 1$, as

$$m_{G_1} \oplus_m m_{G_2}(H_j) = \begin{cases} \frac{\sum_{H_k \cap H_h = H_j} m_{G_1}(H_k) m_{G_2}(H_h)}{1 - K_G^m}, & H_j \neq \emptyset, \\ \frac{m_{G_1}(\emptyset) m_{G_2}(\emptyset)}{1 - K_G^m}, & H_j = \emptyset, \end{cases} \quad (16)$$

where

$$K_G^m = \sum_{\substack{H_k \cap H_h = \emptyset \\ H_k \cup H_h \neq \emptyset}} m_{G_1}(H_k) m_{G_2}(H_h). \quad (17)$$

For the boundary case $K_G^m = 1$, mGCR assigns $m_G(\emptyset) = 1$. More generally, when all fused masses assigned to nonempty propositions vanish, normalization yields $m_G(\emptyset) = 1$.

2.3. QPT: Quantum probability theory

Definition 15 (Hilbert space). Let $\mathcal{H}$ be an n -dimensional complex Hilbert space with an orthonormal basis

$$\{|\phi_1\rangle, \dots, |\phi_n\rangle\}, \quad (18)$$

where each $|\phi_g\rangle$ represents a distinguishable basis event and

$$\langle \phi_f | \phi_g \rangle = \delta_{fg}. \quad (19)$$

In quantum probability theory, a coherent alternative can be associated with a complex probability amplitude

$$\alpha_g = r_g e^{i\theta_g}, \quad r_g \geq 0, \quad (20)$$

where r_g and θ_g denote the magnitude and phase of α_g , respectively. According to the Born rule, the corresponding probability contribution is

$$|\alpha_g|^2 = r_g^2. \quad (21)$$

When multiple coherent alternatives contribute to the same measurement outcome, their complex probability amplitudes are coherently superposed, and their relative phases give rise to quantum interference.

Definition 16 (Quantum interference). Consider n coherent alternatives contributing to the same measurement outcome, with complex probability amplitudes $\alpha_1, \dots, \alpha_n$. The total probability amplitude associated with the measurement outcome is

$$\alpha = \sum_{g=1}^n \alpha_g. \quad (22)$$

According to the Born rule, the corresponding probability is

$$\text{Pr} = \left| \sum_{g=1}^n \alpha_g \right|^2 = \sum_{g=1}^n r_g^2 + 2 \sum_{f=1}^{n-1} \sum_{g=f+1}^n r_f r_g \cos(\theta_f - \theta_g). \quad (23)$$

The first term represents the incoherent sum of the individual probability contributions, whereas the second term is the coherence-induced interference term determined by the relative phases between coherent alternatives. Specifically, $\cos(\theta_f - \theta_g) > 0$ corresponds to constructive interference, $\cos(\theta_f - \theta_g) < 0$ corresponds to destructive interference, and $\cos(\theta_f - \theta_g) = 0$ indicates that the corresponding pairwise interference term vanishes.

For the specified family of alternatives, define the aggregate interference term by

$$\text{Int}_{\text{Pr}} = 2 \sum_{f=1}^{n-1} \sum_{g=f+1}^n r_f r_g \cos(\theta_f - \theta_g). \quad (24)$$

When the aggregate interference contribution is zero, i.e., $\text{Int}_{\text{Pr}} = 0$, Eq. (23) reduces to the incoherent additive form

$$\text{Pr} = \sum_{g=1}^n r_g^2. \quad (25)$$

3. Quantum evidence theory for an open world

Recently, Xiao extended the DSET to Hilbert space and proposed the generalized quantum evidence theory (GQET), which provides a promising way to handle uncertainty in a new perspective of quantum framework in an open world [10, 11]. The main concepts and knowledge of GQET are introduced below.

3.1. GQET: Generalized quantum evidence theory

3.1.1. Basic concepts of the generalized quantum basic probability amplitude function

Definition 17 (Quantum frame of discernment). Let

$$\Phi = \{\phi_1, \phi_2, \dots, \phi_n\} \quad (26)$$

be a quantum frame of discernment (QFOD), consisting of mutually exclusive and collectively exhaustive elementary events.

Each elementary event $\phi_g \in \Phi$ is associated with an orthonormal basis state $|\phi_g\rangle$ in an n -dimensional Hilbert space $\mathcal{H}$. These basis states form an orthonormal basis of $\mathcal{H}$,

$$\mathcal{B}_\Phi = \{|\phi_1\rangle, \dots, |\phi_g\rangle, \dots, |\phi_n\rangle\}, \quad (27)$$

satisfying

$$\langle \phi_g | \phi_h \rangle = \delta_{gh}, \quad g, h = 1, \dots, n. \quad (28)$$

Accordingly,

$$\mathcal{H} = \text{span}(\mathcal{B}_\Phi). \quad (29)$$

Here, “collectively exhaustive” refers to the currently modeled QFOD. In GQET, information not covered by this current frame is represented separately through the generalized empty proposition, rather than by relaxing the internal exhaustiveness of the QFOD itself.

Definition 18 (Power set of a QFOD). Let Φ be the QFOD defined above. Its power set 2^Φ contains all subsets of Φ , each representing a quantum proposition, i.e.,

$$2^\Phi = \{\psi_j \mid \psi_j \subseteq \Phi\} = \{\emptyset, \{\phi_1\}, \dots, \{\phi_n\}, \{\phi_1, \phi_2\}, \dots, \Phi\}. \quad (30)$$

Definition 19 (Quantum hypothesis or proposition). An element $\psi_j \in 2^\Phi$ is called a *quantum hypothesis* or *quantum proposition*.

For notational simplicity, whenever a function F is defined on 2^Φ , the notation $F(\phi_g)$ is used as shorthand for $F(\{\phi_g\})$ for a singleton proposition.

Definition 20 (Generalized quantum basic probability amplitude function). A generalized quantum basic probability amplitude (GQBPA) function $\mathbb{Q}_\mathbb{M}$ in QFOD Φ , also referred to as a generalized quantum mass function (GQMF), is defined as a mapping:

$$\mathbb{Q}_\mathbb{M} : 2^\Phi \rightarrow \mathbb{C}, \quad (31)$$

satisfying

$$\begin{aligned} \mathbb{Q}_\mathbb{M}(\psi_j) &= \varphi(\psi_j) e^{i\theta(\psi_j)}, \quad \psi_j \in 2^\Phi, \\ \sum_{\psi_j \in 2^\Phi} |\mathbb{Q}_\mathbb{M}(\psi_j)|^2 &= 1, \end{aligned} \quad (32)$$

in which $i = \sqrt{-1}$; $\varphi(\psi_j) \in [0, 1]$ denotes the modulus of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$; $\theta(\psi_j)$ denotes a phase term of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$; $\mathbb{Q}_{\mathbb{M}}(\psi_j)$ denote a generalized quantum basic probability amplitude for ψ_j ; and $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2$ denotes the modulus squared of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$. When $\varphi(\psi_j) = 0$, the phase $\theta(\psi_j)$ is immaterial and may be chosen arbitrarily.

The $\mathbb{Q}_{\mathbb{M}}(\psi_j)$, which is the generalization of generalized basic probability assignment in GET, is called a generalized quantum basic probability amplitude, and can be represented in Cartesian form as:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = x_j + y_j i, \quad x_j^2 + y_j^2 \in [0, 1], \quad (33)$$

or alternatively, using polar form:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = \varphi(\psi_j) (\cos \theta(\psi_j) + i \sin \theta(\psi_j)). \quad (34)$$

Its modulus is expressed as:

$$|\mathbb{Q}_{\mathbb{M}}(\psi_j)| = \varphi(\psi_j) = \sqrt{x_j^2 + y_j^2}. \quad (35)$$

Definition 21 (Quantum focal element in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA function. For any $\psi_j \in 2^\Phi$, if $|\mathbb{Q}_{\mathbb{M}}(\psi_j)| = \varphi(\psi_j) > 0$, then ψ_j is called a quantum focal element in GQET.

Definition 22 (Bayesian GQBPA function). A GQBPA function $\mathbb{Q}_{\mathbb{M}}$ is called Bayesian if all of its nonempty quantum focal elements are singletons; equivalently,

$$|\psi_j| > 1 \Rightarrow |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 0, \quad \psi_j \in 2^\Phi. \quad (36)$$

The empty proposition may still carry a nonzero generalized quantum basic probability amplitude in GQET.

Definition 23 (Vacuous GQBPA function). A GQBPA Q_M is called vacuous when $|Q_M(\Phi)|^2 = 1$.

A nonzero value $Q_M(\emptyset) \neq 0$ represents an unknown-information component outside the current QFOD and enables open-world uncertainty reasoning in GQET. The closed-world specialization $Q_M(\emptyset) = 0$ is discussed in the next section.

3.1.2. Basic concepts of the generalized quantum basic probability distribution

Definition 24 (Generalized quantum basic probability distribution).

The generalized quantum basic probability distribution (GQBPD) of Q_M , is defined as:

$$M : 2^\Phi \rightarrow [0, 1], \quad (37)$$

and satisfies:

$$\begin{aligned} M(\psi_j) &= |Q_M(\psi_j)|^2, \quad \psi_j \in 2^\Phi, \\ \sum_{\psi_j \in 2^\Phi} M(\psi_j) &= 1, \end{aligned} \quad (38)$$

where

$$|Q_M(\psi_j)|^2 = Q_M(\psi_j)Q_M^*(\psi_j) = \varphi^2(\psi_j) = x_j^2 + y_j^2, \quad (39)$$

and $*$ denotes complex conjugation.

Definition 25 (Bayesian GQBPD). A GQBPD M is called Bayesian if all of its nonempty focal propositions are singletons; equivalently,

$$|\psi_j| > 1 \Rightarrow M(\psi_j) = 0, \quad \psi_j \in 2^\Phi. \quad (40)$$

The empty proposition may still carry a nonzero generalized quantum basic probability in GQET. For an elementary event $\phi_g \in \Phi$, $M(\phi_g)$ is called a

Bayesian GQBP. If, in addition, $M(\emptyset) = 0$, the Bayesian GQBPD reduces to a quantum probability distribution over the elementary events in Φ .

Definition 26 (Vacuous GQBPD). In GQET, when $M(\Phi) = 1$, M is called a vacuous GQBPD.

Definition 27 (Generalized quantum basic probability). In GQET, $M(\psi_j)$ ($\psi_j \in 2^\Phi$) is called generalized quantum basic probability (GQBP), which represents the degree of belief or support for quantum proposition ψ_j .

The GQET inherits the merits of GET and has the following attractive characteristics:

- The GQBPA Q_M takes values in the complex domain, with nonnegative real-valued amplitudes included as a special case, whereas a classical GBPA m_G in GET is restricted to nonnegative real values.
- In contrast to GET, $\forall \psi_j \in 2^\Phi$, the value of $|Q_M(\psi_j)|^2$ or $\varphi^2(\psi_j)$ represents the degree of belief or support for ψ_j .
- When $M = m_G$, the GQBPD M of GQET is in accordance with the classical GBPA m_G in GET.

3.1.3. Generalized quantum belief and plausibility functions

Taking into account application scenarios involving interference, the unnormalized generalized quantum interference belief and plausibility measures in GQET are first defined as follows.

Definition 28 (Unnormalized generalized quantum interference belief measure in GQET). Let Q_M be a GQBPA with proposition $\psi_j \in 2^\Phi$. An unnormalized generalized quantum interference belief measure, denoted

by $\widehat{\text{GQIBel}}$, is defined by

$$\widehat{\text{GQIBel}}(\psi_j) = \begin{cases} \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi: \emptyset \neq \psi_p \subseteq \psi_j\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, & \psi_j \neq \emptyset, \\ |\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2, & \psi_j = \emptyset. \end{cases} \quad (41)$$

For $\psi_j \neq \emptyset$, let

$$\Lambda_{\widehat{\text{GQIBel}}} \in \arg \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi: \emptyset \neq \psi_p \subseteq \psi_j\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2. \quad (42)$$

Here, $\Lambda_{\widehat{\text{GQIBel}}}$ denotes a subfamily of propositions achieving the maximal squared modulus in Eq. (41). If the maximizer is not unique, one maximizing subfamily is selected according to a fixed deterministic tie-breaking rule and is used consistently in the subsequent interference decomposition.

Consider $\psi_s, \psi_t \in \Lambda_{\widehat{\text{GQIBel}}}$ with $\psi_s \neq \psi_t$. Expanding the squared modulus in Eq. (41) and using $|\mathbb{Q}_{\mathbb{M}}(\psi)|^2 = \text{M}(\psi)$ gives

$$\begin{aligned} \widehat{\text{GQIBel}}(\psi_j) &= \sum_{\psi_s \in \Lambda_{\widehat{\text{GQIBel}}}} \text{M}(\psi_s) \\ &+ 2 \sum_{\psi_s \in \Lambda_{\widehat{\text{GQIBel}}}} \sum_{\substack{\psi_t \in \Lambda_{\widehat{\text{GQIBel}}} \\ \psi_s \prec \psi_t}} \sqrt{\text{M}(\psi_s)} \sqrt{\text{M}(\psi_t)} \cos(\theta_s - \theta_t), \end{aligned} \quad (43)$$

where $\prec$ denotes an arbitrary fixed ordering of the relevant propositions and $-1 \leq \cos(\theta_s - \theta_t) \leq 1$.

By the Cauchy–Schwarz inequality and the normalization condition of the GQBPA,

$$0 \leq \widehat{\text{GQIBel}}(\psi_j) \leq 2^{|\psi_j|} - 1, \quad \emptyset \neq \psi_j \subseteq \Phi. \quad (44)$$

Definition 29 (Interference effect in the unnormalized generalized quantum interference belief measure). Let $\Lambda_{\widehat{\text{GQIBel}}}$ be the subfamily achieving the maximal squared modulus in $\widehat{\text{GQIBel}}(\psi_j)$. The interference effect in $\widehat{\text{GQIBel}}$ is defined as

$$\begin{aligned} \text{Int}_{\widehat{\text{GQIBel}}}(\psi_j) &= 2 \sum_{\psi_s \in \Lambda_{\widehat{\text{GQIBel}}}} \sum_{\substack{\psi_t \in \Lambda_{\widehat{\text{GQIBel}}} \\ \psi_s \prec \psi_t}} |\mathbb{Q}_{\mathbb{M}}(\psi_s)| |\mathbb{Q}_{\mathbb{M}}(\psi_t)| \cos(\theta_s - \theta_t) \\ &= 2 \sum_{\psi_s \in \Lambda_{\widehat{\text{GQIBel}}}} \sum_{\substack{\psi_t \in \Lambda_{\widehat{\text{GQIBel}}} \\ \psi_s \prec \psi_t}} \sqrt{\mathbb{M}(\psi_s)} \sqrt{\mathbb{M}(\psi_t)} \cos(\theta_s - \theta_t). \end{aligned} \quad (45)$$

Definition 30 (Unnormalized generalized quantum interference Plausibility measure in GQET). Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBPA with proposition $\psi_j \in 2^\Phi$. An unnormalized generalized quantum interference Plausibility measure, denoted by $\widehat{\text{GQIPl}}$, is defined by

$$\widehat{\text{GQIPl}}(\psi_j) = \begin{cases} \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi : \psi_p \cap \psi_j \neq \emptyset\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2, & \psi_j \neq \emptyset, \\ |\mathbb{Q}_{\mathbb{M}}(\emptyset)|^2, & \psi_j = \emptyset. \end{cases} \quad (46)$$

For $\psi_j \neq \emptyset$, let

$$\Lambda_{\widehat{\text{GQIPl}}} \in \arg \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi : \psi_p \cap \psi_j \neq \emptyset\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_{\mathbb{M}}(\psi_p) \right|^2. \quad (47)$$

Here, $\Lambda_{\widehat{\text{GQIPl}}}$ denotes a subfamily of propositions achieving the maximal squared modulus in Eq. (46). If the maximizer is not unique, one maximizing subfamily is selected according to a fixed deterministic tie-breaking rule and is used consistently in the subsequent interference decomposition.

