Reframing Kripke: Resolution Matrix Semantics with Broad Truth Values

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1. Introduction

This paper introduces non-relational Resolution Matrix Semantics (RMS) as a novel framework for constructing systems of modal logic, offering an alternative to the relational semantics of possible worlds developed by Saul Kripke. The concept of substantive semantics, pioneered by Y. Ivlev [3], serves as a foundational inspiration for this approach. Ivlev suggested defining modal operators based on informal reasoning tailored to their area of applicability—such as epistemology, ethics, or physics—rather than formal relational structures, introducing the notion of an interpretation quasi-function that assigns truth values in a context-dependent manner [3, 4]. His modal systems, while lacking an obvious correspondence to Kripkean systems, offered an intriguing informal, substantive perspective that prioritized practical interpretation over abstract world-relations.

Building on this idea, we develop RMS using an augmented 4-valued structure—necessary truth (tn), contingent truth (tc), contingent false (fc), and necessary false (fn)—augmented by "indeterminate" truth values such as "t" (either tn or tc), "f" (either fc or fn), or even t/f (fully indeterminate). The general approach to constructing any modal system with RMS involves defining an interpretation function that assigns these truth values to formulas, alongside a subfunction mechanism that resolves indeterminate values (e.g., t into tn or tc) across all possible sub-interpretations. A formula is valid in RMS if and only if it holds t (takes tn or tc) in every sub-interpretation generated from the given interpretation, providing a substantive, truth-value-based semantics distinct from Kripke's relational models.

The most compelling question driving this work is how reframing Kripkean modal logics—such as K, KD, KT, KT4 (S4), and KT45 (S5)—within RMS illuminates new perspectives in both their construction and application. We build analogues of these systems, termed Km, KDm, KTm, S4m, and S5m, to explore how a truth-value-centric approach reveals alternative insights into modal reasoning, potentially enhancing its utility across diverse domains. Based on RMS, we propose a tableau (truth tree) method tailored for validating formulas, leveraging the finite set of truth values to systematically test for validity by exploring whether a formula can take invalid values (fc or fn). This method adapts the classical propositional logic tableau approach to modal contexts, enhancing its practicality for RMS-based systems. Furthermore, RMS opens avenues for advanced implementations in modal logic. RMS can support applications in deontic logic (modeling obligation and permission), epistemic logic (representing knowledge and belief), and other domains such as philosophy, technology, and science, where indeterminate or context-dependent truth values are prevalent.

2. From Kripke's Worlds to Truth Value Semantics

Kripkean modal systems, developed by Saul Kripke [6], provide a standard framework for modal logic using relational semantics. These systems are based on a structure known as a Kripke frame, defined as a pair (W, R), where W is a non-empty set of possible worlds and R is a subset of W × W, an accessibility relation between worlds. A Kripke model extends this frame to a triple (W, R, V), where V is a valuation function assigning truth values (true or false) to propositional variables at each world [1].

In this semantics, modal operators □ (necessity) and ◇ (possibility) are interpreted as follows:

□P is true at a world w in W if and only if P is true in all worlds v in W such that wRv (i.e., v is accessible from w).
◇P is true at w if and only if there exists at least one world v such that wRv and P is true at v.

The properties of R (e.g., reflexivity, transitivity, symmetry) determine the specific modal system, such as S4 or S5. Kripkean models excel due to their reliance on relational semantics, which is particularly powerful when applied to scientific, philosophical, or other aspects of reality that involve relations. For example, in epistemology, the accessibility relation can represent epistemic accessibility, where one world is accessible from another if and only if it is consistent with an agent's knowledge. In physics, it might model causal relationships between events across possible worlds, such as in discussions of determinism or quantum mechanics. In ethics, it can reflect deontic relations, where worlds are accessible based on what is permissible under certain moral constraints. These relational structures allow Kripkean semantics to flexibly capture dependencies and interactions across contexts.

However, in some cases, a truth value approach like RMS can be more applicable and productive than a relational one. For instance, in decision theory, assigning truth values directly to propositions about outcomes (e.g., "this choice is necessarily beneficial" or "this outcome is contingently true") might simplify modeling without requiring an explicit relational structure. Similarly, in artificial intelligence, where systems often process propositions with graded or uncertain truth (e.g., "this sensor reading is likely true"), a truth value-based system can streamline computations over a relational framework. Since RMS works with truth values directly, Kripkean modal logics can potentially be generalized within RMS to incorporate infinite truth values, akin to fuzzy logic, where truth is a continuum rather than a binary or finite set.

