Some Formulas and an Equivalence Between Two Sets of Axioms in Axiomatic Propositional Logic

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Abstract. We derive some formulas in axiomatic propositional logic and we show that two sets of different axioms are equivalent. The outputs were manually derived through logical deductions and subsequently typeset in LaTeX, without the use of automated computational tools.

Construct a new propositional calculus formal system **P**' from **P**. Axiom (A3) is changed to :

$$\text{(A4) } (\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha).$$

Prove that the set of internal theorems of **P**' and **P** are identical. Axioms of **P** are:
(A1) $\alpha \to (\beta \to \alpha)$.
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.
(A3) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.
There is only one rule of deduction, namely *modus ponens* (abbreviated as MP). It says: from α and $\alpha \to \beta$, β is a direct consequence.

Lemma 1: $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

To prove that $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$.

(1) $\alpha \to \beta$	hypothesis
(2) $\beta \to \gamma$	hypothesis
(3) $(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$	axiom (A1)
(4) $\alpha \to (\beta \to \gamma)$	(2),(3) MP
(5) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	axiom (A2)
(6) $(\alpha \to \beta) \to (\alpha \to \gamma)$	(4),(5) MP
(7) $\alpha \to \gamma$	(1),(6) MP

Thus, by (1)-(7), $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$. Therefore, by the Deduction Theorem, $\alpha \to \beta \vdash (\beta \to \gamma) \to (\alpha \to \gamma)$, and, again by the Deduction Theorem, $\vdash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$.

Lemma 2: $\alpha \to \alpha$

(1) $(\alpha \to ((\alpha \to \alpha) \to \alpha))$	axiom (A1)
(2) $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$	axiom (A2)
(3) $\alpha \to (\alpha \to \alpha)$	axiom (A1)
(4) $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$	(1),(2) MP
(5) $\alpha \to \alpha$	(3),(4) MP

Lemma 3: $\neg\alpha \to (\alpha \to \beta)$

(1)	$\neg\alpha$	hypothesis
(2)	$\neg\alpha \to (\neg\beta \to \neg\alpha)$	axiom (A1)
(3)	$(\neg\beta \to \neg\alpha)$	(1),(2) MP
(4)	$(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$	axiom (A3)
(5)	$\alpha \to \beta$	(3),(4) MP

Thus, by (1)-(5), $\neg\alpha \vdash \alpha \to \beta$. Therefore, by the Deduction Theorem, $\vdash \neg\alpha \to (\alpha \to \beta)$.

Lemma 4: $\neg\neg\alpha \to \alpha$

(1)	$\neg\neg\alpha$	hypothesis
(2)	$\neg\neg\alpha \to (\neg\alpha \to \neg\neg\neg\alpha)$	Lemma 3
(3)	$\neg\alpha \to \neg\neg\neg\alpha$	(1),(2) MP
(4)	$(\neg\alpha \to \neg\neg\neg\alpha) \to (\neg\neg\alpha \to \alpha)$	axiom (A3)
(5)	$\neg\neg\alpha \to \alpha$	(3),(4) MP
(6)	α	(1),(5) MP

Thus, by (1)-(6), $\neg\neg\alpha \vdash \alpha$. Therefore, by the Deduction Theorem, $\vdash \neg\neg\alpha \to \alpha$.

Lemma 5: $\alpha \to \neg\neg\alpha$

(1)	$\neg\neg\neg\alpha \to \neg\alpha$	Lemma 4
(2)	$(\neg\neg\neg\alpha \to \neg\alpha) \to (\alpha \to \neg\neg\alpha)$	axiom (A3)
(3)	$\alpha \to \neg\neg\alpha$	(1),(2) MP

Lemma 6: $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$

(1)	$\alpha \to \beta$	hypothesis
(2)	$\neg\neg\alpha \to \alpha$	Lemma 4
(3)	$(\neg\neg\alpha \to \alpha) \to ((\alpha \to \beta) \to (\neg\neg\alpha \to \beta))$	Lemma 1
(4)	$(\alpha \to \beta) \to (\neg\neg\alpha \to \beta)$	(2),(3) MP
(5)	$\neg\neg\alpha \to \beta$	(1),(4) MP
(6)	$\beta \to \neg\neg\beta$	Lemma 5
(7)	$(\neg\neg\alpha \to \beta) \to ((\beta \to \neg\neg\beta) \to (\neg\neg\alpha \to \neg\neg\beta))$	Lemma 1
(8)	$(\beta \to \neg\neg\beta) \to (\neg\neg\alpha \to \neg\neg\beta)$	(5),(7) MP
(9)	$\neg\neg\alpha \to \neg\neg\beta$	(6),(8) MP
(10)	$(\neg\neg\alpha \to \neg\neg\beta) \to (\neg\beta \to \neg\alpha)$	axiom (A3)
(11)	$\neg\beta \to \neg\alpha$	(9),(10) MP