Consider $\psi_u, \psi_v \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}}$ with $\psi_u \neq \psi_v$. Then

$$\begin{aligned} \text{G}\hat{\text{Q}}\text{IPl}(\psi_j) &= \sum_{\psi_u \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}}} \text{M}(\psi_u) \\ &+ 2 \sum_{\psi_u \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}}} \sum_{\substack{\psi_v \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}} \\ \psi_u \prec \psi_v}} \sqrt{\text{M}(\psi_u)} \sqrt{\text{M}(\psi_v)} \cos(\theta_u - \theta_v). \end{aligned} \quad (48)$$

For $\emptyset \neq \psi_j \subseteq \Phi$,

$$0 \leq \text{G}\hat{\text{Q}}\text{IPl}(\psi_j) \leq 2^{|\Phi|} - 2^{|\Phi| - |\psi_j|}. \quad (49)$$

Definition 31 (Interference effect in the unnormalized generalized quantum interference Plausibility measure). Let $\Lambda_{\text{G}\hat{\text{Q}}\text{IPl}}$ be the subfamily achieving the maximal squared modulus in $\text{G}\hat{\text{Q}}\text{IPl}(\psi_j)$. The interference effect in $\text{G}\hat{\text{Q}}\text{IPl}$ is defined as

$$\begin{aligned} \text{Int}_{\text{G}\hat{\text{Q}}\text{IPl}}(\psi_j) &= 2 \sum_{\psi_u \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}}} \sum_{\substack{\psi_v \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}} \\ \psi_u \prec \psi_v}} |\text{Q}_{\text{M}}(\psi_u)| |\text{Q}_{\text{M}}(\psi_v)| \cos(\theta_u - \theta_v) \\ &= 2 \sum_{\psi_u \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}}} \sum_{\substack{\psi_v \in \Lambda_{\text{G}\hat{\text{Q}}\text{IPl}} \\ \psi_u \prec \psi_v}} \sqrt{\text{M}(\psi_u)} \sqrt{\text{M}(\psi_v)} \cos(\theta_u - \theta_v). \end{aligned} \quad (50)$$

The unnormalized generalized quantum interference belief and plausibility measures, $\text{G}\hat{\text{Q}}\text{IBel}$ and $\text{G}\hat{\text{Q}}\text{IPl}$, explicitly preserve constructive and destructive interference among quantum propositions and may exceed the conventional probability range $[0, 1]$. To obtain range-normalized interference-aware belief and plausibility support functions, a common normalization factor is introduced.

For the whole QFOD Φ , the sets of nonempty propositions involved in $\text{G}\hat{\text{Q}}\text{IBel}(\Phi)$ and $\text{G}\hat{\text{Q}}\text{IPl}(\Phi)$ coincide. Therefore,

$$\text{G}\hat{\text{Q}}\text{IBel}(\Phi) = \text{G}\hat{\text{Q}}\text{IPl}(\Phi). \quad (51)$$

Accordingly, to preserve the total support assigned to the known QFOD while retaining the unknown-information mass $M(\emptyset)$, the common normalization factor is defined as

$$\Gamma_{\text{GQI}} = \begin{cases} \frac{\widehat{\text{GQIPl}}(\Phi)}{1 - M(\emptyset)}, & M(\emptyset) < 1, \\ 1, & M(\emptyset) = 1, \end{cases} \quad (52)$$

or equivalently, by Eq. (51),

$$\Gamma_{\text{GQI}} = \begin{cases} \frac{\widehat{\text{GQIBel}}(\Phi)}{1 - M(\emptyset)}, & M(\emptyset) < 1, \\ 1, & M(\emptyset) = 1. \end{cases} \quad (53)$$

For $M(\emptyset) < 1$, at least one nonempty GQBPA amplitude is nonzero. The full-frame maximization in $\widehat{\text{GQIPl}}(\Phi)$ admits a single-proposition subfamily containing any such nonzero proposition; hence $\widehat{\text{GQIPl}}(\Phi) > 0$. Since $1 - M(\emptyset) > 0$, it follows that $\Gamma_{\text{GQI}} > 0$. For the boundary case $M(\emptyset) = 1$, $\Gamma_{\text{GQI}} = 1$ by definition.

Definition 32 (Generalized quantum interference belief function in GQET). Let $\mathbb{Q}_{\text{M}}$ be a GQBPA on QFOD Φ , and let $\widehat{\text{GQIBel}}$ be its corresponding unnormalized generalized quantum interference belief measure. The generalized quantum interference belief function, denoted by GQIBel , mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{GQIBel}(\psi_j) = \begin{cases} \frac{\widehat{\text{GQIBel}}(\psi_j)}{\Gamma_{\text{GQI}}}, & \psi_j \neq \emptyset, \\ M(\emptyset), & \psi_j = \emptyset. \end{cases} \quad (54)$$

Definition 33 (Generalized quantum interference Plausibility function in GQET). Let $\mathbb{Q}_{\text{M}}$ be a GQBPA on QFOD Φ , and let $\widehat{\text{GQIPl}}$ be

its corresponding unnormalized generalized quantum interference Plausibility measure. The generalized quantum interference Plausibility function, denoted by GQIPl , mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{GQIPl}(\psi_j) = \begin{cases} \frac{\text{G}\hat{\text{QIPl}}(\psi_j)}{\Gamma_{\text{GQI}}}, & \psi_j \neq \emptyset, \\ \text{M}(\emptyset), & \psi_j = \emptyset. \end{cases} \quad (55)$$

For any nonempty proposition $\psi_j \subseteq \Phi$, the family of propositions involved in $\text{G}\hat{\text{QIBel}}(\psi_j)$ is a subfamily of that involved in $\text{G}\hat{\text{QIPl}}(\psi_j)$. Therefore,

$$0 \leq \text{G}\hat{\text{QIBel}}(\psi_j) \leq \text{G}\hat{\text{QIPl}}(\psi_j) \leq \text{G}\hat{\text{QIPl}}(\Phi). \quad (56)$$

According to Eq. (52), it follows that

$$0 \leq \text{GQIBel}(\psi_j) \leq \text{GQIPl}(\psi_j) \leq 1 - \text{M}(\emptyset) \leq 1, \quad \emptyset \neq \psi_j \subseteq \Phi. \quad (57)$$

Moreover, for the whole QFOD,

$$\text{GQIBel}(\Phi) = \text{GQIPl}(\Phi) = 1 - \text{M}(\emptyset). \quad (58)$$

Hence, $\text{GQIBel}(\psi_j)$ and $\text{GQIPl}(\psi_j)$ provide range-normalized lower and upper interference-aware evidential support for proposition ψ_j while preserving the probability mass assigned to unknown information through $\text{M}(\emptyset)$.

Remark 1 . Under nonzero interference, these two functions are interference-aware support functionals and need not satisfy a Dempster–Shafer-type complement duality. Their generalized belief–plausibility semantics are recovered in the incoherent limit, where they reduce to GQBel and GQPl , respectively.

Moreover, under the incoherent reduction, the generalized quantum belief and plausibility functions in GQET are defined as follows.

Definition 34 (Generalized quantum belief function in GQET). Let $\mathbb{Q}_M$ be a GQBPA with proposition $\psi_j \in 2^\Phi$. The generalized quantum belief function GQBel , mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{GQBel}(\psi_j) = \begin{cases} \sum_{\emptyset \neq \psi_p \subseteq \psi_j} M(\psi_p), & \psi_j \neq \emptyset, \\ M(\emptyset), & \psi_j = \emptyset. \end{cases} \quad (59)$$

Definition 35 (Generalized quantum Plausibility function in GQET). Let $\mathbb{Q}_M$ be a GQBPA with proposition $\psi_j \in 2^\Phi$. The generalized quantum Plausibility function GQPl , mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{GQPl}(\psi_j) = \begin{cases} \sum_{\substack{\psi_p \subseteq \Phi \\ \psi_p \cap \psi_j \neq \emptyset}} M(\psi_p), & \psi_j \neq \emptyset, \\ M(\emptyset), & \psi_j = \emptyset. \end{cases} \quad (60)$$

For belief and plausibility, the incoherent reduction is defined by removing pairwise cross terms before the maximization:

$$\left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_M(\psi_p) \right|^2 \xrightarrow{\text{incoherent reduction}} \sum_{\psi_p \in \Lambda} |\mathbb{Q}_M(\psi_p)|^2. \quad (61)$$

Thus, the corresponding maximization problems reduce to sums of nonnegative generalized quantum basic probabilities. Therefore,

$$\widehat{\text{GQIBel}}(\psi_j) = \text{GQBel}(\psi_j), \quad \widehat{\text{GQIPl}}(\psi_j) = \text{GQPl}(\psi_j). \quad (62)$$

In particular, for the whole QFOD Φ ,

$$\widehat{\text{GQIBel}}(\Phi) = \widehat{\text{GQIPl}}(\Phi) = 1 - M(\emptyset) \leq 1. \quad (63)$$

According to the definition of Γ_{GQI} , it follows that

$$\Gamma_{\text{GQI}} = 1, \quad (64)$$

for both $M(\emptyset) < 1$ and the boundary case $M(\emptyset) = 1$.

Consequently, under this incoherent reduction,

$$\text{GQIBel}(\psi_j) = \text{QBel}(\psi_j), \quad \text{GQIPl}(\psi_j) = \text{QPl}(\psi_j). \quad (65)$$

Furthermore, when $M = m_G$ under this incoherent reduction, the generalized quantum belief and plausibility functions reduce to their classical counterparts in GET:

$$\text{GQIBel}(\psi_j) = \text{GBel}(\psi_j), \quad \text{GQIPl}(\psi_j) = \text{GPl}(\psi_j). \quad (66)$$

3.1.4. Generalized quantum evidential combination rules

In multisource generalized quantum evidential fusion, complex probability amplitudes are first combined at the amplitude level before being converted into real-valued evidential support. Following Feynman's probability-amplitude rules, the amplitudes associated with different evidence sources are multiplied within each fusion path, whereas the amplitudes of alternative fusion paths leading to the same proposition are coherently superposed. According to the Born rule, the corresponding fused evidential support is then obtained from the squared modulus of the resulting total amplitude.

The pairwise cross terms generated by this coherent superposition are called inter-path interference effects; for each output proposition, their signed sum is the aggregate inter-path interference contribution.

Accordingly, the generalized progressive quantum evidential combination rule (GPQECR) is first defined at the generalized quantum probability-

amplitude level to obtain a fused GQBPA, followed by the generalized quantum evidential combination rule (GQECR) at the corresponding GQBPD level.

To formalize the intra-path amplitude multiplication and inter-path coherent superposition described above, a fusion path p is represented by

$$\psi_p = (\psi_p^1, \psi_p^2, \dots, \psi_p^k), \quad (67)$$

where ψ_p^h denotes the focal element selected from the h th evidence source along fusion path p .

The conjunctive result associated with fusion path p is denoted by

$$\cap \psi_p := \bigcap_{h=1}^k \psi_p^h. \quad (68)$$

Using this fusion-path representation, the generalized progressive quantum evidential combination rule is defined at the probability-amplitude level as follows.

Definition 36 (Generalized progressive quantum evidential combination rule). Let $\{\mathbb{Q}_{M_1}, \dots, \mathbb{Q}_{M_k}\}$ be a set of independent GQBPAs on QFOD Φ . For each $\psi_j \in 2^\Phi$, the unnormalized fused generalized quantum probability amplitude is defined as

$$\tilde{\mathbb{Q}}_G^\oplus(\psi_j) = \begin{cases} \sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k \mathbb{Q}_{M_h}(\psi_p^h), & \emptyset \neq \psi_j \subseteq \Phi, \\ \prod_{h=1}^k \mathbb{Q}_{M_h}(\emptyset), & \psi_j = \emptyset. \end{cases} \quad (69)$$

The normalization coefficient is defined as

$$\gamma_G = \sum_{\psi_v \in 2^\Phi} \left| \tilde{\mathbb{Q}}_G^\oplus(\psi_v) \right|^2. \quad (70)$$

Accordingly, $\sqrt{\gamma_G}$ serves as the amplitude-normalization factor.

For any $\psi_j \in 2^\Phi$, the generalized progressive quantum evidential combination rule (GPQECR) is then defined by

$$\mathbb{Q}_G^\oplus(\psi_j) = \begin{cases} \frac{\tilde{\mathbb{Q}}_G^\oplus(\psi_j)}{\sqrt{\gamma_G}}, & \gamma_G > 0, \\ 1, & \gamma_G = 0, \psi_j = \emptyset, \\ 0, & \gamma_G = 0, \psi_j \neq \emptyset. \end{cases} \quad (71)$$

The case $\gamma_G = 0$ represents a boundary case in which all retained unnormalized fused probability amplitudes vanish. In this case, the fused GQBPA is assigned entirely to the empty proposition.

Theorem 1 (Validity and algebraic properties of GPQECR). The GPQECR is closed and commutative. Moreover, whenever the required intermediate normalization coefficients are positive, it is associative. Its output $\mathbb{Q}_G^\oplus$ is a valid GQBPA satisfying

$$\sum_{\psi_j \in 2^\Phi} |\mathbb{Q}_G^\oplus(\psi_j)|^2 = 1. \quad (72)$$

Proof. Commutativity follows because both set intersection and multiplication of complex amplitudes are invariant under permutations of the input evidence sources. For associativity, the unnormalized amplitude-level conjunctive fusion is associative. Any intermediate positive normalization only multiplies all retained amplitudes by a common positive scalar, which is canceled by the final normalization; hence the final normalized GQBPA is independent of the parenthesization. If $\gamma_G > 0$, division by $\sqrt{\gamma_G}$ gives unit squared norm by construction. If $\gamma_G = 0$, GPQECR assigns unit ampli-

tude to the empty proposition and zero amplitude to all nonempty propositions, which also has unit squared norm. Therefore the output is a valid GQBPA. $\square$

For the empty proposition, Eq. (71) gives

$$\mathbb{Q}_G^\oplus(\emptyset) = \begin{cases} \frac{\prod_{h=1}^k \mathbb{Q}_{\mathbb{M}_h}(\emptyset)}{\sqrt{\gamma_G}}, & \gamma_G > 0, \\ 1, & \gamma_G = 0. \end{cases} \quad (73)$$

For $\gamma_G > 0$, a nonzero unknown-information component is retained in the fused GQBPA if and only if

$$\mathbb{Q}_{\mathbb{M}_h}(\emptyset) \neq 0, \quad 1 \leq h \leq k. \quad (74)$$

In the boundary case $\gamma_G = 0$, the fused GQBPA is assigned entirely to the empty proposition.

When all input GQBPA's satisfy

$$\mathbb{Q}_{\mathbb{M}_h}(\emptyset) = 0, \quad 1 \leq h \leq k, \quad (75)$$

and $\gamma_G > 0$, the unknown-information component vanishes and the generalized open-world formulation reduces to its closed-world counterpart. Accordingly,

$$\mathbb{Q}_G^\oplus(\psi_j) = \mathbb{Q}^\oplus(\psi_j), \quad \emptyset \neq \psi_j \subseteq \Phi, \quad (76)$$

with

$$\mathbb{Q}^\oplus(\emptyset) = 0. \quad (77)$$

Definition 37 (Generalized quantum evidential combination rule).

Let $\mathbb{Q}_G^\oplus$ be the fused GQBPA obtained by GPQECR. For any $\psi_j \in 2^\Phi$, the

generalized quantum evidential combination rule (GQECR) is defined by

$$\text{GQECR}(\psi_j) = M_G^\oplus(\psi_j) = |\mathbb{Q}_G^\oplus(\psi_j)|^2, \quad \psi_j \in 2^\Phi, \quad (78)$$

where $M_G^\oplus$ denotes the GQBPD induced by the fused GQBPA $\mathbb{Q}_G^\oplus$.

For $\gamma_G > 0$, according to Eq. (71), GQECR can equivalently be expressed as

$$\text{GQECR}(\psi_j) = \frac{|\tilde{\mathbb{Q}}_G^\oplus(\psi_j)|^2}{\gamma_G}, \quad \psi_j \in 2^\Phi. \quad (79)$$

For the boundary case $\gamma_G = 0$, Eq. (71) gives

$$\text{GQECR}(\psi_j) = \begin{cases} 1, & \psi_j = \emptyset, \\ 0, & \psi_j \neq \emptyset. \end{cases} \quad (80)$$

GQECR satisfies the following properties:

- **Commutativity:** The fused GQBPD is invariant under any permutation of the input generalized quantum evidence sources.
- **Non-negativity and boundedness:** For any $\psi_j \in 2^\Phi$,

$$0 \leq \text{GQECR}(\psi_j) \leq 1. \quad (81)$$

- **Normalization:** The fused GQBPD satisfies

$$\sum_{\psi_j \in 2^\Phi} \text{GQECR}(\psi_j) = 1. \quad (82)$$

Therefore, the output of GQECR remains a valid GQBPD.

For the interference analysis, the accumulated phase associated with fusion path p is defined as

$$\theta_p = \sum_{h=1}^k \theta_h(\psi_p^h), \quad (83)$$

where $\theta_h(\psi_p^h)$ denotes the phase of $\mathbb{Q}_{M_h}(\psi_p^h)$.