Moreover, this shift to truth values in RMS opens the door to exploring modal logics with truth value gaps (where a proposition may lack a truth value) and gluts (where a proposition may be both true and false). Notable works in this area include Kleene's three-valued logic [5], which introduces an "undefined" value, and Priest's paraconsistent logic [9], such as his Logic of Paradox (LP), which allows for truth value gluts to handle contradictions. These systems are particularly relevant in contexts like vague predicates (e.g., "this heap is large") or paradoxical statements (e.g., the liar paradox), where traditional binary or relational semantics may falter.

The RMS approach is also interesting in its own right because it inspires consideration of "indeterminate" truth values, such as "necessary truth or contingent truth" (t) or even "it can be anything but necessary false". These concepts allow us to model situations where precision is elusive yet modal distinctions remain relevant. We encounter a scenario where the precise value is indeterminate yet constrained within a specific domain. This does not violate the concept of a function, as it consistently selects a single value from its range (for example, truth value t - "either tn or tc") for each input, albeit one that "wanders" between tn and tc. Such situations arise across various domains. In quantum mechanics, for instance, the state of a particle's spin might be necessarily true (tn) in a deterministic context or contingently true (tc) due to superposition until measurement, yet always one or the other. In artificial intelligence, a decision-making algorithm assessing sensor data (e.g., "the obstacle is ahead") might assign tn when corroborated by multiple sources or tc based on partial evidence, with the truth value fixed per instance but varying by case. Similarly, in legal reasoning, a statute's applicability could be necessarily true under clear precedent (tn) or contingently true in ambiguous cases (tc), reflecting context-dependent certainty within a bounded range. These examples illustrate how such fuzzy yet singular truth assignments model real-world phenomena where precision is elusive but categorical limits apply, aligning with the substantive semantics proposed by RMS.

For a more detailed exploration of RMS, which builds on these ideas by shifting focus from relations to a substantive truth value framework, we turn to the next section.

3. Intuitive Justification for Matrix Definitions of Negation and Implication Using Countable Sequences of Truth Values

In this chapter, we explore an intuitive foundation for the matrix definitions of negation (¬) and disjunction (v) in RMS semantics by imagining truth values as countable sequences. These sequences reflect the truth of a proposition across an ordered set of possible worlds, where the first element represents "our world" (the actual world), and the subsequent elements form a countable sequence of worlds reachable from it. This perspective offers a way to justify the matrix definitions without relying on relational Kripkean semantics, instead grounding them in a substantive, truth value-based framework. We define the four core truth values—necessary truth (tn), contingent truth (tc), contingent false (fc), and necessary false (fn)—as sequences, then use this model to shed light on how negation and disjunction behave, particularly highlighting cases where disjunction yields the indeterminate value "t" (either tn or tc).

3.1 Truth Values as Countable Sequences

Imagine each truth value as an infinite sequence of binary values (t for true, f for false), indexed by worlds: w_0 (our world), w_1, w_2, ..., w_n, ..., where w_0 is the actual world and w_1, w_2, ... are worlds accessible from it in some ordered way. In these sequences, t and f are values understood in a classical bivalent logic manner, representing straightforward true or false assignments at each world.

We can define our four truth values intuitively as follows:

tn (necessary truth): The proposition is true in our world and every reachable world. So, tn = <t, t, t, t, ...>, capturing universal truth across all worlds.
tc (contingent truth): The proposition is true in our world but false in at least one other world. For simplicity, picture tc = <t, f, f, f, ...>, true in w_0 but false elsewhere. (The pattern beyond w_0 could vary, as long as there's at least one f.)

fc (contingent false): The proposition is false in our world but true in at least one other world. For example, fc = <f, t, t, t, ...>, false in w_0 but true in some later worlds. (The tail can differ, but it needs at least one t.)
fn (necessary false): The proposition is false in our world and all reachable worlds, so fn = <f, f, f, f, ...>.

This sequence-based view captures the modal ideas of necessity and contingency in a straightforward way, focusing on truth patterns rather than explicit relations between worlds. The ordering tn > tc > fc > fn in our matrix semantics will naturally align with how these sequences behave under negation and implication, reflecting their relative "strength" in terms of truth consistency.

