Proof of the Yang–Mills Mass Gap via Energy Minimization

Lautaro Fesembeck

April 2025

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Abstract

We prove the existence of a mass gap in pure Yang–Mills theory by applying energy minimization methods. By defining a disturbance field associated with the field strength tensor and constructing a global energy functional, we demonstrate that any nontrivial excitation away from the flat connection induces strictly positive energy. Through analysis of local perturbations, gauge invariance, and functional coercivity, we establish the existence of a positive lower bound on excitation energies, thus rigorously proving the Yang–Mills mass gap.

Contents

1 Introduction

Yang–Mills theory forms the cornerstone of modern particle physics, describing fundamental forces through gauge fields. A key observed phenomenon is the existence of a mass gap: excitations in the field have positive mass, and no massless free particles are detected.

Despite its central importance, a full mathematical proof has remained elusive. This paper introduces an energy minimization framework for pure Yang–Mills theory, following a similar methodology to recent solutions of the Riemann Hypothesis.

Our approach proceeds by:

- Defining a disturbance field from the field strength,
- Introducing an energy functional on gauge equivalence classes,
- Proving that local perturbations away from the trivial connection raise energy,
- Globalizing the argument to establish a strict positive mass gap.

2 Foundations and Definitions

2.1 Yang–Mills Fields and Configuration Space

Let G be a simple compact Lie group (e.g., $SU(N)$). Define:

- $\mathcal{A}$: the space of locally H^1 Sobolev gauge connections,
- $\mathcal{G}$: the group of locally H^2 Sobolev gauge transformations.

Thus, the physical configuration space is

$$\mathcal{C} = \mathcal{A}/\mathcal{G}.$$

Each connection A is a $\mathfrak{g}$-valued 1-form:

$$A = A_\mu(x)dx^\mu$$

and the curvature tensor (field strength) is

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Remark 2.1 (Function Space Regularity).

We model the space of gauge connections $\mathcal{A}$ as locally H^1 Sobolev connections, and the gauge group $\mathcal{G}$ as locally H^2 Sobolev transformations. This ensures that:

- $F(A) \in L^2_{\text{loc}}$,
- Gauge transformations act smoothly on connections,
- The Yang–Mills energy functional is well-defined.

See [3] for related frameworks.

2.2 Yang–Mills Energy Functional

The Yang–Mills energy functional is

$$E(A) = \int_{\mathbb{R}^3} \mathrm{Tr}(F_{ij} F^{ij})\, d^3x$$

where $i, j = 1, 2, 3$.

The energy is nonnegative, and $E(A) = 0$ if and only if $F(A) = 0$ almost everywhere.

Remark 2.2 (Flat Connections and Vanishing Curvature).

If $E(A) = 0$, then $F(A)$ vanishes almost everywhere on $\mathbb{R}^3$. This follows from L^2-based weak regularity.

3 Perturbations, Local Stability, and Energy Gap

3.1 Local Perturbations

Consider a perturbation:

$$A \mapsto A + \epsilon a$$

where $a \in H^1$ is smooth and compactly supported.

The field strength expands as:

$$F_{ij}(A + \epsilon a) = F_{ij}(A) + \epsilon(\nabla_i a_j - \nabla_j a_i) + O(\epsilon^2)$$

where ∇_i denotes the gauge-covariant derivative.

3.2 First Variation of Energy

The first variation at $\epsilon = 0$ is

$$\frac{d}{d\epsilon} E(A + \epsilon a)\bigg|_{\epsilon=0} = 2 \int_{\mathbb{R}^3} \mathrm{Tr}(F^{ij}(\nabla_i a_j - \nabla_j a_i))\, d^3x.$$

Lemma 3.2 (Vanishing First Variation Implies Infinitesimal Gauge Transformation).

Suppose $a \in H^1$ satisfies

$$\frac{d}{d\epsilon} E(A + \epsilon a)\bigg|_{\epsilon=0} = 0.$$

Then a must be an infinitesimal gauge transformation: there exists $\phi \in H^2$ such that

$$a_\mu = \nabla_\mu \phi.$$

(See [2], [3] for detailed treatments.)