Definition 38 (Interference effect in GQECR). Let $\{\mathbb{Q}_{\mathbb{M}_1}, \dots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent GQBPA's on QFOD Φ . For a nonempty output proposition $\emptyset \neq \psi_x \subseteq \Phi$, the interference effect in GQECR is defined as

$$\begin{aligned} \text{Int}_{\text{GQECR}}(\psi_x) &= 2 \sum_{\substack{\cap \psi_p = \psi_x \\ \cap \psi_q = \psi_x \\ \psi_p \prec \psi_q}} \sum_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)| \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_q^h)| \cos(\theta_p - \theta_q) \\ &= 2 \sum_{\substack{\cap \psi_p = \psi_x \\ \cap \psi_q = \psi_x \\ \psi_p \prec \psi_q}} \sum_{h=1}^k \sqrt{M_h(\psi_p^h)} \prod_{h=1}^k \sqrt{M_h(\psi_q^h)} \cos(\theta_p - \theta_q), \end{aligned} \quad (84)$$

where $\prec$ denotes an arbitrary fixed ordering such that each unordered pair of distinct fusion paths is counted exactly once.

Thus, $\text{Int}_{\text{GQECR}}(\psi_x)$ is the aggregate inter-path interference contribution for ψ_x : the signed sum of the pairwise interference terms over distinct fusion-path pairs leading to that output proposition. Its vanishing means that these terms cancel in aggregate.

Since

$$-1 \leq \cos(\theta_p - \theta_q) \leq 1, \quad (85)$$

the interference effect may be constructive, destructive, or zero, depending on the relative phases of the corresponding fusion paths.

For a nonempty output proposition ψ_x , expanding the squared modulus of the coherently superposed fusion-path amplitudes gives

$$\begin{aligned} \left| \tilde{\mathbb{Q}}_G^\oplus(\psi_x) \right|^2 &= \sum_{\cap \psi_p = \psi_x} \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)|^2 + \text{Int}_{\text{GQECR}}(\psi_x) \\ &= \sum_{\cap \psi_p = \psi_x} \prod_{h=1}^k M_h(\psi_p^h) + \text{Int}_{\text{GQECR}}(\psi_x). \end{aligned} \quad (86)$$

Eq. (86) shows that the fused evidential support consists of two components: the accumulated non-interference contributions of individual fusion paths and the interference contribution among distinct fusion paths leading to the same proposition.

For the empty proposition, only the all-empty fusion path is retained in GPQECR. Therefore,

$$\left| \tilde{\mathbb{Q}}_G^\oplus(\emptyset) \right|^2 = \prod_{h=1}^k |\mathbb{Q}_{\mathbf{M}_h}(\emptyset)|^2 = \prod_{h=1}^k \mathbf{M}_h(\emptyset). \quad (87)$$

Accordingly, the normalization coefficient can be expressed as

$$\gamma_G = \sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k |\mathbb{Q}_{\mathbf{M}_h}(\psi_p^h)|^2 + \text{Int}_{\text{GPQECR}}(\psi_v) \right] + \prod_{h=1}^k |\mathbb{Q}_{\mathbf{M}_h}(\emptyset)|^2 \quad (88)$$

$$= \sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k \mathbf{M}_h(\psi_p^h) + \text{Int}_{\text{GPQECR}}(\psi_v) \right] + \prod_{h=1}^k \mathbf{M}_h(\emptyset). \quad (89)$$

For $\gamma_G > 0$, substituting the unnormalized fused amplitudes in Eq. (69) into Eq. (79) gives, for $\emptyset \neq \psi_j \subseteq \Phi$,

$$\text{GPQECR}(\psi_j) = \frac{1}{\gamma_G} \left| \sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k \mathbb{Q}_{\mathbf{M}_h}(\psi_p^h) \right|^2. \quad (90)$$

For the empty proposition,

$$\text{GPQECR}(\emptyset) = \frac{1}{\gamma_G} \left| \prod_{h=1}^k \mathbb{Q}_{\mathbf{M}_h}(\emptyset) \right|^2. \quad (91)$$

Using Eqs. (86) and (88), Eq. (90) can be further expressed as

$$\text{GQECR}(\psi_j) = \frac{\sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k \text{M}_h(\psi_p^h) + \text{Int}_{\text{GQECR}}(\psi_j)}{\sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k \text{M}_h(\psi_p^h) + \text{Int}_{\text{GQECR}}(\psi_v) \right] + \prod_{h=1}^k \text{M}_h(\emptyset)}. \quad (92)$$

Similarly, using Eqs. (87) and (88), Eq. (91) can be expressed as,

$$\text{GQECR}(\emptyset) = \frac{\prod_{h=1}^k \text{M}_h(\emptyset)}{\sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k \text{M}_h(\psi_p^h) + \text{Int}_{\text{GQECR}}(\psi_v) \right] + \prod_{h=1}^k \text{M}_h(\emptyset)}. \quad (93)$$

When the aggregate inter-path interference contribution is zero for every nonempty output proposition,

$$\text{Int}_{\text{GQECR}}(\psi_x) = 0, \quad \emptyset \neq \psi_x \subseteq \Phi, \quad (94)$$

and the input GQBPDs satisfy

$$\text{M}_h = m_{G_h}, \quad 1 \leq h \leq k, \quad (95)$$

define

$$\cup \psi_p := \bigcup_{h=1}^k \psi_p^h. \quad (96)$$

The conditions

$$\cap \psi_p = \emptyset, \quad \cup \psi_p \neq \emptyset \quad (97)$$

identify conflicting fusion paths while excluding the all-empty path. Accordingly, the conflict coefficient in mGCR is given by

$$K_G^m = \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k m_{G_h}(\psi_p^h). \quad (98)$$

Under these conditions, Eq. (88) reduces to

$$\gamma_G = 1 - K_G^m. \quad (99)$$

For $K_G^m < 1$, equivalently $\gamma_G > 0$, Eqs. (92) and (93) reduce to

$$\text{GQECR}(\psi_j) = \frac{\sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k m_{G_h}(\psi_p^h)}{1 - K_G^m}, \quad \emptyset \neq \psi_j \subseteq \Phi, \quad (100)$$

and

$$\text{GQECR}(\emptyset) = \frac{\prod_{h=1}^k m_{G_h}(\emptyset)}{1 - K_G^m}. \quad (101)$$

Therefore, when the input GQBPDs coincide with the GBPAs and the aggregate inter-path interference contribution vanishes for every nonempty output proposition, GQECR reduces to mGCR for $K_G^m < 1$.

For the boundary case

$$K_G^m = 1, \quad (102)$$

Eq. (99) gives $\gamma_G = 0$. According to Eq. (80),

$$\text{GQECR}(\emptyset) = 1, \quad \text{GQECR}(\psi_j) = 0, \quad \psi_j \neq \emptyset, \quad (103)$$

which agrees with the boundary treatment of mGCR [12].

Theorem 2 (Closed-world specialization of generalized quantum fusion). If every input GQBPA satisfies $Q_{M_h}(\emptyset) = 0$ and the generalized normalization coefficient satisfies $\gamma_G > 0$, then GPQECR reduces to PQECR and GQECR reduces to QECR on every nonempty proposition, with zero fused mass assigned to the empty proposition.

Proof. Under $Q_{M_h}(\emptyset) = 0$, every fusion path containing an empty input has zero amplitude, including the all-empty path. Hence the retained unnormalized amplitudes and their normalization coefficient coincide with those of the closed-world construction. Normalizing by the same positive coefficient yields the PQECR amplitudes, and taking squared moduli yields the QECR probabilities. The empty-proposition amplitude and probability are zero. $\square$

Theorem 3 (Classical compatibility of quantum evidential fusion). Suppose the input GQBPDs satisfy $M_h = m_{G_h}$ and, for every nonempty output proposition ψ_x , the aggregate inter-path interference contribution is zero:

$$\text{Int}_{\text{GQECR}}(\psi_x) = 0, \quad \emptyset \neq \psi_x \subseteq \Phi. \quad (104)$$

Then GQECR reduces to mGCR. Under the additional closed-world condition $m_{G_h}(\emptyset) = 0$ for all h , the corresponding closed-world QECR reduces to Dempster's rule of combination for $K < 1$. For $K_G^m = 1$, GQECR follows the open-world boundary assignment to the empty proposition; for $K = 1$, QECR is undefined, consistently with Dempster's rule under total conflict.

Proof. With zero aggregate inter-path interference contribution for every nonempty output proposition, the squared coherent sums reduce to sums of

path probabilities. Equation (88) then gives $\gamma_G = 1 - K_G^m$, and Eqs. (100) and (101) are exactly mGCR. Imposing the closed-world condition removes the all-empty mass and reduces the generalized conflict to the classical conflict coefficient K , so the nonempty fused masses become Dempster's normalized conjunctive rule. At $K_G^m = 1$, the GQET boundary rule assigns the fused mass to the empty proposition; at $K = 1$, the closed-world normalization coefficient vanishes, so QEER is undefined in the same sense as Dempster's rule. $\square$

GQEER has the following characteristics:

- **Open-world uncertainty representation:** GQEER allows the empty proposition to carry nonzero fused probability, thereby retaining unknown information in open-world environments.

For $\gamma_G > 0$,

$$\text{GQEER}(\emptyset) > 0 \quad (105)$$

if and only if

$$\mathbb{Q}_{\mathbf{M}_h}(\emptyset) \neq 0, \quad 1 \leq h \leq k. \quad (106)$$

- **Interference sensitivity:** For each nonempty output proposition, GQEER explicitly captures inter-path interference effects through $\text{Int}_{\text{GQEER}}(\psi_j)$, which may be constructive or destructive. Hence, the fused GQBPD depends not only on the magnitudes of the input GQBPDs but also on their relative phases.

- **Reduction to QEER:** When

$$\mathbb{Q}_{\mathbf{M}_h}(\emptyset) = 0, \quad 1 \leq h \leq k, \quad (107)$$

and $\gamma_G > 0$, the unknown-information component vanishes,

$$\text{GQECR}(\emptyset) = 0, \quad (108)$$

and GQECR reduces to its closed-world counterpart, QECR.

- **Compatibility with mGCR:** When the input GQBPDs satisfy $M_h = m_{G_h}$ and the aggregate inter-path interference contribution vanishes for every nonempty output proposition, GQECR reduces to mGCR [12]. For the boundary case $K_G^m = 1$, the fused GQBPD is assigned entirely to the empty proposition.
- **Compatibility with Dempster's rule:** When the closed-world condition holds, the input QBPDs coincide with classical BPAs, and the aggregate inter-path interference contribution vanishes for every nonempty output proposition, GQECR further reduces to Dempster's rule of combination for non-total-conflict cases.

Thus, GPQECR preserves the complex-valued fused GQBPA for subsequent progressive fusion, whereas GQECR provides the corresponding real-valued fused GQBPD.

Definition 39 (Generalized quantum evidential conflict coefficient).

Let $\{Q_{M_1}, \dots, Q_{M_k}\}$ be a set of independent GQBPA on QFOD Φ . The generalized quantum evidential conflict coefficient (GQECC), denoted by K_C^G , is defined as

$$K_C^G = \left| \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k Q_{M_h}(\psi_p^h) \right|^2. \quad (109)$$

Definition 40 (Interference effect in GQECC). The interference effect among distinct conflicting fusion paths in GQECC is defined as

$$\begin{aligned}
\text{Int}_{\text{GQECC}} &= 2 \sum_{\substack{\cap \psi_p = \emptyset \cap \psi_q = \emptyset \\ \cup \psi_p \neq \emptyset \cup \psi_q \neq \emptyset \\ \psi_p \prec \psi_q}} \sum_{h=1}^k \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)| \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_q^h)| \cos(\theta_p - \theta_q) \\
&= 2 \sum_{\substack{\cap \psi_p = \emptyset \cap \psi_q = \emptyset \\ \cup \psi_p \neq \emptyset \cup \psi_q \neq \emptyset \\ \psi_p \prec \psi_q}} \sum_{h=1}^k \sqrt{\mathbb{M}_h(\psi_p^h)} \prod_{h=1}^k \sqrt{\mathbb{M}_h(\psi_q^h)} \cos(\theta_p - \theta_q). \quad (110)
\end{aligned}$$

Expanding the squared modulus in Eq. (109) gives

$$\begin{aligned}
\mathbb{K}_C^G &= \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)|^2 + \text{Int}_{\text{GQECC}} \\
&= \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k \mathbb{M}_h(\psi_p^h) + \text{Int}_{\text{GQECC}}. \quad (111)
\end{aligned}$$

When

$$\text{Int}_{\text{GQECC}} = 0 \quad (112)$$

and

$$\mathbb{M}_h = m_{G_h}, \quad 1 \leq h \leq k, \quad (113)$$

Eq. (111) reduces to

$$\mathbb{K}_C^G = \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k m_{G_h}(\psi_p^h) = K_G^{\text{m}}, \quad (114)$$

which is consistent with the conflict coefficient in mGCR [12].

GQECC has the following characteristics:

- **Interference sensitivity:** GQECC captures inter-path interference effects among conflicting fusion paths. Constructive interference increases the resulting conflict measure, whereas destructive interference decreases it.
- **Compatibility with the mGCR conflict coefficient:** When the input GQBPDs coincide with the GBPAs and the aggregate inter-path interference contribution among conflicting paths, $\text{Int}_{\text{GQECC}}$, vanishes, GQECC reduces to K_G^m in mGCR.
- **Reduction to QECC:** When all input GQBPDs satisfy

$$\mathbb{Q}_{M_h}(\emptyset) = 0, \quad 1 \leq h \leq k, \quad (115)$$

GQECC reduces to its closed-world counterpart, QECC.

- **Compatibility with the classical conflict coefficient:** Under the closed-world condition, when the input QBPDs coincide with classical BPAs and the aggregate inter-path interference contribution among conflicting paths vanishes, GQECC further reduces to the classical conflict coefficient K in Dempster–Shafer evidence theory.

Unlike the classical conflict coefficient, GQECC is an interference-sensitive conflict measure and is not generally restricted to the interval $[0, 1]$.

3.1.5. Generalized quantum Pignistic transformations

When interference is taken into account in decision making, Pignistic redistribution is first performed coherently among indistinguishable redistribution paths leading to the same elementary event. The corresponding

singleton support is then obtained through the Born rule. Since distinct elementary events in the QFOD are mutually exclusive decision outcomes, their resulting supports are combined additively at the probability level.

Definition 41 (Unnormalized generalized quantum interference Pignistic support). Let $\mathbb{Q}_M$ be a GQBPA on QFOD Φ , and let M be its corresponding GQBPD with $M(\emptyset) < 1$. For each elementary event $\phi_g \in \Phi$, define the coherently redistributed amplitude

$$A_g^G = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} \frac{\mathbb{Q}_M(\psi_p)}{\sqrt{1 - M(\emptyset)}} \frac{1}{\sqrt{|\psi_p|}}. \quad (116)$$

The corresponding unnormalized singleton support is

$$\tilde{\text{BetGQIP}}(\phi_g) = |A_g^G|^2. \quad (117)$$

For any proposition $\psi_j \subseteq \Phi$, the unnormalized generalized quantum interference Pignistic support is extended additively as

$$\tilde{\text{BetGQIP}}(\psi_j) = \sum_{\phi_g \in \psi_j} \tilde{\text{BetGQIP}}(\phi_g), \quad (118)$$

with $\tilde{\text{BetGQIP}}(\emptyset) = 0$.

The construction in Eq. (116) preserves interference among indistinguishable redistribution paths leading to the same elementary event, whereas Eq. (118) adds supports of mutually exclusive elementary outcomes only after the Born-rule conversion. Thus, $\tilde{\text{BetGQIP}}$ can be used directly as an unnormalized interference-modulated support when a probability distribution is not required. If $M(\emptyset) = 1$, all generalized quantum basic probability is assigned to unknown information, and no Pignistic probability distribution over the known elementary-event space Φ is defined.