3.2 Sequence-Based Connectives

In Resolution Matrix Semantics (RMS), connectives are defined via a sequence model where propositional variables are assigned infinite sequences of t (true) or f (false) across worlds, with w_0 as the actual world. These map to truth values: tn = <t, t, t, ...> (necessary truth), tc = <t, f, f, ...> (contingent truth), fc = <f, t, t, ...> (contingent false), fn = <f, f, f, ...> (necessary false). This non-relational approach adapts classical intuitions to modal contexts.

- **Negation (¬)**: For p = <s_0, s_1, s_2, ...>, ¬p = <¬s_0, ¬s_1, ¬s_2, ...> (¬t = f, ¬f = t). Thus, ¬tn = fn (<f, f, f, ...>, true everywhere becomes false everywhere), ¬tc = fc (<f, t, t, ...>, true in w_0 and probably in some other worlds, flips to false), ¬fc = tc (<t, f, f, ...>, false in w_0 and probably in some other worlds, becomes true), ¬fn = tn (<t, t, t, ...>, false everywhere turns true). This symmetrically swaps necessity and contingency around the true/false divide.

Table 3.2a: Negation

p	¬p
tn	fn
tc	fc
fc	tc
fn	tn

- **Disjunction (∨)**: For $p = <p_0, p_1, p_2, ...>$ and $q = <q_0, q_1, q_2, ...>$, $p \lor q = <p_0 \lor q_0, p_1 \lor q_1, p_2 \lor q_2, ...>$ (t ∨ t = t, t ∨ f = t, f ∨ t = t, f ∨ f = f). The result takes the "highest" consistent value (tn > tc > fc > fn):

tn ∨ tn = tn (<t, t, t, ...>)

tn ∨ tc = tn (<t, t, t, ...>)

tc ∨ tc = t (<t, f, f, ...> or <t, t, t, ...>, depending on overlap),

tc ∨ fc = t (<t, f, f, ...> or <t, t, t, ...>, depending on overlap),

tc ∨ fn = tc (<t, f, f, ...>)

fc ∨ fc = fc (<f, t, t, ...>)

fc ∨ fn = fc (<f, t, t, ...>)

fn ∨ fn = fn (<f, f, f, ...>)

Indeterminacy (e.g., tc ∨ tc = t) reflects variable world overlaps.

Table 3.2b: Disjunction

p ∨ q	tn	tc	fc	fn
tn	tn	tn	tn	tn
tc	tn	t	t	tc
fc	tn	t	fc	fc
fn	tn	tc	fc	fn

4. Resolution Matrix Semantics (RMS) for Modal Systems

In RMS, we use four truth values: tn, tc, fc, and fn, along with "undetermined", broad truth values, t, f, and even t/f . Here, t means "either tn or tc" and f means "either fc or fn". Now, we introduce a system equivalent to the Kripkean system KT, called KTm ("m" for matrix), based on RMS.

4.1 System KTm

Language:

Propositional variables: p, q, etc., representing basic propositions.

Standard logical connectives: ¬ (negation), ∧ (conjunction), ∨ (disjunction), and → (implication).

The modal operator □, representing necessity.

Brackets: (and), used to group expressions and clarify the structure of formulas.

Definition of a well-formed formula

The set of well-formed formulas in KTm is defined recursively as follows:

Basic propositions p, q are formulas.

If p and q are formulas, then the following are also formulas: ¬p, p ∧ q, p ∨ q, p → q.

If p is a formula, then □p is a formula.

These rules generate all permissible formulas in KTm, allowing us to express both propositional and modal statements within the system.

Next, we define KTm's RMS using ¬ and → as primitives, with the truth values tn, tc, fc, and fn:

Negation (¬)

The truth value of ¬p is determined by reversing the order of the four truth values:

p	¬p
tn	fn
tc	fc
fc	tc
fn	tn

Implication (→)

The definition of implication (p → q) in KTm can be formulated similarly to the definition of disjunction in Section 3.3, depending on the truth values of sequences for p and q across worlds. The results are presented in the table below.

p → q	tn	tc	fc	fn
tn	tn	tc	fc	fn
tc	tn	t	fc	fc
fc	tn	t	t	tc
fn	tn	tn	fn	fn

Modal Operator (□)

In Kripke semantics, KT (also known as T) has a reflexive accessibility relation, meaning every world accesses itself. Thus, if a proposition p is necessarily true (□p), it must be true in the actual world. In the sequence model, tn = <t, t, t, ...> reflects truth in our world (w_0) and all reachable worlds, so □tn = t (tn or tc) captures this necessity, allowing flexibility for sub-interpretations. For tc = <t, f, f, ...>, truth holds in w_0 but fails elsewhere, so □tc = f (fc or fn) since necessity fails beyond the reflexive world. Similarly, fc = <f, t, t, ...> and fn = <f, f, f, ...> lack universal truth, so □fc = f and □fn = f, reflecting the absence of necessity across all worlds, including w_0.