Thus, by (1)-(11), $\alpha \to \beta \vdash \neg\beta \to \neg\alpha$. Therefore, by the Deduction Theorem, $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$.

Lemma 7: $\neg\beta \to ((\neg\beta \to \alpha) \to \alpha)$

(1) $\neg\beta$	hypothesis
(2) $\neg\beta \to \alpha$	hypothesis
(3) α	(1),(2) MP

Thus, by (1)-(3), $\neg\beta, \neg\beta \to \alpha \vdash \alpha$. Therefore, by the Deduction Theorem, $\neg\beta \vdash (\neg\beta \to \alpha) \to \alpha$, and, again by the Deduction Theorem, $\vdash \neg\beta \to ((\neg\beta \to \alpha) \to \alpha)$.

Lemma 8: $\neg\beta \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$

(1) $\neg\beta$	hypothesis
(2) $\neg\beta \to ((\neg\beta \to \alpha) \to \alpha)$	Lemma 7
(3) $(\neg\beta \to \alpha) \to \alpha$	(1),(2) MP
(4) $((\neg\beta \to \alpha) \to \alpha) \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$	Lemma 6
(5) $\neg\alpha \to \neg(\neg\beta \to \alpha)$	(3),(4) MP

Thus, by (1)-(5), $\neg\beta \vdash \neg\alpha \to \neg(\neg\beta \to \alpha)$. Therefore, by the Deduction Theorem, $\vdash \neg\beta \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$.

Lemma 9: $(\beta \to \alpha) \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$

(1) $\beta \to \alpha$	hypothesis
(2) $\neg\alpha$	hypothesis
(3) $(\beta \to \alpha) \to (\neg\alpha \to \neg\beta)$	Lemma 6
(4) $\neg\alpha \to \neg\beta$	(1),(3) MP
(5) $\neg\beta$	(2),(4) MP
(6) $\neg\beta \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$	Lemma 8
(7) $\neg\alpha \to \neg(\neg\beta \to \alpha)$	(5),(6) MP
(8) $\neg(\neg\beta \to \alpha)$	(2),(7) MP

Thus, by (1)-(8), $\beta \to \alpha, \neg\alpha \vdash \neg(\neg\beta \to \alpha)$. Therefore, by the Deduction Theorem, $\beta \to \alpha \vdash \neg\alpha \to \neg(\neg\beta \to \alpha)$, and, again by the Deduction Theorem, $\vdash (\beta \to \alpha) \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$.

Lemma 10: $(\beta \to \alpha) \to ((\neg\beta \to \alpha) \to \alpha)$

(1) $\beta \to \alpha$	hypothesis
(2) $(\beta \to \alpha) \to (\neg\alpha \to \neg(\neg\beta \to \alpha))$	Lemma 9
(3) $\neg\alpha \to \neg(\neg\beta \to \alpha)$	(1),(2) MP
(4) $(\neg\alpha \to \neg(\neg\beta \to \alpha)) \to ((\neg\beta \to \alpha) \to \alpha)$	axiom (A3)
(5) $(\neg\beta \to \alpha) \to \alpha$	(3),(4) MP

Thus, by (1)-(5), $\beta \to \alpha \vdash (\neg\beta \to \alpha) \to \alpha$. Therefore, by the Deduction Theorem, $\vdash (\beta \to \alpha) \to ((\neg\beta \to \alpha) \to \alpha)$.

To derive (A4) from (A1), (A2) and (A3).