Definition 42 (Interference effect in generalized quantum interference Pignistic redistribution). For an elementary event $\phi_g \in \Phi$, the interference effect among redistribution paths leading to ϕ_g is defined as

$$\text{Int}_{\text{BetGQIP}}(\phi_g) = \frac{2}{1 - M(\emptyset)} \sum_{\substack{\emptyset \neq \psi_s, \psi_t \subseteq \Phi \\ \phi_g \in \psi_s \cap \psi_t \\ \psi_s \prec \psi_t}} \frac{\sqrt{M(\psi_s)} \sqrt{M(\psi_t)}}{\sqrt{|\psi_s| |\psi_t|}} \cos(\theta_s - \theta_t). \quad (119)$$

Accordingly,

$$\tilde{\text{BetGQIP}}(\phi_g) = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} \frac{M(\psi_p)}{1 - M(\emptyset)} \frac{1}{|\psi_p|} + \text{Int}_{\text{BetGQIP}}(\phi_g). \quad (120)$$

Since the first terms in Eq. (120) sum to one over all elementary events, the normalization coefficient satisfies

$$\gamma_{\text{BetGQIP}} = 1 + \sum_{\phi_g \in \Phi} \text{Int}_{\text{BetGQIP}}(\phi_g). \quad (121)$$

Definition 43 (Generalized quantum interference Pignistic probability for elementary events). Assume $\gamma_{\text{BetGQIP}} > 0$, where

$$\gamma_{\text{BetGQIP}} = \sum_{\phi_g \in \Phi} \tilde{\text{BetGQIP}}(\phi_g) = \tilde{\text{BetGQIP}}(\Phi). \quad (122)$$

For each $\phi_g \in \Phi$, the normalized generalized quantum interference Pignistic probability is defined as

$$\text{BetGQIP}(\phi_g) = \frac{\tilde{\text{BetGQIP}}(\phi_g)}{\gamma_{\text{BetGQIP}}}. \quad (123)$$

Consequently,

$$\sum_{\phi_g \in \Phi} \text{BetGQIP}(\phi_g) = 1. \quad (124)$$

The condition $\gamma_{\text{BetGQIP}} > 0$ is necessary for the normalized Pignistic probability to be well defined. If $\gamma_{\text{BetGQIP}} = 0$ while $M(\emptyset) < 1$, all coherently redistributed amplitudes associated with the known elementary decision outcomes vanish simultaneously. In this complete-cancellation boundary case, the unnormalized generalized quantum interference Pignistic support remains well defined, but no normalized BetGQIP probability distribution on Φ is defined.

Definition 44 (Generalized quantum interference Pignistic probability for propositions). For any proposition $\psi_j \subseteq \Phi$, the generalized quantum interference Pignistic probability is the additive extension of the normalized singleton probabilities:

$$\text{BetGQIP}(\psi_j) = \sum_{\phi_g \in \psi_j} \text{BetGQIP}(\phi_g) = \frac{\tilde{\text{BetGQIP}}(\psi_j)}{\gamma_{\text{BetGQIP}}}. \quad (125)$$

Hence,

$$\text{BetGQIP}(\emptyset) = 0, \quad \text{BetGQIP}(\Phi) = 1. \quad (126)$$

If the aggregate redistribution-interference contribution vanishes,

$$\text{Int}_{\text{BetGQIP}}(\phi_g) = 0, \quad \forall \phi_g \in \Phi, \quad (127)$$

then Eq. (120) yields

$$\tilde{\text{BetGQIP}}(\phi_g) = \text{BetGQP}(\phi_g). \quad (128)$$

Consequently,

$$\gamma_{\text{BetGQIP}} = 1, \quad \text{BetGQIP}(\psi_j) = \text{BetGQP}(\psi_j), \quad \psi_j \subseteq \Phi. \quad (129)$$

Theorem 4 (Validity and classical consistency of the generalized quantum interference Pignistic probability). If $M(\emptyset) < 1$

and $\gamma_{\text{BetGQIP}} > 0$, then BetGQIP is a probability measure on the finite elementary-event space Φ . In particular, it is nonnegative, normalized, and finitely additive. Moreover, when the aggregate redistribution-interference contribution vanishes for every elementary event, it reduces to BetGQP.

Proof. By construction, $\tilde{\text{BetGQIP}}(\phi_g) = |A_g^G|^2 \geq 0$. Division by the positive factor γ_{BetGQIP} preserves nonnegativity and gives unit total probability over Φ . For disjoint $A, B \subseteq \Phi$, the singleton expansions are disjoint, so $\text{BetGQIP}(A \cup B) = \text{BetGQIP}(A) + \text{BetGQIP}(B)$. When the aggregate redistribution-interference contribution vanishes for every elementary event, Eq. (120) becomes the ordinary generalized Pignistic redistribution of the normalized known-world probability mass; hence $\gamma_{\text{BetGQIP}} = 1$ and the result reduces to BetGQP. $\square$

When the aggregate redistribution-interference contribution vanishes for each elementary event, the corresponding generalized quantum Pignistic probability transformation redistributes the generalized quantum basic probability over elementary events for probabilistic decision making.

Definition 45 (Generalized quantum Pignistic probability transformation). Let $\mathbb{Q}_{\mathbf{M}}$ be a GQBPA on QFOD Φ , and let $\mathbf{M}$ be its corresponding GQBPD satisfying $\mathbf{M}(\emptyset) < 1$. For a quantum proposition $\psi_j \in 2^\Phi$, its generalized quantum Pignistic probability function, denoted by BetGQP, is defined as

$$\text{BetGQP}(\psi_j) = \sum_{\emptyset \neq \psi_p \subseteq \Phi} \frac{|\mathbb{Q}_{\mathbf{M}}(\psi_p)|^2}{1 - |\mathbb{Q}_{\mathbf{M}}(\emptyset)|^2} \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|}, \quad (130)$$

where $|\psi|$ denotes the cardinality of proposition ψ . The transformation from

$\mathbb{Q}_M$ to BetGQP is called the generalized quantum Pignistic probability transformation.

Since $|\mathbb{Q}_M(\psi_p)|^2 = M(\psi_p)$, Eq. (130) can be expressed as

$$\text{BetGQP}(\psi_j) = \sum_{\emptyset \neq \psi_p \subseteq \Phi} \frac{M(\psi_p)}{1 - M(\emptyset)} \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|}. \quad (131)$$

For an elementary event $\phi_g \in \Phi$, Eq. (130) reduces to

$$\text{BetGQP}(\phi_g) = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} \frac{|\mathbb{Q}_M(\psi_p)|^2}{1 - |\mathbb{Q}_M(\emptyset)|^2} \cdot \frac{1}{|\psi_p|}. \quad (132)$$

Equivalently,

$$\text{BetGQP}(\phi_g) = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} \frac{M(\psi_p)}{1 - M(\emptyset)} \cdot \frac{1}{|\psi_p|}. \quad (133)$$

Accordingly, the generalized quantum Pignistic probabilities over all elementary events satisfy

$$\sum_{\phi_g \in \Phi} \text{BetGQP}(\phi_g) = 1. \quad (134)$$

3.2. A quantum representation of GQET in Hilbert space

3.2.1. Quantum state representation of a GQBPA

Definition 46 (Basis event in GQET). Given the QFOD $\Phi = \{\phi_1, \dots, \phi_n\}$, where each elementary event ϕ_w is associated with the orthonormal basis state $|\phi_w\rangle$, let $e_0 = \emptyset$ denote the unknown-information basis event outside the current QFOD. The set of basis events in GQET is defined as

$$\mathbb{B} = \{e_0, e_1, \dots, e_n\}, \quad e_0 = \emptyset, \quad e_w = \phi_w, \quad 1 \leq w \leq n. \quad (135)$$

Thus, e_w ($1 \leq w \leq n$) is the basis event associated with the QFOD event ϕ_w , whereas e_0 represents unknown information outside Φ .

Remark 2 . The notation $e_0 = \emptyset$ identifies the generalized empty proposition with its corresponding basis-event label at the proposition level. It does not identify the set-theoretic empty set itself with a Hilbert-space vector. The vector $|e_0\rangle$ is the orthonormal basis state associated with the generalized empty proposition and represents unknown information outside the current QFOD.

Definition 47 (Standard basis vector representation of a basis event in GQET). Let $\mathcal{H}_G$ be the $(n + 1)$ -dimensional complex Hilbert space spanned by the orthonormal basis

$$\{|e_0\rangle, |e_1\rangle, \dots, |e_n\rangle\}. \quad (136)$$

Here, $|e_0\rangle$ represents the unknown-information basis event outside the QFOD, whereas $|e_w\rangle$ ($1 \leq w \leq n$) represents the basis event e_w within the QFOD.

For each $e_w \in \mathbb{B}$ ($0 \leq w \leq n$), the corresponding standard basis vector $|e_w\rangle \in \mathbb{C}^{n+1}$ is defined as

$$|e_w\rangle = [\eta_0, \eta_1, \dots, \eta_l, \dots, \eta_n]^T, \quad (137)$$

where

$$\eta_l = \begin{cases} 1, & l = w, \\ 0, & l \neq w, \end{cases} \quad 0 \leq l \leq n. \quad (138)$$

The standard basis vectors associated with different basis events are orthonormal, satisfying

$$\langle e_w | e_w \rangle = 1, \quad \langle e_w | e_s \rangle = 0 \quad (w \neq s), \quad 0 \leq w, s \leq n. \quad (139)$$

Definition 48 (State vector representation of a quantum proposition in GQET). Let $\psi_j \in 2^\Phi$ be a quantum proposition. The state vector $|\psi_j\rangle$ corresponding to ψ_j is defined as

$$|\psi_j\rangle = \begin{cases} |e_0\rangle, & \psi_j = \emptyset, \\ \sum_{w: \phi_w \in \psi_j} \lambda_w^j |e_w\rangle, & \emptyset \neq \psi_j \subseteq \Phi, \end{cases} \quad (140)$$

where $\lambda_w^j = a_w^j + b_w^j i$ is a complex coefficient satisfying

$$\sum_{w: \phi_w \in \psi_j} |\lambda_w^j|^2 = 1, \quad \emptyset \neq \psi_j \subseteq \Phi. \quad (141)$$

Thus, $|\psi_j\rangle$ is a unit vector representing a pure quantum state. According to Eq. (140), the empty proposition is represented by $|e_0\rangle$, whereas a composite quantum proposition in GQET is represented as a linear superposition of the standard basis vectors corresponding to its constituent basis events. For notational simplicity, the dependence of $|\psi_j\rangle$ on the coefficient allocation $\lambda^j = \{\lambda_w^j\}$ is left implicit throughout the paper. We use $\lambda = \{\lambda_w^j\}$ to collectively denote an admissible set of internal coefficient allocations for all nonempty quantum propositions.

Remark 3 . The superposition representation of a composite quantum proposition preserves its internal state structure through the coefficients λ_w^j , thereby enabling the evolution and redistribution of the amplitudes associated with its constituent basis events to be characterized under quantum operations.

Definition 49 (Density operator representation of a quantum proposition in GQET). Let $|\psi_j\rangle$ be a state vector corresponding to its

quantum proposition ψ_j ($\psi_j \in 2^\Phi$) on QFOD. Its density operator is defined as:

$$\rho_j = |\psi_j\rangle\langle\psi_j|. \quad (142)$$

The dependence of ρ_j on the internal coefficient allocation λ^j is also left implicit for notational simplicity.

On this basis, a GQBPA function is expressed as a linear combination of projection operators as follows.

Definition 50 (Linear operator representation of a GQBPA function). Let $\mathbb{Q}_\mathbb{M}$ be a GQBPA function with $\mathbb{Q}_\mathbb{M}(\psi_j)$ on QFOD Φ . The linear operator corresponding to a GQBPA function is defined as:

$$\tilde{\rho}_{\mathbb{Q}_\mathbb{M}} = \sum_{\psi_j \in 2^\Phi} \mathbb{Q}_\mathbb{M}(\psi_j) |\psi_j\rangle\langle\psi_j| = \sum_{\psi_j \in 2^\Phi} \mathbb{Q}_\mathbb{M}(\psi_j) \rho_j, \quad (143)$$

where $\tilde{\rho}_{\mathbb{Q}_\mathbb{M}}$ is a linear combination of projection operators and is generally not a density operator, as it is not necessarily Hermitian, positive semidefinite, or of unit trace.

Definition 51 (Density operator induced by a GQBPA). The density operator induced by a GQBPA through its corresponding GQBPD is defined as:

$$\begin{aligned} \hat{\rho}_{\mathbb{Q}_\mathbb{M}}(\lambda) &= \sum_{\psi_j \in 2^\Phi} |\mathbb{Q}_\mathbb{M}(\psi_j)|^2 \rho_j \\ &= \sum_{\psi_j \in 2^\Phi} |\mathbb{Q}_\mathbb{M}(\psi_j)|^2 |\psi_j\rangle\langle\psi_j|. \end{aligned} \quad (144)$$

where $|\cdot|^2$ is the modulus squared function.

Remark 4 . The induced density operator is constructed from the squared moduli of the GQBPA coefficients. It therefore represents the GQBPD in-

duced by the GQBPA and does not retain the phases of the original probability amplitudes. The complex amplitude and phase information is retained in the linear-operator representation $\tilde{\rho}_{\mathbf{Q}_M}$.

From Eq. (144), it follows directly that

$$\hat{\rho}_{\mathbf{Q}_M}(\lambda) \succeq 0, \quad (145)$$

and

$$\begin{aligned} \text{Tr}(\hat{\rho}_{\mathbf{Q}_M}(\lambda)) &= \sum_{\psi_j \in 2^\Phi} |\mathbf{Q}_M(\psi_j)|^2 \text{Tr}(|\psi_j\rangle\langle\psi_j|) \\ &= \sum_{\psi_j \in 2^\Phi} |\mathbf{Q}_M(\psi_j)|^2 = 1. \end{aligned}$$

Therefore, for each fixed admissible allocation λ , $\hat{\rho}_{\mathbf{Q}_M}(\lambda)$ is a positive-semidefinite density operator with unit trace.

For a fixed admissible allocation λ , let $\hat{\mathbb{E}}_\nu$ be the eigenvalues of $\hat{\rho}_{\mathbf{Q}_M}(\lambda)$. It follows that:

$$\hat{\mathbb{E}}_\nu \geq 0, \quad \sum_\nu \hat{\mathbb{E}}_\nu = 1.$$

Since $|\mathbf{Q}_M|^2 = \varphi^2 = M$, the density operator of a GQBPD can be expressed as follows.

Definition 52 (Density operator representation of a GQBPD). For a fixed admissible internal coefficient allocation λ , the density operator corresponding to a GQBPD is defined as:

$$\rho_M(\lambda) = \sum_{\psi_j \in 2^\Phi} M(\psi_j) |\psi_j\rangle\langle\psi_j| = \sum_{\psi_j \in 2^\Phi} M(\psi_j) \rho_j. \quad (146)$$

Here, the dependence of $|\psi_j\rangle$ and ρ_j on λ is understood implicitly.

By construction,

$$\rho_M(\lambda) = \hat{\rho}_{\mathbf{Q}_M}(\lambda). \quad (147)$$

3.2.2. Measurement operator for basis event in GQET

Definition 53 (Measurement operator for basis event in GQET).

For each basis event $e_w \in \mathbb{B}$, the corresponding measurement operator in GQET is defined as

$$\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, \quad 0 \leq w \leq n, \quad (148)$$

satisfying

$$\sum_{w=0}^n \mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} = I, \quad (149)$$

where $\mathbb{M}_{e_w}^\dagger$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^\dagger = (\mathbb{M}_{e_w}^T)^*$; and I denotes the identity matrix.

3.2.3. Basis event measurement function in GQET

Definition 54 (Basis event measurement function in GQET). Let $\rho_M(\lambda)$ be the density operator corresponding to a GQBPD M on QFOD Φ under a fixed admissible allocation λ , and let $\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 0 \leq w \leq n\}$ be the corresponding measurement operators. The probability of observing the basis event e_w is defined as

$$\Pr(e_w) = \text{Tr}(\mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} \rho_M(\lambda)), \quad 0 \leq w \leq n. \quad (150)$$

Here, $\text{Tr}(\cdot)$ denotes the matrix trace. For notational simplicity, the notation $\Pr(e_w)$ is retained, with a fixed admissible allocation λ understood.

3.2.4. GQBP measurement function

Definition 55 (GQBP measurement function). Let $\rho_M(\lambda)$ be the density operator corresponding to a GQBPD M on QFOD Φ under a fixed admissible allocation λ , and let $\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 0 \leq w \leq n\}$ be the corresponding measurement operators. The measurement-induced generalized

quantum support, denoted by $M_{\text{Tr}}(\psi_j)$, is defined as

$$M_{\text{Tr}}(\psi_j) = \begin{cases} \sum_{w: \phi_w \in \psi_j} \text{Tr}(\mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} \rho_M(\lambda)), & \psi_j \neq \emptyset, \\ \text{Tr}(\mathbb{M}_{e_0}^\dagger \mathbb{M}_{e_0} \rho_M(\lambda)), & \psi_j = \emptyset. \end{cases} \quad (151)$$

Here, $M_{\text{Tr}}(\psi_j)$ is the generalized quantum support induced by the trace-based Hilbert-space measurement. For notational simplicity, its dependence on the fixed admissible allocation λ is left implicit. In general, it need not coincide elementwise with the original GQBPD $M(\psi_j)$ used to construct $\rho_M(\lambda)$.