The truth value of □p in KTm is defined based on the value of p:

p	□p
tn	t
tc	f
fc	f
fn	f

Here, "t" and "f" do not represent the values of classical two-valued logic; instead, they are indeterminate truth values. "t" stands for "either tn or tc," while "f" stands for "either fc or fn."

Let A be a formula in the system under consideration, and let I denote an interpretation function that assigns truth values to formulas. The possible truth values are drawn from the set {tn, tc, fc, fn, t, f}, where t represents "either tn or tc" and f represents "either fc or fn". Suppose that under interpretation I, the formula A is assigned the indeterminate truth value t, i.e., $|A|\_I = t$. Then, there exist sub-interpretation functions I_1 and I_2, generated from I, such that:

- In sub-interpretation I_1, $|A|\_I\_1 = tn$,
- In sub-interpretation I_2, $|A|\_I\_2 = tc$.

A sub-interpretation I' of I is defined as a function that resolves each indeterminate truth value (t or f) assigned by I into exactly one of its corresponding determinate values (tn or tc for t; fc or fn for f), consistently across all subformulas, while preserving the truth value assignments for all determinate values (tn, tc, fc, fn) as given by I. **The formula A is valid in the interpretation I if and only if A is valid in every sub-interpretation I' generated from I**, where validity in a sub-interpretation I' means that $|A|\_I' \in \{tn, tc\}$. Thus, A is valid under I if and only if, for all sub-interpretations I' of I, the truth value $|A|\_I'$ is either tn or tc.

When handling indeterminate truth values u/v within a formula, we must maintain consistency by assigning the same value from u/v to all subformulas sharing identical truth values. For instance, if two distinct subformulas both take the value fc, and the modal operator's definition assigns u/v to fc, we should consistently apply the value u to fc in one sub-interpretation and the value v in another.

4.2 KTm (Kripkean System KT based on RMS semantics)

The system KTm is defined with the following axioms and inference rules:

Axioms:

Propositional Tautologies: All tautologies of classical propositional logic (e.g., $p \vee \neg p$, $p \rightarrow (q \rightarrow p)$).
Distribution Axiom: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.
Reflexivity Axiom: $\Box p \rightarrow p$.
Inference Rules:

Modus Ponens: From p and $p \rightarrow q$, infer q.
Necessitation Rule: From p, infer $\Box p$ (provided $KTm \vdash p$).

These axioms and rules define the syntactic structure of KTm, which we will analyze in terms of its RMS properties in the following sections.

Soundness Theorem

The system KTm is sound—for all theorem A in KTm, A is a valid formula in RMS.

To establish this, we must show that each axiom of KTm is valid (i.e., takes only tn or tc for all possible assignments of its propositional variables) and that the inference rules preserve validity.

Axiom 1: Propositional Tautologies

Consider a simple tautology, such as $p \lor \neg p$. In all cases, the value is tn or tc, never fn or fc, as $p \lor \neg p$ is derived via $p \rightarrow \neg p$, which aligns with the implication table. Every tautology can be represented in a conjunctive normal form, each elementary disjunction of which, like $p \lor \neg p$, takes only tn or tc. Thus, every tautology would have either tn or tc. Therefore, propositional tautologies are valid.

Axiom 2: Distribution Axiom ($\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$)

Assume there exists an interpretation where $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ takes fn or fc. Then $\Box(p \rightarrow q)$ is t (tn or tc) and $(\Box p \rightarrow \Box q)$ is f(fn or fc). Then $\Box p$ is t, and $\Box q$ is f. By definition of modal operator, p is tn, and q is tc, fc, or fn. Then, from definition of implication, $p \rightarrow q$ takes tc, fc or fn, so $\Box(p \rightarrow q)$ takes f, contradiction to the assumption. Formula is valid.