(1)	$\neg\alpha \to \neg\beta$	hypothesis
(2)	$\neg\alpha \to \beta$	hypothesis
(3)	$(\neg\alpha \to \beta) \to (\neg\beta \to \neg\neg\alpha)$	Lemma 6
(4)	$\neg\beta \to \neg\neg\alpha$	(2),(3) MP
(5)	$\neg\neg\alpha \to \alpha$	Lemma 4
(6)	$(\neg\beta \to \neg\neg\alpha) \to ((\neg\neg\alpha \to \alpha) \to (\neg\beta \to \alpha))$	Lemma 1
(7)	$(\neg\neg\alpha \to \alpha) \to (\neg\beta \to \alpha)$	(4),(6) MP
(8)	$\neg\beta \to \alpha$	(5),(7) MP
(9)	$(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$	axiom (A3)
(10)	$\beta \to \alpha$	(1),(9) MP
(11)	$(\beta \to \alpha) \to ((\neg\beta \to \alpha) \to \alpha)$	Lemma 10
(12)	$(\neg\beta \to \alpha) \to \alpha$	(10),(11) MP
(13)	α	(8),(12) MP

Thus, by (1)-(13), $\neg\alpha \to \neg\beta, \neg\alpha \to \beta \vdash \alpha$. Therefore, by the Deduction Theorem, $\neg\alpha \to \neg\beta \vdash (\neg\alpha \to \beta) \to \alpha$, and, again by the Deduction Theorem, $\vdash (\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha)$.

To derive (A3) from (A1), (A2) and (A4).

(1)	$\neg\alpha \to \neg\beta$	hypothesis
(2)	β	hypothesis
(3)	$(\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha)$	axiom (A4)
(4)	$\beta \to (\neg\alpha \to \beta)$	axiom (A1)
(5)	$\neg\alpha \to \beta$	(2),(4) MP
(6)	$(\neg\alpha \to \beta) \to \alpha$	(1),(3) MP
(7)	α	(5),(6) MP

Thus, by (1)-(7), $\neg\alpha \to \neg\beta, \beta \vdash \alpha$. Therefore, by the Deduction Theorem, $\neg\alpha \to \neg\beta \vdash \beta \to \alpha$, and, again by the Deduction Theorem, $\vdash (\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.

Lemma 11: $\neg(\alpha \to \beta) \to \alpha$

(1)	$\neg(\alpha \to \beta)$	hypothesis
(2)	$\neg\alpha \to (\alpha \to \beta)$	Lemma 3
(3)	$(\neg\alpha \to (\alpha \to \beta)) \to (\neg(\alpha \to \beta) \to \neg\neg\alpha)$	Lemma 6
(4)	$\neg(\alpha \to \beta) \to \neg\neg\alpha$	(2),(3) MP
(5)	$\neg\neg\alpha$	(1),(4) MP
(6)	$\neg\neg\alpha \to \alpha$	Lemma 4
(7)	α	(5),(6) MP

Thus, by (1)-(7), $\neg(\alpha \to \beta) \vdash \alpha$. Therefore, by the Deduction Theorem, $\vdash \neg(\alpha \to \beta) \to \alpha$.

Lemma 12: $\neg(\alpha \to \neg\beta) \to \beta$

(1)	$\neg(\alpha \to \neg\beta)$	hypothesis
(2)	$\neg\beta \to (\alpha \to \neg\beta)$	Axiom (A1)
(3)	$(\neg\beta \to (\alpha \to \neg\beta)) \to (\neg(\alpha \to \neg\beta) \to \neg\neg\beta)$	Lemma 6
(4)	$\neg(\alpha \to \neg\beta) \to \neg\neg\beta$	(2),(3) MP
(5)	$\neg\neg\beta$	(1),(4) MP
(6)	$\neg\neg\beta \to \beta$	Lemma 4
(7)	β	(5),(6) MP

Thus, by (1)-(7), $\neg(\alpha \to \neg\beta) \vdash \beta$. Therefore, by the Deduction Theorem, $\vdash \neg(\alpha \to \neg\beta) \to \beta$.