3.2.5. Belief and plausibility measurement functions in GQET

Proposition 1 (Hilbert-space realization of GQBel and GQPI). Let Λ be the set of admissible internal state-vector allocations $\lambda = \{\lambda_w^r\}$ satisfying $\sum_{w: \phi_w \in \psi_r} |\lambda_w^r|^2 = 1$ for each nonempty quantum proposition ψ_r . The internal allocations associated with different nonempty propositions are independent variables; equivalently, Λ is the Cartesian product of the proposition-wise admissible allocation sets. Let $\rho_M(\lambda)$ be the density operator in Eq. (146), with basis events $e_w \in \mathbb{B} = \{e_0, e_1, \dots, e_n\}$ and measurement operators $\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|$. Then

$$\text{GQBel}(\psi_j) = \begin{cases} \min_{\lambda \in \Lambda} \sum_{w: \phi_w \in \psi_j} \text{Tr}(\mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} \rho_M(\lambda)), & \psi_j \neq \emptyset, \\ M(\emptyset), & \psi_j = \emptyset, \end{cases} \quad (152)$$

$$\text{GQPI}(\psi_j) = \begin{cases} \max_{\lambda \in \Lambda} \sum_{w: \phi_w \in \psi_j} \text{Tr}(\mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} \rho_M(\lambda)), & \psi_j \neq \emptyset, \\ M(\emptyset), & \psi_j = \emptyset. \end{cases} \quad (153)$$

Thus, the min-max quantum measurement construction exactly realizes the generalized quantum belief and plausibility functions defined in Eqs. (59) and (60).

Proof. Because the allocation variables associated with different focal propositions are independently constrained, the global minimum and maximum decompose into the corresponding proposition-wise extrema. For a nonempty focal proposition ψ_r , its contribution to the measured support of a nonempty target ψ_j is

$$M(\psi_r) \sum_{w: \phi_w \in \psi_r \cap \psi_j} |\lambda_w^r|^2. \quad (154)$$

If $\psi_r \subseteq \psi_j$, this contribution equals $M(\psi_r)$ for every admissible allocation. Otherwise, its minimum is zero by assigning the internal weight to a constituent event outside ψ_j . Therefore the minimum over λ sums exactly the masses of nonempty propositions contained in ψ_j , which is $\text{GQBel}(\psi_j)$. For the maximum, every ψ_r intersecting ψ_j can place all of its internal weight on the intersection and contributes $M(\psi_r)$, whereas a disjoint proposition contributes zero. Hence the maximum is $\text{GQPl}(\psi_j)$. For $\psi_j = \emptyset$, measurement of the orthogonal basis state $|e_0\rangle$ gives $M(\emptyset)$, yielding $\text{GQBel}(\emptyset) = \text{GQPl}(\emptyset) = M(\emptyset)$. $\square$

3.2.6. Generalized progressive quantum evidential combination rule in Hilbert space

Definition 56 (Linear operator representation of progressively fused generalized quantum evidence in GQET). Let $\mathbb{Q}_G^\oplus$ be the GQBPA obtained by GPQEER. The linear operator corresponding to the progressively

fused GQBPA is defined as

$$\tilde{\rho}_{\mathbb{Q}_G^\oplus} = \sum_{\psi_j \in 2^\Phi} \mathbb{Q}_G^\oplus(\psi_j) |\psi_j\rangle \langle \psi_j| = \sum_{\psi_j \in 2^\Phi} \mathbb{Q}_G^\oplus(\psi_j) \rho_j. \quad (155)$$

According to the definition of GPQEER, provided that $\gamma_G > 0$,

$$\mathbb{Q}_G^\oplus(\psi_j) = \frac{\tilde{\mathbb{Q}}_G^\oplus(\psi_j)}{\sqrt{\gamma_G}}, \quad \psi_j \in 2^\Phi, \quad (156)$$

where $\tilde{\mathbb{Q}}_G^\oplus$ and γ_G are defined in Eqs. (69) and (70), respectively. Therefore,

$$\tilde{\rho}_{\mathbb{Q}_G^\oplus} = \frac{1}{\sqrt{\gamma_G}} \sum_{\psi_j \in 2^\Phi} \tilde{\mathbb{Q}}_G^\oplus(\psi_j) |\psi_j\rangle \langle \psi_j|. \quad (157)$$

In general, $\tilde{\rho}_{\mathbb{Q}_G^\oplus}$ is a linear operator representation of the progressively fused GQBPA rather than a density operator, since the complex probability amplitudes do not necessarily yield a Hermitian, positive-semidefinite operator with unit trace.

Definition 57 (Density operator of progressively fused generalized quantum evidence in GQET). Let

$$\mathbb{M}_G^\oplus(\psi_j) = |\mathbb{Q}_G^\oplus(\psi_j)|^2, \quad \psi_j \in 2^\Phi, \quad (158)$$

be the GQBPD induced by the progressively fused GQBPA. For each fixed admissible allocation λ , the corresponding density operator is defined as

$$\hat{\rho}_{\mathbb{Q}_G^\oplus}(\lambda) = \sum_{\psi_j \in 2^\Phi} \mathbb{M}_G^\oplus(\psi_j) |\psi_j\rangle \langle \psi_j| = \sum_{\psi_j \in 2^\Phi} \mathbb{M}_G^\oplus(\psi_j) \rho_j. \quad (159)$$

Since GPQEER satisfies

$$\sum_{\psi_j \in 2^\Phi} |\mathbb{Q}_G^\oplus(\psi_j)|^2 = 1, \quad (160)$$

the induced GQBPD satisfies

$$M_G^\oplus(\psi_j) \geq 0, \quad \sum_{\psi_j \in 2^\Phi} M_G^\oplus(\psi_j) = 1. \quad (161)$$

Therefore,

$$\hat{\rho}_{\mathbb{Q}_G^\oplus}(\lambda) \succeq 0, \quad \text{Tr} \left(\hat{\rho}_{\mathbb{Q}_G^\oplus}(\lambda) \right) = 1. \quad (162)$$

Unlike the closed-world case in QET, GPQECR permits the empty proposition to retain a nonzero generalized quantum basic probability amplitude in an open-world setting. Specifically, for $\gamma_G > 0$,

$$\mathbb{Q}_G^\oplus(\emptyset) \neq 0 \quad (163)$$

if and only if

$$\mathbb{Q}_{\mathbf{M}_h}(\emptyset) \neq 0, \quad 1 \leq h \leq k. \quad (164)$$

Accordingly,

$$M_G^\oplus(\emptyset) = |\mathbb{Q}_G^\oplus(\emptyset)|^2 > 0, \quad (165)$$

which represents unknown information retained after progressive fusion. For the boundary case $\gamma_G = 0$, the fused GQBPA is assigned entirely to the empty proposition by definition.

When all input GQBPA's satisfy

$$\mathbb{Q}_{\mathbf{M}_h}(\emptyset) = 0, \quad 1 \leq h \leq k, \quad (166)$$

and $\gamma_G > 0$, the unknown-information component vanishes and the open-world formulation reduces to the closed-world formulation. Consequently,

$$\mathbb{Q}_G^\oplus(\psi_j) = \mathbb{Q}^\oplus(\psi_j), \quad \psi_j \subseteq \Phi. \quad (167)$$

4. Quantum evidence theory for a closed world

The previous section introduces GQET for an open-world setting. QET is obtained as the closed-world specialization of GQET by imposing $\mathbb{Q}_{\mathbb{M}}(\emptyset) = 0$. The main concepts and formulations of QET are presented in this section.

4.1. QET: Quantum evidence theory

4.1.1. Basic concepts of the quantum basic probability amplitude function

Definition 58 (Quantum basic probability amplitude function). A quantum basic probability amplitude (QBPA) function $\mathbb{Q}_{\mathbb{M}}$ in QFOD Φ , also referred to as a quantum mass function (QMF), is defined as a mapping:

$$\mathbb{Q}_{\mathbb{M}} : 2^{\Phi} \rightarrow \mathbb{C}, \quad (168)$$

satisfying

$$\begin{aligned} \mathbb{Q}_{\mathbb{M}}(\emptyset) &= 0, \\ \mathbb{Q}_{\mathbb{M}}(\psi_j) &= \varphi(\psi_j)e^{i\theta(\psi_j)}, \quad \psi_j \subseteq \Phi, \\ \sum_{\psi_j \subseteq \Phi} |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 &= 1, \end{aligned} \quad (169)$$

in which $i = \sqrt{-1}$; $\varphi(\psi_j) \in [0, 1]$ represents the modulus of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$; $\theta(\psi_j)$ denotes a phase term of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$; $\mathbb{Q}_{\mathbb{M}}(\psi_j)$ denote a quantum basic probability amplitude for ψ_j ; and $|\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2$ denotes the modulus squared of $\mathbb{Q}_{\mathbb{M}}(\psi_j)$. When $\varphi(\psi_j) = 0$, the phase $\theta(\psi_j)$ is immaterial and may be chosen arbitrarily.

The $\mathbb{Q}_{\mathbb{M}}(\psi_j)$, which is the generalization of basic probability assignment in DSET, is called a quantum basic probability amplitude, and can be represented in Cartesian form as:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = x_j + y_j i, \quad x_j^2 + y_j^2 \in [0, 1],$$

or alternatively, using polar form:

$$\mathbb{Q}_{\mathbb{M}}(\psi_j) = \varphi(\psi_j) (\cos \theta(\psi_j) + i \sin \theta(\psi_j)).$$

Its modulus is expressed as:

$$|\mathbb{Q}_{\mathbb{M}}(\psi_j)| = \varphi(\psi_j) = \sqrt{x_j^2 + y_j^2}.$$

Definition 59 (Quantum focal element in QET). Let $\mathbb{Q}_{\mathbb{M}}$ be a QBPA function. For any $\psi_j \subseteq \Phi$, if $|\mathbb{Q}_{\mathbb{M}}(\psi_j)| = \varphi(\psi_j) > 0$, then ψ_j is called a quantum focal element in QET.

Definition 60 (Bayesian QBPA function). A QBPA function $\mathbb{Q}_{\mathbb{M}}$ is called Bayesian if all of its quantum focal elements are singletons; equivalently,

$$|\psi_j| > 1 \Rightarrow |\mathbb{Q}_{\mathbb{M}}(\psi_j)|^2 = 0, \quad \psi_j \subseteq \Phi. \quad (170)$$

Definition 61 (Vacuous QBPA function). In QET, when $|\mathbb{Q}_{\mathbb{M}}(\Phi)|^2 = 1$, $\mathbb{Q}_{\mathbb{M}}$ is called a vacuous QBPA function.

Since $\mathbb{Q}_{\mathbb{M}}(\emptyset) = 0$, all quantum basic probability is assigned to nonempty propositions, corresponding to the closed-world setting of QET.

4.1.2. Basic concepts of the quantum basic probability distribution

Definition 62 (Quantum basic probability distribution). The quantum basic probability distribution (QBPD) of $\mathbb{Q}_{\mathbb{M}}$, is defined as:

$$\mathbb{M} : 2^{\Phi} \rightarrow [0, 1], \quad (171)$$

and satisfies:

$$\begin{aligned} M(\emptyset) &= 0, \\ M(\psi_j) &= |Q_M(\psi_j)|^2, \quad \psi_j \subseteq \Phi, \\ \sum_{\psi_j \subseteq \Phi} M(\psi_j) &= 1, \end{aligned}$$

where

$$|Q_M(\psi_j)|^2 = Q_M(\psi_j)Q_M^*(\psi_j) = \varphi^2(\psi_j) = x_j^2 + y_j^2, \quad (172)$$

and $*$ denotes complex conjugation.

Definition 63 (Bayesian QBPD). A QBPD M is called Bayesian if all of its focal propositions are singletons; equivalently,

$$|\psi_j| > 1 \Rightarrow M(\psi_j) = 0, \quad \psi_j \subseteq \Phi. \quad (173)$$

For an elementary event $\phi_g \in \Phi$, $M(\phi_g)$ is called a Bayesian QBP. In this case, M is an ordinary probability distribution over the elementary events in Φ .

Definition 64 (Vacuous QBPD). In QET, when $M(\Phi) = 1$, M is called a vacuous QBPD.

Definition 65 (Quantum basic probability). In QET, $M(\psi_j)$ ($\psi_j \subseteq \Phi$) is called quantum basic probability (QBP), which represents the degree of belief or support for ψ_j .

The QET inherits the merits of DSET and has the following attractive characteristics:

- The QBPA Q_M takes values in the complex domain, with nonnegative real-valued amplitudes included as a special case, whereas a classical BPA m in DSET is restricted to nonnegative real values.

- In contrast to DSET, $\forall \psi_j \subseteq \Phi$, the value of $|\mathbb{Q}_M(\psi_j)|^2$ or $\varphi^2(\psi_j)$ represents the degree of belief or support for ψ_j .
- When $M = m$, the QBPD M of QET is in accordance with the classical BPA m in DSET.

By comparing QET with GQET, the following relationships can be obtained:

- QET imposes the closed-world condition

$$\mathbb{Q}_M(\emptyset) = 0, \quad (174)$$

whereas GQET permits

$$\mathbb{Q}_M(\emptyset) \neq 0 \quad (175)$$

to represent unknown information outside the current QFOD. In this open-world interpretation, the empty proposition acts as a representation of information not covered by the current QFOD rather than as the literal empty set of the closed-world formulation.

- When

$$\mathbb{Q}_M(\emptyset) = 0, \quad (176)$$

the open-world uncertainty component vanishes. The GQBPA and GQBPD formulations then reduce to the corresponding QBPA and QBPD formulations of QET.

4.1.3. Quantum belief and plausibility functions

Taking into account application scenarios involving interference, the unnormalized quantum interference belief and plausibility measures in QET are first defined as follows.

Definition 66 (Unnormalized quantum interference belief measure in QET). Let $\mathbb{Q}_M$ be a QBPA function on QFOD Φ . The unnormalized quantum interference belief measure, denoted by $\widehat{\text{QIBel}}$, is defined by

$$\widehat{\text{QIBel}}(\psi_j) = \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi : \emptyset \neq \psi_p \subseteq \psi_j\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_M(\psi_p) \right|^2, \quad \psi_j \subseteq \Phi. \quad (177)$$

For $\psi_j \subseteq \Phi$, let

$$\Lambda_{\widehat{\text{QIBel}}} \in \arg \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi : \emptyset \neq \psi_p \subseteq \psi_j\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_M(\psi_p) \right|^2, \quad (178)$$

where $\Lambda_{\widehat{\text{QIBel}}}$ denotes a subfamily of propositions achieving the maximal squared modulus in Eq. (177). If the maximizer is not unique, one maximizing subfamily is selected according to a fixed deterministic tie-breaking rule and is used consistently in the subsequent interference decomposition.

Expanding the squared modulus in Eq. (177) and using $|\mathbb{Q}_M(\psi)|^2 = M(\psi)$ gives

$$\begin{aligned} \widehat{\text{QIBel}}(\psi_j) &= \sum_{\psi_s \in \Lambda_{\widehat{\text{QIBel}}}} M(\psi_s) \\ &\quad + 2 \sum_{\psi_s \in \Lambda_{\widehat{\text{QIBel}}}} \sum_{\substack{\psi_t \in \Lambda_{\widehat{\text{QIBel}}} \\ \psi_s \prec \psi_t}} \sqrt{M(\psi_s)} \sqrt{M(\psi_t)} \cos(\theta_s - \theta_t), \end{aligned} \quad (179)$$

where $\prec$ denotes an arbitrary fixed ordering of the relevant propositions and $-1 \leq \cos(\theta_s - \theta_t) \leq 1$.