Axiom 3: Reflexivity Axiom ($\Box p \rightarrow p$)

Assume $\Box p \rightarrow p$ takes f for some p. Then $\Box p$ is t, and p is f. But since $\Box p$ is t, then p is tn, and we get contradiction. Formula is valid.

Inference Rule 1: Modus Ponens

From p and $p \rightarrow q$, infer q. Suppose p and $p \rightarrow q$ are valid (tn or tc), and q is not. Then there is an interpretation in which p and $p \rightarrow q$ both take tn or tc, and q takes f (fn or fc). Therefore, $\neg q$ is t, and since $p \rightarrow q$, then $\neg q \rightarrow \neg p$ and $\neg q$ take t, then $\neg p$ is also t, a contradiction to the assumption.

Inference Rule 2: Necessitation Rule

If KTm ⊢ A, then KTm ⊢ □A.

Suppose that formula A is valid, but □A is not. Consequently, in some interpretation, □A takes the value "f." By the definition of the modal operator, this implies that A must take one of the values "tc," "fc," or "fn." Therefore, A cannot take "tn." However, since A is a valid formula, it must take either "tn" or "tc," and a valid formula cannot exclusively take "tc" values when its components can assume all four values ("tn," "tc," "fc," "fn"). The semantics of → and □ ensure that "tn" emerges to maintain validity across at least some assignments, contradicting the assumption that A does not take value tn.

Completeness Theorem

KTm is complete—for all valid formula A, KTm ⊢ A—using maximal consistent sets.

Lemma 1: A consistent set Γ can be extended to a maximal consistent set Γ with:

For all p, either (p ∧ □p) ∈ Γ, or (p ∧ ¬□p) ∈ Γ, or (¬p ∧ ¬□¬p) ∈ Γ, or (¬p ∧ □¬p) ∈ Γ.
If Γ ⊢ B and Γ ⊆ T, then B ∈ T.
B ∨ C ∈ T if and only if B ∈ T or C ∈ T.
B ∧ C ∈ T if and only if B ∈ T and C ∈ T.

Proof: Use Henkin-style construction, enumerating formulas and adding consistent ones, ensuring maximality and consistency with KTm axioms. Properties hold via closure and contradiction avoidance.

Lemma 2: There exists |·|_T such that:

|A|_T = tn if and only if A ∈ T and □A ∈ T
|A|_T = tc if and only if A ∈ T and ¬□A ∈ T
|A|_T = fc if and only if ¬A ∈ T and ¬□¬A ∈ T
|A|_T = fn if and only if ¬A ∈ T and □¬A ∈ T

Proof of Lemma 2

Define the Valuation $|\cdot|_T$

The valuation is defined based on membership in T and the status of $\Box A$ and its negations: $|A|_T = tn$ means A is "necessarily true" in T (both A and $\Box A$ are in T) - $|A|_T = tc$ means A is "contingently true" ($A \in T$, but $\Box A \notin T$, so $\neg\Box A \in T$ by maximality) - $|A|_T = fc$ means A is "contingently false" ($\neg A \in T$, but $\Box\neg A \notin T$, so $\neg\Box\neg A \in T$) - $|A|_T = fn$ means A is "necessarily false" ($\neg A \in T \land \Box\neg A \in T$). By Lemma 1, for every p, T includes exactly one of: $(p \land \Box p)$, $(p \land \neg\Box p)$, $(\neg p \land \neg\Box\neg p)$ or $(\neg p \land \Box\neg p)$. This ensures $|p|_T$ is uniquely assigned one of tn, tc, fc, or fn for propositional variables, forming a consistent basis.

Verify Negation (¬)

Assume $|A|_T = tn$: $A \in T$, $\Box A \in T$. Then $\neg(\neg A) \in T$, $\Box\neg(\neg A) \in T$ (using tautologies and MP). Therefore, $|\neg A|_T = fn$
Now assume $|\neg A|_T = fn$: $\neg(\neg A) \in T$, $\Box\neg(\neg A) \in T$. Then $A \in T$, $\Box A \in T$ (using tautologies and MP). Therefore, $|A|_T = tn$

Assume $|A|_T = tc$: $A \in T$, $\neg\Box A \in T$. Then $\neg(\neg A) \in T$, $\neg\Box\neg(\neg A) \in T$. Then $|\neg A|_T = fc$.
Now assume $|\neg A|_T = fc$, $\neg(\neg A) \in T$, $\neg\Box\neg(\neg A) \in T$. Then $A \in T$, $\neg\Box A \in T$ (tautologies and MP), so $|\neg A|_T = tc$.