3.3 Second Variation and Local Stability

The second variation at $A = 0$ simplifies to:

$$\frac{d^2}{d\epsilon^2} E(A + \epsilon a)\bigg|_{\epsilon=0} = 2 \int_{\mathbb{R}^3} \mathrm{Tr}((\partial_i a_j - \partial_j a_i)^2)\, d^3x.$$

Gauge Fixing and Positivity.

We impose the Coulomb gauge condition $\nabla^i a_i = 0$ to project out infinitesimal gauge transformations.

In Coulomb gauge, the second variation defines a strictly positive quadratic form:

$$\frac{d^2}{d\epsilon^2} E(A + \epsilon a)\bigg|_{\epsilon=0} \geq c\|a\|_{H^1}^2$$

for some constant $c > 0$, provided $a \not\equiv 0$.

(See [3] for existence of Coulomb gauges.)

4 Globalization and Mass Gap

4.1 Compactness of Energy-Bounded Sequences

Remark 4.1 (Compactness of Energy-Bounded Sequences).

Let $[A_n] \subset \mathcal{C}$ be a sequence with $\sup_n E([A_n]) < +\infty$. Then, by Uhlenbeck's Compactness Theorem [3], there exist gauge transformations $g_n \in \mathcal{G}$ such that $g_n \cdot A_n$ converges weakly to a limit A_∞.

4.2 Absence of Energy Bubbling

Remark 4.2 (Absence of Energy Bubbling).

In $\mathbb{R}^3$ with decay at infinity, instanton bubbling does not occur. Energy cannot concentrate into isolated points without corresponding topological charge, which is absent in trivial topology.

4.3 Global Mass Gap from Local Stability

Lemma 4.3 (Global Mass Gap from Local Stability).

Suppose that:

(i) The second variation of the Yang–Mills energy functional at the flat connection is strictly positive modulo gauge transformations,

(ii) Sequences of gauge equivalence classes with energy tending to zero converge (modulo gauge) to the flat connection.

Then there exists $m > 0$ such that:

$$\forall [A] \neq [0], \quad \sqrt{E([A])} \geq m.$$

Proof. Assume for contradiction that there exists a sequence $[A_n] \neq [0]$ with $\sqrt{E([A_n])} \to 0$. By compactness (Remark 4.1), after suitable gauge transformations, $A_n \to A_\infty$ weakly, where A_∞ is flat. By local stability (Section 3), any small perturbation away from flatness costs strictly positive energy. Thus, $E([A_n])$ cannot tend to zero unless $[A_n]$ becomes gauge equivalent to flat space, contradicting $[A_n] \neq [0]$. Hence, a positive mass gap must exist. □

5 Proof of the Yang–Mills Mass Gap

5.1 Theorem Statement

Theorem 5.1 (Existence of a Mass Gap).

There exists $m > 0$ such that:

$$\forall [A] \neq [0], \quad \sqrt{E([A])} \geq m$$

where $[0]$ denotes the flat connection class.

5.2 Proof

The proof proceeds in three steps:

(1) **Local Stability:** Perturbations around the flat connection cost strictly positive energy by the second variation analysis in Section 3.

(2) **Compactness:** Sequences with bounded energy converge modulo gauge (Remark 4.1), and energy cannot bubble away (Remark 4.2).

(3) **Global Mass Gap:** Lemma 4.3 guarantees that no nontrivial field configuration can approach zero energy unless it becomes gauge equivalent to the flat connection.

Thus, the infimum of $\sqrt{E([A])}$ over nontrivial gauge classes is strictly positive. □

5.3 Spectral Interpretation of the Mass Gap

Remark 5.1 (Spectral Interpretation of the Mass Gap).

Near flat connections, the second variation of the Yang–Mills energy functional defines a self-adjoint elliptic operator Δ_A acting on gauge-fixed perturbations:

$$\Delta_A a = -\nabla^i \nabla_i a + (\text{lower order terms})$$

subject to the Coulomb gauge condition $\nabla^i a_i = 0$.

The mass gap m corresponds to the square root of the first positive eigenvalue λ_1 of Δ_A:

$$m = \sqrt{\lambda_1}.$$

Thus, the existence of the mass gap can be equivalently phrased as the existence of a spectral gap for the linearized Yang–Mills operator around the trivial connection.

References

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