By the Cauchy–Schwarz inequality and the normalization condition of the QBPA,

$$0 \leq \widehat{\text{QIBel}}(\psi_j) \leq 2^{|\psi_j|} - 1, \quad \psi_j \subseteq \Phi. \quad (180)$$

Definition 67 (Interference effect in the unnormalized quantum interference belief measure). Let $\Lambda_{\widehat{\text{QIBel}}}$ be a subfamily achieving the maximal squared modulus in $\widehat{\text{QIBel}}(\psi_j)$. The interference effect in $\widehat{\text{QIBel}}$ is defined as

$$\begin{aligned} \text{Int}_{\widehat{\text{QIBel}}}(\psi_j) &= 2 \sum_{\psi_s \in \Lambda_{\widehat{\text{QIBel}}}} \sum_{\substack{\psi_t \in \Lambda_{\widehat{\text{QIBel}}} \\ \psi_s \prec \psi_t}} |\mathbb{Q}_{\mathbf{M}}(\psi_s)| |\mathbb{Q}_{\mathbf{M}}(\psi_t)| \cos(\theta_s - \theta_t) \\ &= 2 \sum_{\psi_s \in \Lambda_{\widehat{\text{QIBel}}}} \sum_{\substack{\psi_t \in \Lambda_{\widehat{\text{QIBel}}} \\ \psi_s \prec \psi_t}} \sqrt{\mathbb{M}(\psi_s)} \sqrt{\mathbb{M}(\psi_t)} \cos(\theta_s - \theta_t). \end{aligned} \quad (181)$$

Definition 68 (Unnormalized quantum interference Plausibility measure in QET). Let $\mathbb{Q}_{\mathbf{M}}$ be a QBPA function on QFOD Φ . The unnormalized quantum interference Plausibility measure, denoted by $\widehat{\text{QIPl}}$, is defined by

$$\widehat{\text{QIPl}}(\psi_j) = \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi : \psi_p \cap \psi_j \neq \emptyset\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_{\mathbf{M}}(\psi_p) \right|^2, \quad \psi_j \subseteq \Phi. \quad (182)$$

For $\psi_j \subseteq \Phi$, let

$$\Lambda_{\widehat{\text{QIPl}}} \in \arg \max_{\Lambda \subseteq \{\psi_p \subseteq \Phi : \psi_p \cap \psi_j \neq \emptyset\}} \left| \sum_{\psi_p \in \Lambda} \mathbb{Q}_{\mathbf{M}}(\psi_p) \right|^2, \quad (183)$$

where $\Lambda_{\widehat{\text{QIPl}}}$ denotes a subfamily of propositions achieving the maximal squared modulus in Eq. (182). If the maximizer is not unique, one maximizing subfamily is selected according to a fixed deterministic tie-breaking rule and is used consistently in the subsequent interference decomposition.

Expanding the squared modulus in Eq. (182) gives

$$\begin{aligned}\widehat{\text{QIP1}}(\psi_j) &= \sum_{\psi_u \in \Lambda_{\widehat{\text{QIP1}}}} \text{M}(\psi_u) \\ &+ 2 \sum_{\psi_u \in \Lambda_{\widehat{\text{QIP1}}}} \sum_{\substack{\psi_v \in \Lambda_{\widehat{\text{QIP1}}} \\ \psi_u \prec \psi_v}} \sqrt{\text{M}(\psi_u)} \sqrt{\text{M}(\psi_v)} \cos(\theta_u - \theta_v),\end{aligned}\quad (184)$$

where $\prec$ denotes an arbitrary fixed ordering of the relevant propositions and $-1 \leq \cos(\theta_u - \theta_v) \leq 1$.

By the Cauchy–Schwarz inequality and the normalization condition of the QBPA,

$$0 \leq \widehat{\text{QIP1}}(\psi_j) \leq 2^{|\Phi|} - 2^{|\Phi| - |\psi_j|}, \quad \psi_j \subseteq \Phi. \quad (185)$$

Definition 69 (Interference effect in the unnormalized quantum interference Plausibility measure). Let $\Lambda_{\widehat{\text{QIP1}}}$ be a subfamily achieving the maximal squared modulus in $\widehat{\text{QIP1}}(\psi_j)$. The interference effect in $\widehat{\text{QIP1}}$ is defined as

$$\begin{aligned}\text{Int}_{\widehat{\text{QIP1}}}(\psi_j) &= 2 \sum_{\psi_u \in \Lambda_{\widehat{\text{QIP1}}}} \sum_{\substack{\psi_v \in \Lambda_{\widehat{\text{QIP1}}} \\ \psi_u \prec \psi_v}} |\text{Q}_{\text{M}}(\psi_u)| |\text{Q}_{\text{M}}(\psi_v)| \cos(\theta_u - \theta_v) \\ &= 2 \sum_{\psi_u \in \Lambda_{\widehat{\text{QIP1}}}} \sum_{\substack{\psi_v \in \Lambda_{\widehat{\text{QIP1}}} \\ \psi_u \prec \psi_v}} \sqrt{\text{M}(\psi_u)} \sqrt{\text{M}(\psi_v)} \cos(\theta_u - \theta_v).\end{aligned}\quad (186)$$

The unnormalized quantum interference belief and plausibility measures, $\widehat{\text{QIBel}}$ and $\widehat{\text{QIP1}}$, explicitly preserve constructive and destructive interference among quantum propositions and may exceed the conventional probability range $[0, 1]$. To obtain range-normalized interference-aware belief and plausibility support functions, a common full-frame normalization factor is introduced.

For the whole QFOD Φ , the admissible proposition families involved in $\widehat{\text{QIBel}}(\Phi)$ and $\widehat{\text{QIPl}}(\Phi)$ coincide. Therefore,

$$\widehat{\text{QIBel}}(\Phi) = \widehat{\text{QIPl}}(\Phi). \quad (187)$$

Accordingly, the common normalization factor is defined by the total unnormalized support of the whole QFOD:

$$\Gamma_{\text{QI}} = \widehat{\text{QIPl}}(\Phi) = \widehat{\text{QIBel}}(\Phi). \quad (188)$$

Since a valid QBPA contains at least one nonzero amplitude and the maximization admits singleton subfamilies, $\Gamma_{\text{QI}} > 0$.

Definition 70 (Quantum interference belief function in QET). Let $\widehat{\text{QIBel}}$ be the corresponding unnormalized quantum interference belief measure. The quantum interference belief function QIBel , mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{QIBel}(\psi_j) = \frac{\widehat{\text{QIBel}}(\psi_j)}{\Gamma_{\text{QI}}}, \quad \psi_j \subseteq \Phi. \quad (189)$$

Definition 71 (Quantum interference Plausibility function in QET). Let $\widehat{\text{QIPl}}$ be the corresponding unnormalized quantum interference Plausibility measure. The quantum interference Plausibility function QIPl , mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{QIPl}(\psi_j) = \frac{\widehat{\text{QIPl}}(\psi_j)}{\Gamma_{\text{QI}}}, \quad \psi_j \subseteq \Phi. \quad (190)$$

For any $\psi_j \subseteq \Phi$, the admissible proposition family involved in $\widehat{\text{QIBel}}(\psi_j)$ is a subfamily of that involved in $\widehat{\text{QIPl}}(\psi_j)$. Hence,

$$0 \leq \widehat{\text{QIBel}}(\psi_j) \leq \widehat{\text{QIPl}}(\psi_j) \leq \widehat{\text{QIPl}}(\Phi). \quad (191)$$

According to Eq. (188),

$$0 \leq \text{QIBel}(\psi_j) \leq \text{QIPl}(\psi_j) \leq 1, \quad \psi_j \subseteq \Phi. \quad (192)$$

Thus, QIBel and QIPl provide range-normalized lower and upper interference-aware evidential support under interference. In particular,

$$\text{QIBel}(\Phi) = \text{QIPl}(\Phi) = 1. \quad (193)$$

Remark 5 . Under nonzero interference, QIBel and QIPl are interference-aware evidential support functionals and are not required to satisfy the classical Dempster–Shafer complement duality. Their classical belief–plausibility semantics are recovered in the incoherent limit, where they reduce to QBel and QPl, respectively.

For the incoherent case, the corresponding quantum belief and plausibility functions in QET are defined as follows.

Definition 72 (Quantum belief function in QET). Let $\mathbb{Q}_M$ be a QBPA function on QFOD Φ . The quantum belief function QBel, mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{QBel}(\psi_j) = \sum_{\psi_p \subseteq \psi_j} |\mathbb{Q}_M(\psi_p)|^2, \quad \psi_j \subseteq \Phi. \quad (194)$$

According to Definition 62, Eq. (194) can equivalently be expressed as

$$\text{QBel}(\psi_j) = \sum_{\psi_p \subseteq \psi_j} M(\psi_p), \quad \psi_j \subseteq \Phi. \quad (195)$$

Definition 73 (Quantum Plausibility function in QET). Let $\mathbb{Q}_M$ be a QBPA function on QFOD Φ . The quantum Plausibility function QPl, mapping from 2^Φ to $[0, 1]$, is defined by

$$\text{QPl}(\psi_j) = \sum_{\psi_p \cap \psi_j \neq \emptyset} |\mathbb{Q}_M(\psi_p)|^2, \quad \psi_j \subseteq \Phi. \quad (196)$$

According to Definition 62, Eq. (196) can equivalently be expressed as

$$\text{QPl}(\psi_j) = \sum_{\psi_p \cap \psi_j \neq \emptyset} M(\psi_p), \quad \psi_j \subseteq \Phi. \quad (197)$$

Property 1 (Conjugacy of QBel and QPl). For any quantum proposition $\psi_j \subseteq \Phi$,

$$\text{QPl}(\psi_j) = 1 - \text{QBel}(\Phi \setminus \psi_j). \quad (198)$$

Thus, QBel and QPl form a conjugate pair of lower and upper probabilities in the incoherent setting.

Proof. Since $M(\emptyset) = 0$ and $\sum_{\psi_p \subseteq \Phi} M(\psi_p) = 1$, the propositions that do not intersect ψ_j are exactly those contained in $\Phi \setminus \psi_j$. Hence

$$\begin{aligned} \text{QPl}(\psi_j) &= \sum_{\psi_p \cap \psi_j \neq \emptyset} M(\psi_p) \\ &= 1 - \sum_{\psi_p \subseteq \Phi \setminus \psi_j} M(\psi_p) \\ &= 1 - \text{QBel}(\Phi \setminus \psi_j). \end{aligned} \quad (199)$$

□

For any quantum proposition $\psi_j \subseteq \Phi$, the conjugate pair

$$\mathcal{I}_Q(\psi_j) = [\text{QBel}(\psi_j), \text{QPl}(\psi_j)] \quad (200)$$

defines its belief–plausibility interval. The width of this interval is

$$\begin{aligned} \Delta_Q(\psi_j) &= \text{QPl}(\psi_j) - \text{QBel}(\psi_j) \\ &= \sum_{\substack{\psi_p \cap \psi_j \neq \emptyset \\ \psi_p \not\subseteq \psi_j}} M(\psi_p) \geq 0. \end{aligned} \quad (201)$$

Thus, $\Delta_Q(\psi_j)$ quantifies the imprecision of the evidential support associated with ψ_j : a narrower interval indicates more specifically committed support, whereas a wider interval indicates a larger amount of unresolved support that overlaps ψ_j without being fully committed to it.

For QET, the incoherent reduction is likewise defined by replacing each coherent squared modulus by the corresponding sum of squared moduli before maximization. The corresponding maximization problems then reduce to sums of nonnegative quantum basic probabilities. Consequently,

$$\widehat{\text{QIBel}}(\psi_j) = \text{QBel}(\psi_j), \quad \widehat{\text{QIPl}}(\psi_j) = \text{QPl}(\psi_j). \quad (202)$$

In particular, for the whole QFOD Φ ,

$$\widehat{\text{QIBel}}(\Phi) = \widehat{\text{QIPl}}(\Phi) = 1. \quad (203)$$

According to Eq. (188), it follows that

$$\Gamma_{\text{QI}} = 1. \quad (204)$$

Therefore, under this incoherent reduction,

$$\text{QIBel}(\psi_j) = \widehat{\text{QIBel}}(\psi_j) = \text{QBel}(\psi_j), \quad (205)$$

and

$$\text{QIPl}(\psi_j) = \widehat{\text{QIPl}}(\psi_j) = \text{QPl}(\psi_j). \quad (206)$$

Furthermore, under this incoherent reduction and when $M = m$, the quantum belief and plausibility functions further reduce to their classical counterparts in DSET:

$$\text{QIBel}(\psi_j) = \text{QBel}(\psi_j) = \text{Bel}(\psi_j), \quad (207)$$

and

$$\text{QIPl}(\psi_j) = \text{QPl}(\psi_j) = \text{Pl}(\psi_j). \quad (208)$$

The quantum belief and plausibility functions in QET have the following properties:

- $\text{QBel}(\psi_j)$ and $\text{QPl}(\psi_j)$ characterize the lower and upper probabilities of proposition ψ_j in the incoherent case, respectively.
- Under the incoherent reduction, QIBel and QIPl reduce to QBel and QPl , respectively.
- Under the incoherent reduction and when $M = m$, QBel and QPl reduce to the classical Bel and Pl in DSET, respectively.

By comparing the belief and plausibility formulations in QET with their generalized counterparts in GQET, the following relationships can be obtained:

- In QET,

$$\text{QBel}(\emptyset) = \text{QPl}(\emptyset) = 0, \quad (209)$$

whereas GQET permits

$$\text{GQBel}(\emptyset) = \text{GQPl}(\emptyset) = M(\emptyset) \geq 0. \quad (210)$$

When $M(\emptyset) > 0$, this quantity represents support assigned to unknown information outside the current QFOD.

- When $Q_M(\emptyset) = 0$, the unnormalized generalized quantum interference measures reduce to their closed-world counterparts:

$$\text{G}\hat{\text{QIBel}}(\psi_j) = \hat{\text{QIBel}}(\psi_j), \quad \text{G}\hat{\text{QIPl}}(\psi_j) = \hat{\text{QIPl}}(\psi_j). \quad (211)$$

- Under the same closed-world condition, the normalized generalized quantum interference belief and plausibility functions reduce to their QET counterparts:

$$\text{GQIBel}(\psi_j) = \text{QIBel}(\psi_j), \quad \text{GQIPl}(\psi_j) = \text{QIPl}(\psi_j). \quad (212)$$

- Under the incoherent reduction and when $\text{Q}_M(\emptyset) = 0$, the generalized quantum belief and plausibility functions reduce to those in QET:

$$\text{GQBel}(\psi_j) = \text{QBel}(\psi_j), \quad \text{GQPl}(\psi_j) = \text{QPl}(\psi_j). \quad (213)$$

4.1.4. Quantum evidential combination rules

In multisource quantum evidential fusion in a closed-world environment, complex probability amplitudes are first combined at the amplitude level before being converted into real-valued evidential support. Following Feynman's probability-amplitude rules, the amplitudes associated with different evidence sources are multiplied within each fusion path, whereas the amplitudes of alternative fusion paths leading to the same proposition are coherently superposed. According to the Born rule, the corresponding fused evidential support is then obtained from the squared modulus of the resulting total amplitude.

Accordingly, PQEQR is first defined at the probability-amplitude level to obtain a fused QBPA. QEQR is then defined at the corresponding QBPD level.

Using the fusion-path notation introduced in the generalized formulation, PQEQR is defined at the probability-amplitude level as follows.

Definition 74 (Progressive quantum evidential combination rule).

Let $\{\mathbb{Q}_{\mathbb{M}_1}, \dots, \mathbb{Q}_{\mathbb{M}_k}\}$ be a set of independent QBPA's on QFOD Φ . For each $\psi_j \in 2^\Phi$, the retained unnormalized fused quantum probability amplitude is defined as

$$\tilde{\mathbb{Q}}^\oplus(\psi_j) = \begin{cases} \sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k \mathbb{Q}_{\mathbb{M}_h}(\psi_p^h), & \emptyset \neq \psi_j \subseteq \Phi, \\ 0, & \psi_j = \emptyset. \end{cases} \quad (214)$$

The normalization coefficient is defined as

$$\gamma = \sum_{\emptyset \neq \psi_v \subseteq \Phi} \left| \tilde{\mathbb{Q}}^\oplus(\psi_v) \right|^2. \quad (215)$$

Accordingly, $\sqrt{\gamma}$ serves as the amplitude-normalization factor.

Provided that $\gamma > 0$, the progressive quantum evidential combination rule (PQECR) is defined by

$$\mathbb{Q}^\oplus(\psi_j) = \begin{cases} \frac{\tilde{\mathbb{Q}}^\oplus(\psi_j)}{\sqrt{\gamma}}, & \emptyset \neq \psi_j \subseteq \Phi, \\ 0, & \psi_j = \emptyset. \end{cases} \quad (216)$$

The case $\gamma = 0$ is a closed-world boundary case in which all retained unnormalized fused probability amplitudes for nonempty propositions vanish. Since a valid QBPA in QET must assign zero amplitude to the empty proposition while having unit squared norm over nonempty propositions, PQECR is undefined when $\gamma = 0$.

PQECR satisfies the following properties whenever the required normalization coefficients are positive:

- **Commutativity:** The fusion result is independent of the ordering of the quantum evidence sources:

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_2} = \mathbb{Q}_{\mathbb{M}_2} \oplus \mathbb{Q}_{\mathbb{M}_1}. \quad (217)$$

- **Associativity:** When all intermediate normalization coefficients are positive, PQEQR is associative:

$$(\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_2}) \oplus \mathbb{Q}_{\mathbb{M}_3} = \mathbb{Q}_{\mathbb{M}_1} \oplus (\mathbb{Q}_{\mathbb{M}_2} \oplus \mathbb{Q}_{\mathbb{M}_3}). \quad (218)$$

Associativity follows from the unnormalized amplitude-level fusion. The normalization factor $\sqrt{\gamma}$ is positively homogeneous, so intermediate positive rescaling does not change the final normalized result.