Theorem 1: $\neg((\alpha \to \beta) \to \neg(\beta \to \gamma)) \to (\alpha \to \gamma)$

(1)	$\neg((\alpha \to \beta) \to \neg(\beta \to \gamma))$	hypothesis
(2)	$\neg((\alpha \to \beta) \to \neg(\beta \to \gamma)) \to (\beta \to \gamma)$	Lemma 12
(3)	$\beta \to \gamma$	(1),(2) MP
(4)	$\neg((\alpha \to \beta) \to \neg(\beta \to \gamma)) \to (\alpha \to \beta)$	Lemma 11
(5)	$\alpha \to \beta$	(1),(4) MP
(6)	$(\beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$	Axiom (A1)
(7)	$\alpha \to (\beta \to \gamma)$	(3),(6) MP
(8)	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$	Axiom (A2)
(9)	$(\alpha \to \beta) \to (\alpha \to \gamma)$	(7),(8) MP
(10)	$\alpha \to \gamma$	(5),(9) MP

Thus, by (1)-(10), $\neg((\alpha \to \beta) \to \neg(\beta \to \gamma)) \vdash \alpha \to \gamma$. So, by the Deduction Theorem, $\vdash \neg((\alpha \to \beta) \to \neg(\beta \to \gamma)) \to (\alpha \to \gamma)$.

Theorem 2: $(\alpha \to \neg\beta) \to (\neg(\beta \to \gamma) \to \neg\alpha)$

(1)	$\alpha \to \neg\beta$	hypothesis
(2)	$\neg(\beta \to \gamma)$	hypothesis
(3)	$\neg(\beta \to \gamma) \to \beta$	Lemma 11
(4)	β	(2),(3) MP
(5)	$(\alpha \to \neg\beta) \to (\neg\neg\alpha \to (\alpha \to \neg\beta))$	Axiom (A1)
(6)	$\neg\neg\alpha \to (\alpha \to \neg\beta)$	(1),(5) MP
(7)	$(\neg\neg\alpha \to (\alpha \to \neg\beta)) \to ((\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \neg\beta))$	Axiom (A2)
(8)	$\neg\neg\alpha \to \alpha$	Lemma 4
(9)	$(\neg\neg\alpha \to \alpha) \to (\neg\neg\alpha \to \neg\beta)$	(6),(7) MP
(10)	$\neg\neg\alpha \to \neg\beta$	(8),(9) MP
(11)	$(\neg\neg\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$	Axiom (A3)
(12)	$\beta \to \neg\alpha$	(10),(11) MP
(13)	$\neg\alpha$	(4),(12) MP

Thus, by (1)-(13), $\alpha \to \neg\beta, \neg(\beta \to \gamma) \vdash \neg\alpha$. So, by the Deduction Theorem, $\alpha \to \neg\beta \vdash \neg(\beta \to \gamma) \to \neg\alpha$ and, again by the Deduction Theorem $\vdash (\alpha \to \neg\beta) \to (\neg(\beta \to \gamma) \to \neg\alpha)$.

Lemma 13: $(\neg\alpha \to \neg\neg\beta) \to (\neg\alpha \to \beta)$

(1) $\neg\neg\beta \to \beta$	Lemma 4
(2) $(\neg\neg\beta \to \beta) \to (\neg\alpha \to (\neg\neg\beta \to \beta))$	Axiom (A1)
(3) $\neg\alpha \to (\neg\neg\beta \to \beta)$	(1),(2) MP
(4) $(\neg\alpha \to (\neg\neg\beta \to \beta)) \to ((\neg\alpha \to \neg\neg\beta) \to (\neg\alpha \to \beta))$	Axiom (A2)
(5) $(\neg\alpha \to \neg\neg\beta) \to (\neg\alpha \to \beta)$	(3),(4) MP

Theorem 3: $(\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$

(1) $\neg\beta \to \neg\alpha$	hypothesis
(2) $\neg\beta \to \alpha$	hypothesis
(3) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$	Axiom (A3)
(4) $\alpha \to \beta$	(1),(3) MP
(5) $(\neg\beta \to \alpha) \to (\neg\alpha \to \neg\neg\beta)$	Lemma 6
(6) $\neg\alpha \to \neg\neg\beta$	(2),(5) MP
(7) $(\neg\alpha \to \neg\neg\beta) \to (\neg\alpha \to \beta)$	Lemma 13
(8) $\neg\alpha \to \beta$	(6),(7) MP
(9) $(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$	Lemma 10
(10) $(\neg\alpha \to \beta) \to \beta$	(4),(9) MP
(11) β	(8),(10) MP

Thus, by (1)-(11), $\neg\beta \to \neg\alpha, \neg\beta \to \alpha \vdash \beta$. So, by the Deduction Theorem, $\neg\beta \to \neg\alpha \vdash (\neg\beta \to \alpha) \to \beta$ and, again by the Deduction Theorem $\vdash (\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$.

References

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