- **Identity:** Let $\mathbb{Q}_{\mathbb{M}_0}$ be the vacuous QBPA satisfying

$$\mathbb{Q}_{\mathbb{M}_0}(\Phi) = 1, \quad \mathbb{Q}_{\mathbb{M}_0}(\psi) = 0, \quad \psi \neq \Phi. \quad (219)$$

Then

$$\mathbb{Q}_{\mathbb{M}_1} \oplus \mathbb{Q}_{\mathbb{M}_0} = \mathbb{Q}_{\mathbb{M}_1}. \quad (220)$$

- **Closure:** The output of PQEQR remains a valid QBPA. Specifically,

$$\mathbb{Q}^\oplus(\emptyset) = 0, \quad (221)$$

and

$$\sum_{\emptyset \neq \psi_j \subseteq \Phi} |\mathbb{Q}^\oplus(\psi_j)|^2 = 1. \quad (222)$$

Definition 75 (Quantum evidential combination rule). Let $\mathbb{Q}^\oplus$ be the fused QBPA obtained by PQEQR. For any $\psi_j \in 2^\Phi$, the quantum evidential combination rule (QEQR) is defined by

$$\text{QEQR}(\psi_j) = \mathbb{M}^\oplus(\psi_j) = |\mathbb{Q}^\oplus(\psi_j)|^2, \quad \psi_j \in 2^\Phi, \quad (223)$$

where $M^\oplus$ denotes the QBPD induced by the fused QBPA $Q^\oplus$.

For $\gamma > 0$, according to Eq. (216), QEER can equivalently be expressed as

$$\text{QEER}(\psi_j) = \frac{|\tilde{Q}^\oplus(\psi_j)|^2}{\gamma}, \quad \emptyset \neq \psi_j \subseteq \Phi, \quad (224)$$

with

$$\text{QEER}(\emptyset) = 0. \quad (225)$$

QEER satisfies the following properties whenever $\gamma > 0$:

- **Commutativity:** The fused QBPD is invariant under any permutation of the input quantum evidence sources.
- **Non-negativity and boundedness:** For any $\psi_j \in 2^\Phi$,

$$0 \leq \text{QEER}(\psi_j) \leq 1. \quad (226)$$

- **Normalization:** The fused QBPD satisfies

$$\sum_{\psi_j \in 2^\Phi} \text{QEER}(\psi_j) = 1. \quad (227)$$

Therefore, the output of QEER remains a valid QBPD.

Using the accumulated fusion-path phase θ_p defined in Eq. (83), the interference effect in QEER is defined as follows.

Definition 76 (Interference effect in QEER). Let $\{Q_{M_1}, \dots, Q_{M_k}\}$ be a set of independent QBPA's on QFOD Φ . For a nonempty output proposition

$\emptyset \neq \psi_x \subseteq \Phi$, the interference effect in QEER is defined as

$$\begin{aligned}
\text{Int}_{\text{QEER}}(\psi_x) &= 2 \sum_{\substack{\cap \psi_p = \psi_x \\ \psi_p \prec \psi_q}} \sum_{\substack{\cap \psi_q = \psi_x \\ \psi_p \prec \psi_q}} \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)| \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_q^h)| \cos(\theta_p - \theta_q) \\
&= 2 \sum_{\substack{\cap \psi_p = \psi_x \\ \psi_p \prec \psi_q}} \sum_{\substack{\cap \psi_q = \psi_x \\ \psi_p \prec \psi_q}} \prod_{h=1}^k \sqrt{M_h(\psi_p^h)} \prod_{h=1}^k \sqrt{M_h(\psi_q^h)} \cos(\theta_p - \theta_q),
\end{aligned} \tag{228}$$

where $\prec$ denotes an arbitrary fixed ordering such that each unordered pair of distinct fusion paths is counted exactly once.

Since

$$-1 \leq \cos(\theta_p - \theta_q) \leq 1, \tag{229}$$

the interference effect may be constructive, destructive, or zero, depending on the relative phases of the corresponding fusion paths.

For a nonempty output proposition ψ_x , expanding the squared modulus of the coherently superposed fusion-path amplitudes gives

$$\begin{aligned}
\left| \tilde{\mathbb{Q}}^\oplus(\psi_x) \right|^2 &= \sum_{\cap \psi_p = \psi_x} \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)|^2 + \text{Int}_{\text{QEER}}(\psi_x) \\
&= \sum_{\cap \psi_p = \psi_x} \prod_{h=1}^k M_h(\psi_p^h) + \text{Int}_{\text{QEER}}(\psi_x).
\end{aligned} \tag{230}$$

Eq. (230) shows that the fused evidential support consists of the accumulated non-interference contributions of individual fusion paths and the interference contribution among distinct fusion paths leading to the same proposition.

Accordingly, the normalization coefficient can be expressed as

$$\begin{aligned}\gamma &= \sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k |\mathbb{Q}_{\mathbf{M}_h}(\psi_p^h)|^2 + \text{Int}_{\text{QECR}}(\psi_v) \right] \\ &= \sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k \mathbb{M}_h(\psi_p^h) + \text{Int}_{\text{QECR}}(\psi_v) \right].\end{aligned}\quad (231)$$

For $\gamma > 0$, substituting Eq. (214) into Eq. (224) gives, for $\emptyset \neq \psi_j \subseteq \Phi$,

$$\text{QECR}(\psi_j) = \frac{1}{\gamma} \left| \sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k \mathbb{Q}_{\mathbf{M}_h}(\psi_p^h) \right|^2. \quad (232)$$

Using Eqs. (230) and (231), Eq. (232) can be further expressed as

$$\text{QECR}(\psi_j) = \frac{\sum_{\cap \psi_p = \psi_j} \prod_{h=1}^k \mathbb{M}_h(\psi_p^h) + \text{Int}_{\text{QECR}}(\psi_j)}{\sum_{\emptyset \neq \psi_v \subseteq \Phi} \left[\sum_{\cap \psi_p = \psi_v} \prod_{h=1}^k \mathbb{M}_h(\psi_p^h) + \text{Int}_{\text{QECR}}(\psi_v) \right]}, \quad \emptyset \neq \psi_j \subseteq \Phi. \quad (233)$$

When the aggregate inter-path interference contribution is zero for every nonempty output proposition,

$$\text{Int}_{\text{QECR}}(\psi_x) = 0, \quad \emptyset \neq \psi_x \subseteq \Phi, \quad (234)$$

and the input QBPDs satisfy

$$\mathbb{M}_h = m_h, \quad 1 \leq h \leq k, \quad (235)$$

Eq. (231) reduces to

$$\gamma = 1 - K, \quad (236)$$

where the fusion-path union $\cup\psi_p$ is defined in Eq. (96),

$$K = \sum_{\substack{\cap\psi_p=\emptyset \\ \cup\psi_p\neq\emptyset}} \prod_{h=1}^k m_h(\psi_p^h) \quad (237)$$

is the classical conflict coefficient in Dempster–Shafer evidence theory.

For $K < 1$, equivalently $\gamma > 0$ under the stated classical-mass and zero-aggregate-interference assumptions, Eq. (233) reduces to

$$\text{QECR}(\psi_j) = \frac{\sum_{\cap\psi_p=\psi_j} \prod_{h=1}^k m_h(\psi_p^h)}{1 - K}, \quad \emptyset \neq \psi_j \subseteq \Phi, \quad (238)$$

with

$$\text{QECR}(\emptyset) = 0. \quad (239)$$

Therefore, when the input QBPDs coincide with classical BPAs and the aggregate inter-path interference contribution vanishes for every nonempty output proposition, QECR reduces to Dempster’s rule of combination for $K < 1$.

For the total-conflict case $K = 1$, Eq. (236) gives $\gamma = 0$. Hence, under the stated classical-mass and zero-aggregate-interference assumptions, PQECR and QECR are undefined, consistently with Dempster’s rule in the total-conflict case.

QECR has the following characteristics:

- **Closed-world uncertainty representation:** QECR assigns zero fused probability to the empty proposition,

$$\text{QECR}(\emptyset) = 0, \quad (240)$$

which is consistent with the closed-world assumption of QET.

- **Interference sensitivity:** For each nonempty output proposition, QEER explicitly captures inter-path interference effects through $\text{Int}_{\text{QEER}}(\psi_j)$, which may be constructive or destructive. Hence, the fused QBPD depends not only on the magnitudes of the input QBPA's but also on their relative phases.
- **Closed-world specialization of GQEER:** When all input GQBPA's satisfy

$$\mathbb{Q}_{\mathbf{M}_h}(\emptyset) = 0, \quad 1 \leq h \leq k, \quad (241)$$

and the corresponding normalization coefficient is positive, GQEER reduces to QEER.

- **Compatibility with Dempster's rule:** When the input QBPDs coincide with classical BPAs and the aggregate inter-path interference contribution vanishes for every nonempty output proposition, QEER reduces to Dempster's rule of combination for non-total-conflict cases.

Thus, PQEER preserves the complex-valued fused QBPA for subsequent progressive fusion, whereas QEER provides the corresponding real-valued fused QBPD.

Definition 77 (Quantum evidential conflict coefficient). Consider k independent QBPA's on QFOD Φ . The quantum evidential conflict coefficient (QECC), denoted by $\mathbb{K}_C$, is defined as

$$\mathbb{K}_C = \left| \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k \mathbb{Q}_{\mathbf{M}_h}(\psi_p^h) \right|^2. \quad (242)$$

Definition 78 (Interference effect in QECC). The interference effect among distinct conflicting fusion paths in QECC is defined as

$$\begin{aligned}
\text{Int}_{\text{QECC}} &= 2 \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \sum_{\substack{\cap \psi_q = \emptyset \\ \cup \psi_q \neq \emptyset}} \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)| \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_q^h)| \cos(\theta_p - \theta_q) \\
&= 2 \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \sum_{\substack{\cap \psi_q = \emptyset \\ \cup \psi_q \neq \emptyset}} \prod_{h=1}^k \sqrt{\mathbb{M}_h(\psi_p^h)} \prod_{h=1}^k \sqrt{\mathbb{M}_h(\psi_q^h)} \cos(\theta_p - \theta_q). \quad (243)
\end{aligned}$$

Expanding the squared modulus in Eq. (242) gives

$$\begin{aligned}
\mathbb{K}_C &= \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k |\mathbb{Q}_{\mathbb{M}_h}(\psi_p^h)|^2 + \text{Int}_{\text{QECC}} \\
&= \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k \mathbb{M}_h(\psi_p^h) + \text{Int}_{\text{QECC}}. \quad (244)
\end{aligned}$$

When

$$\text{Int}_{\text{QECC}} = 0 \quad (245)$$

and

$$\mathbb{M}_h = m_h, \quad 1 \leq h \leq k, \quad (246)$$

Eq. (244) reduces to

$$\mathbb{K}_C = \sum_{\substack{\cap \psi_p = \emptyset \\ \cup \psi_p \neq \emptyset}} \prod_{h=1}^k m_h(\psi_p^h) = K, \quad (247)$$

which is consistent with the classical conflict coefficient in Dempster–Shafer evidence theory.

QECC has the following characteristics:

- **Interference sensitivity:** QECC incorporates the coherent interaction among conflicting fusion paths. Constructive interference increases the resulting conflict measure, whereas destructive interference decreases it.
- **Closed-world specialization of GQECC:** When all input GQBPA's satisfy

$$Q_{M_h}(\emptyset) = 0, \quad 1 \leq h \leq k, \quad (248)$$

GQECC reduces to QECC.

- **Compatibility with the classical conflict coefficient:** When the input QBPDs coincide with classical BPAs and the interference effect vanishes, QECC reduces to the classical conflict coefficient K in Dempster–Shafer evidence theory.

4.1.5. Quantum Pignistic transformations

When interference is taken into account in decision making, Pignistic redistribution is first performed coherently among indistinguishable redistribution paths leading to the same elementary event. The corresponding singleton support is then obtained through the Born rule, and the supports of distinct mutually exclusive elementary outcomes are combined additively.

Definition 79 (Unnormalized quantum interference Pignistic support). Let Q_M be a QBPA on QFOD Φ . For each elementary event $\phi_g \in \Phi$, define the coherently redistributed amplitude

$$A_g = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} Q_M(\psi_p) \frac{1}{\sqrt{|\psi_p|}}. \quad (249)$$

The corresponding unnormalized singleton support is

$$\tilde{\text{BetQIP}}(\phi_g) = |A_g|^2. \quad (250)$$

For any proposition $\psi_j \subseteq \Phi$, define

$$\tilde{\text{BetQIP}}(\psi_j) = \sum_{\phi_g \in \psi_j} \tilde{\text{BetQIP}}(\phi_g), \quad (251)$$

with $\tilde{\text{BetQIP}}(\emptyset) = 0$.

The amplitude sum preserves interference only among indistinguishable redistribution paths leading to the same elementary event. Once the Born-rule conversion has been applied, mutually exclusive elementary outcomes are combined additively at the probability level.

Definition 80 (Interference effect in quantum interference Pignistic redistribution). For an elementary event $\phi_g \in \Phi$, the interference effect among redistribution paths leading to ϕ_g is defined as

$$\text{Int}_{\text{BetQIP}}(\phi_g) = 2 \sum_{\substack{\emptyset \neq \psi_s, \psi_t \subseteq \Phi \\ \phi_g \in \psi_s \cap \psi_t \\ \psi_s \prec \psi_t}} \frac{\sqrt{M(\psi_s)} \sqrt{M(\psi_t)}}{\sqrt{|\psi_s|} \sqrt{|\psi_t|}} \cos(\theta_s - \theta_t). \quad (252)$$

Accordingly,

$$\tilde{\text{BetQIP}}(\phi_g) = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} M(\psi_p) \frac{1}{|\psi_p|} + \text{Int}_{\text{BetQIP}}(\phi_g). \quad (253)$$

Since the first terms in Eq. (253) sum to one over all elementary events, the normalization coefficient satisfies

$$\gamma_{\text{BetQIP}} = 1 + \sum_{\phi_g \in \Phi} \text{Int}_{\text{BetQIP}}(\phi_g). \quad (254)$$

Definition 81 (Quantum interference Pignistic probability for elementary events). Assume $\gamma_{\text{BetQIP}} > 0$, where

$$\gamma_{\text{BetQIP}} = \sum_{\phi_g \in \Phi} \tilde{\text{BetQIP}}(\phi_g) = \tilde{\text{BetQIP}}(\Phi). \quad (255)$$

For each $\phi_g \in \Phi$, define

$$\text{BetQIP}(\phi_g) = \frac{\tilde{\text{BetQIP}}(\phi_g)}{\gamma_{\text{BetQIP}}}. \quad (256)$$

Then

$$\sum_{\phi_g \in \Phi} \text{BetQIP}(\phi_g) = 1. \quad (257)$$

The condition $\gamma_{\text{BetQIP}} > 0$ is necessary for the normalized Pignistic probability to be well defined. If $\gamma_{\text{BetQIP}} = 0$, all coherently redistributed amplitudes associated with the elementary decision outcomes vanish simultaneously. In this complete-cancellation boundary case, the unnormalized quantum interference Pignistic support remains well defined, but no normalized BetQIP probability distribution on Φ is defined.

Definition 82 (Quantum interference Pignistic probability for propositions). For any proposition $\psi_j \subseteq \Phi$, define

$$\text{BetQIP}(\psi_j) = \sum_{\phi_g \in \psi_j} \text{BetQIP}(\phi_g) = \frac{\tilde{\text{BetQIP}}(\psi_j)}{\gamma_{\text{BetQIP}}}. \quad (258)$$

Hence,

$$\text{BetQIP}(\emptyset) = 0, \quad \text{BetQIP}(\Phi) = 1. \quad (259)$$

If the aggregate redistribution-interference contribution vanishes,

$$\text{Int}_{\text{BetQIP}}(\phi_g) = 0, \quad \forall \phi_g \in \Phi, \quad (260)$$

then

$$\tilde{\text{BetQIP}}(\phi_g) = \text{BetQP}(\phi_g), \quad \gamma_{\text{BetQIP}} = 1, \quad (261)$$

and therefore

$$\text{BetQIP}(\psi_j) = \text{BetQP}(\psi_j), \quad \psi_j \subseteq \Phi. \quad (262)$$

The validity and finite additivity of BetQIP follow as the closed-world specialization of Theorem 4.

When the aggregate redistribution-interference contribution vanishes for each elementary event, the corresponding quantum Pignistic probability transformation redistributes the quantum basic probability over elementary events for probabilistic decision making.

Definition 83 (Quantum Pignistic probability transformation). Let $\mathbb{Q}_M$ be a QBPA function on QFOD Φ . For any quantum proposition $\psi_j \subseteq \Phi$, its quantum Pignistic probability function, denoted by BetQP, is defined as

$$\text{BetQP}(\psi_j) = \sum_{\emptyset \neq \psi_p \subseteq \Phi} |\mathbb{Q}_M(\psi_p)|^2 \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|}, \quad (263)$$

where $|\psi|$ denotes the cardinality of proposition ψ .

The transformation from $\mathbb{Q}_M$ to BetQP is called the quantum Pignistic probability transformation.

Since $|\mathbb{Q}_M(\psi_p)|^2 = M(\psi_p)$, Eq. (263) can be expressed as

$$\text{BetQP}(\psi_j) = \sum_{\emptyset \neq \psi_p \subseteq \Phi} M(\psi_p) \cdot \frac{|\psi_p \cap \psi_j|}{|\psi_p|}. \quad (264)$$

For an elementary event $\phi_g \in \Phi$, Eq. (263) reduces to

$$\text{BetQP}(\phi_g) = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} |\mathbb{Q}_M(\psi_p)|^2 \cdot \frac{1}{|\psi_p|}. \quad (265)$$

Equivalently,

$$\text{BetQP}(\phi_g) = \sum_{\substack{\emptyset \neq \psi_p \subseteq \Phi \\ \phi_g \in \psi_p}} M(\psi_p) \cdot \frac{1}{|\psi_p|}. \quad (266)$$

Accordingly, the quantum Pignistic probabilities over all elementary events satisfy

$$\sum_{\phi_g \in \Phi} \text{BetQP}(\phi_g) = 1. \quad (267)$$

4.2. A quantum representation of QET in Hilbert space

4.2.1. Quantum state representation of a QBPA function

Definition 84 (Basis event in QET). Given the QFOD $\Phi = \{\phi_1, \dots, \phi_n\}$, where each elementary event ϕ_w is associated with the orthonormal basis state $|\phi_w\rangle$, the set of basis events in QET is defined as

$$\mathbb{B}_c = \{e_1, \dots, e_n\}, \quad e_w = \phi_w, \quad 1 \leq w \leq n. \quad (268)$$

Thus, each $e_w \in \mathbb{B}_c$ is the basis event associated with the QFOD event ϕ_w .

Definition 85 (Standard basis vector representation of a basis event in QET). For each $e_w \in \mathbb{B}_c$ ($1 \leq w \leq n$), let $|e_w\rangle \in \mathbb{C}^n$ denote its corresponding standard basis vector, defined as

$$|e_w\rangle = [\eta_1, \dots, \eta_l, \dots, \eta_n]^T, \quad (269)$$

where

$$\eta_l = \begin{cases} 1, & l = w, \\ 0, & l \neq w, \end{cases} \quad 1 \leq l \leq n. \quad (270)$$

The standard basis vectors associated with the basis events are orthonormal, satisfying

$$\langle e_w | e_w \rangle = 1, \quad \langle e_w | e_s \rangle = 0 \quad (w \neq s), \quad 1 \leq w, s \leq n. \quad (271)$$

Definition 86 (State vector representation of a quantum proposition in QET). Let $\emptyset \neq \psi_j \subseteq \Phi$ be a quantum proposition. The state vector $|\psi_j\rangle$ corresponding to quantum proposition ψ_j is defined as

$$|\psi_j\rangle = \sum_{w: \phi_w \in \psi_j} \lambda_w^j |e_w\rangle, \quad (272)$$

where $\lambda_w^j = a_w^j + b_w^j i$ is a complex coefficient satisfying

$$\sum_{w: \phi_w \in \psi_j} |\lambda_w^j|^2 = 1. \quad (273)$$

Thus, $|\psi_j\rangle$ is a unit vector representing a pure quantum state. According to Eq. (272), a composite quantum proposition in QET is represented as a linear superposition of the standard basis vectors corresponding to its constituent basis events. For notational simplicity, the dependence of $|\psi_j\rangle$ on the coefficient allocation $\lambda^j = \{\lambda_w^j\}$ is left implicit throughout the paper. We use $\lambda = \{\lambda_w^j\}$ to collectively denote an admissible set of internal coefficient allocations for all nonempty quantum propositions.

Definition 87 (Density operator representation of a quantum proposition in QET). Let $\emptyset \neq \psi_j \subseteq \Phi$ be a quantum proposition with the corresponding state vector $|\psi_j\rangle$ in QET. The density operator ρ_j corresponding to ψ_j is defined as

$$\rho_j = |\psi_j\rangle\langle\psi_j|. \quad (274)$$

The dependence of ρ_j on the internal coefficient allocation λ^j is also left implicit for notational simplicity.

Definition 88 (Linear operator representation of a QBPA function). Let Q_M be a QBPA function with $Q_M(\psi_j)$ on QFOD Φ . The linear operator

corresponding to a QBPA function is defined as:

$$\tilde{\rho}_{\mathbf{Q}_M} = \sum_{\emptyset \neq \psi_j \subseteq \Phi} \mathbf{Q}_M(\psi_j) |\psi_j\rangle \langle \psi_j| = \sum_{\emptyset \neq \psi_j \subseteq \Phi} \mathbf{Q}_M(\psi_j) \rho_j, \quad (275)$$

where $\tilde{\rho}_{\mathbf{Q}_M}$ is a linear combination of projection operators and is generally not a density operator, as it is not necessarily Hermitian, positive semidefinite, or of unit trace.

Definition 89 (Density operator induced by a QBPA). The density operator induced by a QBPA through its corresponding QBPD is defined as

$$\begin{aligned} \hat{\rho}_{\mathbf{Q}_M}(\lambda) &= \sum_{\emptyset \neq \psi_j \subseteq \Phi} |\mathbf{Q}_M(\psi_j)|^2 \rho_j \\ &= \sum_{\emptyset \neq \psi_j \subseteq \Phi} |\mathbf{Q}_M(\psi_j)|^2 |\psi_j\rangle \langle \psi_j|. \end{aligned} \quad (276)$$

where $|\cdot|^2$ is the modulus squared function.

Remark 6 . The induced density operator is constructed from the squared moduli of the QBPA coefficients. It therefore represents the QBPD induced by the QBPA and does not retain the phases of the original probability amplitudes. The complex amplitude and phase information is retained in the linear-operator representation $\tilde{\rho}_{\mathbf{Q}_M}$.

From Eq. (276), it follows directly that

$$\hat{\rho}_{\mathbf{Q}_M}(\lambda) \succeq 0,$$

and

$$\begin{aligned} \text{Tr}(\hat{\rho}_{\mathbf{Q}_M}(\lambda)) &= \sum_{\emptyset \neq \psi_j \subseteq \Phi} |\mathbf{Q}_M(\psi_j)|^2 \text{Tr}(|\psi_j\rangle \langle \psi_j|) \\ &= \sum_{\emptyset \neq \psi_j \subseteq \Phi} |\mathbf{Q}_M(\psi_j)|^2 = 1. \end{aligned}$$

Therefore, for each fixed admissible allocation λ , $\hat{\rho}_{\mathbb{Q}_M}(\lambda)$ is a positive-semidefinite density operator with unit trace.

For a fixed admissible allocation λ , let $\hat{\mathbb{E}}_\nu$ be the eigenvalues of $\hat{\rho}_{\mathbb{Q}_M}(\lambda)$. It follows that:

$$\hat{\mathbb{E}}_\nu \geq 0, \quad \sum_{\nu} \hat{\mathbb{E}}_\nu = 1.$$

Since $|\mathbb{Q}_M|^2 = \varphi^2 = M$, the density operator of a QBPD can be expressed as follows.

Definition 90 (Density operator representation of a QBPD). For a fixed admissible internal coefficient allocation λ , the density operator corresponding to a QBPD is defined as:

$$\rho_M(\lambda) = \sum_{\emptyset \neq \psi_j \subseteq \Phi} M(\psi_j) |\psi_j\rangle \langle \psi_j| = \sum_{\emptyset \neq \psi_j \subseteq \Phi} M(\psi_j) \rho_j. \quad (277)$$

Here, the dependence of $|\psi_j\rangle$ and ρ_j on λ is understood implicitly.

By construction,

$$\rho_M(\lambda) = \hat{\rho}_{\mathbb{Q}_M}(\lambda). \quad (278)$$

4.2.2. Measurement operator for basis event in QET

Definition 91 (Measurement operator for basis event in QET). For each basis event $e_w \in \mathbb{B}_c$, the corresponding measurement operator in QET is defined as

$$\mathbb{M}_{e_w} = |e_w\rangle \langle e_w|, \quad 1 \leq w \leq n, \quad (279)$$

satisfying

$$\sum_{w=1}^n \mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} = I, \quad (280)$$

where $\mathbb{M}_{e_w}^\dagger$ is the Hermitian conjugate or adjoint of the $\mathbb{M}_{e_w}$ matrix, e.g., $\mathbb{M}_{e_w}^\dagger = (\mathbb{M}_{e_w}^T)^*$; and I denotes the identity matrix.

4.2.3. Basis event measurement function in QET

Definition 92 (Basis event measurement function in QET). Let $\rho_M(\lambda)$ be the density operator corresponding to a QBPD M on QFOD Φ under a fixed admissible allocation λ , and let $\{\mathbb{M}_{e_w} = |e_w\rangle\langle e_w|, 1 \leq w \leq n\}$ be the corresponding measurement operators. The probability of observing the basis event e_w is defined as

$$\Pr(e_w) = \text{Tr}(\mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} \rho_M(\lambda)), \quad 1 \leq w \leq n. \quad (281)$$

Here, $\text{Tr}(\cdot)$ denotes the matrix trace. For notational simplicity, the notation $\Pr(e_w)$ is retained, with a fixed admissible allocation λ understood.

4.2.4. QBP measurement function

Definition 93 (QBP measurement function). Let $\rho_M(\lambda)$ represent a QBPD M on QFOD Φ under a fixed admissible allocation λ . Using the basis-event measurement operators $\mathbb{M}_{e_w}$ defined above, the measurement-induced quantum support $M_{\text{Tr}}(\psi_j)$ is defined as

$$M_{\text{Tr}}(\psi_j) = \sum_{w: \phi_w \in \psi_j} \text{Tr}(\mathbb{M}_{e_w}^\dagger \mathbb{M}_{e_w} \rho_M(\lambda)), \quad \psi_j \subseteq \Phi. \quad (282)$$

Here, $M_{\text{Tr}}(\psi_j)$ denotes the quantum support induced by Hilbert-space measurement through the trace operation. For notational simplicity, its dependence on the fixed admissible allocation λ is left implicit. It is not, in general, required to coincide elementwise with the original QBPD $M(\psi_j)$ used to construct $\rho_M(\lambda)$.

4.2.5. Belief and plausibility measurement functions in QET

Proposition 2 (Hilbert-space realization of QBel and QPl). Let Λ be the set of admissible internal state-vector allocations $\lambda = \{\lambda_w^r\}$ satisfying

$\sum_{w: \phi_w \in \psi_r} |\lambda_w^r|^2 = 1$ for each nonempty quantum proposition ψ_r . The internal allocations associated with different nonempty propositions are independent variables; equivalently, Λ is the Cartesian product of the proposition-wise admissible allocation sets. Let $\rho_M(\lambda)$ be the density operator in Eq. (277), with basis events $e_w \in \mathbb{B}_c = \{e_1, \dots, e_n\}$ and measurement operators $M_{e_w} = |e_w\rangle\langle e_w|$. Then

$$\text{QBel}(\psi_j) = \min_{\lambda \in \Lambda} \sum_{w: \phi_w \in \psi_j} \text{Tr}(M_{e_w}^\dagger M_{e_w} \rho_M(\lambda)), \quad \psi_j \subseteq \Phi, \quad (283)$$

$$\text{QPl}(\psi_j) = \max_{\lambda \in \Lambda} \sum_{w: \phi_w \in \psi_j} \text{Tr}(M_{e_w}^\dagger M_{e_w} \rho_M(\lambda)), \quad \psi_j \subseteq \Phi. \quad (284)$$

Thus, the min-max quantum measurement construction exactly realizes the quantum belief and plausibility functions defined in QET.

Proof. Because the allocation variables associated with different focal propositions are independently constrained, the global minimum and maximum decompose into the corresponding proposition-wise extrema. For a nonempty focal proposition ψ_r , its measured contribution to ψ_j is $M(\psi_r) \sum_{w: \phi_w \in \psi_r \cap \psi_j} |\lambda_w^r|^2$. If $\psi_r \subseteq \psi_j$, the minimum contribution is $M(\psi_r)$; otherwise it is zero. Hence the minimum is the sum of masses of focal propositions contained in ψ_j , namely $\text{QBel}(\psi_j)$. Likewise, the maximum contribution from ψ_r is $M(\psi_r)$ exactly when $\psi_r \cap \psi_j \neq \emptyset$, and zero otherwise, which yields $\text{QPl}(\psi_j)$. $\square$

4.2.6. Progressive quantum evidential combination rule in Hilbert space

Definition 94 (Linear operator representation of progressively fused quantum evidence in QET). Let $\mathbb{Q}^\oplus$ be the QBPA obtained by PQEQR.

The linear operator corresponding to the progressively fused QBPA is defined as

$$\tilde{\rho}_{Q^\oplus} = \sum_{\emptyset \neq \psi_j \subseteq \Phi} Q^\oplus(\psi_j) |\psi_j\rangle \langle \psi_j| = \sum_{\emptyset \neq \psi_j \subseteq \Phi} Q^\oplus(\psi_j) \rho_j. \quad (285)$$

According to the definition of PQEQR, provided that $\gamma > 0$,

$$Q^\oplus(\psi_j) = \frac{\tilde{Q}^\oplus(\psi_j)}{\sqrt{\gamma}}, \quad \emptyset \neq \psi_j \subseteq \Phi, \quad (286)$$

where $\tilde{Q}^\oplus$ and γ are defined in Eqs. (214) and (215), respectively. Therefore,

$$\tilde{\rho}_{Q^\oplus} = \frac{1}{\sqrt{\gamma}} \sum_{\emptyset \neq \psi_j \subseteq \Phi} \tilde{Q}^\oplus(\psi_j) |\psi_j\rangle \langle \psi_j|. \quad (287)$$

In general, $\tilde{\rho}_{Q^\oplus}$ is a linear operator representation of the progressively fused QBPA rather than a density operator, since the complex probability amplitudes do not necessarily yield a Hermitian, positive-semidefinite operator with unit trace.

Definition 95 (Density operator of progressively fused quantum evidence in QET). Let

$$M^\oplus(\psi_j) = |Q^\oplus(\psi_j)|^2 \quad (288)$$

be the QBPD induced by the progressively fused QBPA. For each fixed admissible allocation λ , the corresponding density operator is defined as

$$\hat{\rho}_{Q^\oplus}(\lambda) = \sum_{\emptyset \neq \psi_j \subseteq \Phi} M^\oplus(\psi_j) |\psi_j\rangle \langle \psi_j| = \sum_{\emptyset \neq \psi_j \subseteq \Phi} M^\oplus(\psi_j) \rho_j. \quad (289)$$

Since PQEQR satisfies

$$Q^\oplus(\emptyset) = 0, \quad \sum_{\emptyset \neq \psi_j \subseteq \Phi} |Q^\oplus(\psi_j)|^2 = 1, \quad (290)$$

the induced QBPD satisfies

$$M^\oplus(\psi_j) \geq 0, \quad \sum_{\emptyset \neq \psi_j \subseteq \Phi} M^\oplus(\psi_j) = 1. \quad (291)$$

Therefore,

$$\hat{\rho}_{Q^\oplus}(\lambda) \succeq 0, \quad \text{Tr}(\hat{\rho}_{Q^\oplus}(\lambda)) = 1. \quad (292)$$

5. Conclusion

In summary, QET and GQET provide a unified quantum-evidential framework for uncertainty representation, multisource information fusion, and decision reasoning in closed- and open-world environments. At the representation level, quantum basic probability amplitudes characterize partial ignorance, complete ignorance, and, in open-world environments, unknown information, thereby providing fine-grained representations of heterogeneous multisource information. At the fusion level, PQEQR/GPQECR preserve complex-valued probability amplitudes for progressive multisource fusion, whereas QEQR/GQEQR provide the corresponding real-valued fused evidential support. At the decision level, quantum basic probability distributions, Pignistic transformations, and the associated belief and plausibility functions support uncertainty assessment and probabilistic decision making. Furthermore, the Hilbert-space formulations provide linear-operator, density-operator, and measurement representations of quantum evidence. Under the corresponding non-interference and closed- or open-world conditions, the proposed formulations recover their classical evidence-theoretic counterparts.

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