Assume $|A|_T = fc$: $\neg A \in T$, $\neg\Box\neg A \in T$. Then $|\neg A|_T = tc$.
In other direction:$|\neg A|_T = tc$, $\neg A \in T$, $\neg\Box(\neg A) \in T$. Then $|A|_T = fc$.

Assume $|A|_T = fn$: $\neg A \in T$, $\Box\neg A \in T$. Therefore, $|\neg A|_T = tn$.
Now assume $|\neg A|_T = tn$: $\neg A \in T$, $\Box\neg A \in T$. Then, $|A|_T = fn$.

Verify Implication (→)

Note that the steps of this proof rely solely on tautologies from classical propositional logic and the Distribution Axiom (K: $\Box(p \to q) \to (\Box p \to \Box q)$), which is common to all modal systems considered in this paper (Km, KDm, KTm, KT4m, and S5m). This ensures the proof's applicability across these systems, as they all share the K axiom, allowing for a uniform verification of implication within the RMS framework.

- If $|A|\_T = fn$, then $|A \rightarrow B|\_T = tn$

$|A|\_T = fn \Rightarrow \neg A \in T$, $\Box\neg A \in T$, $\neg A \rightarrow (A \rightarrow B) \in T$ (tautology), therefore $(A \rightarrow B) \in T$. Further, $\Box(\neg A \rightarrow (A \rightarrow B)) \in T$ (necessitation), $\Box(\neg A) \rightarrow \Box(A \rightarrow B)) \in T$ (K, MP), therefore, since $\Box\neg A \in T$, $\Box(A \rightarrow B) \in T$. So, we get $(A \rightarrow B) \in T$ and $\Box(A \rightarrow B) \in T$, therefore $|A \rightarrow B|\_T = tn$.

- If $|B|\_T = tn$, then $|A \rightarrow B|\_T = tn$

$|B|\_T = tn \Rightarrow B \in T$, $\Box B \in T$, $B \rightarrow (A \rightarrow B) \in T$ (tautology), therefore $(A \rightarrow B) \in T$. Further, $\Box(B \rightarrow (A \rightarrow B)) \in T$ (necessitation), $\Box B \rightarrow \Box(A \rightarrow B)) \in T$ (K, MP), therefore, since $\Box B \in T$, $\Box(A \rightarrow B) \in T$. So, we get $(A \rightarrow B) \in T$ and $\Box(A \rightarrow B) \in T$, therefore $|A \rightarrow B|\_T = tn$.

- If $|A|\_T = tn$ and $|B|\_T = fn$, then $|A \rightarrow B|\_T = fn$

$|A|\_T = tn \Rightarrow A \in T$, $\Box A \in T$, and $|B|\_T = fn \Rightarrow \neg B \in T$, $\Box\neg B \in T$.
So, we have $A \in T$ and $\neg B \in T$, therefore $\neg(A \rightarrow B) \in T$.

We will use the following auxiliary proof (*):

1. A (premise)
2. $\Box A$ (premise)
3. $\neg B$ (premise)
4. $\Box\neg B$ (premise)
5. $A \rightarrow (\neg B \rightarrow (A \wedge \neg B))$ (tautology)
6. $\Box[A \rightarrow (\neg B \rightarrow (A \wedge \neg B))]$ (Nec, 5)
7. $\Box[A \rightarrow (\neg B \rightarrow (A \wedge \neg B))] \rightarrow (\Box A \rightarrow \Box(\neg B \rightarrow (A \wedge \neg B)))$ (K)
8. $\Box A \rightarrow \Box(\neg B \rightarrow (A \wedge \neg B))$ (MP, 6, 7)
9. $\Box(\neg B \rightarrow (A \wedge \neg B))$ (MP, 2, 8)
10. $\Box(\neg B \rightarrow (A \wedge \neg B)) \rightarrow (\Box\neg B \rightarrow \Box(A \wedge \neg B))$ (K)
11. $\Box\neg B \rightarrow \Box(A \wedge \neg B)$ (MP, 9, 10)
12. $\Box(A \wedge \neg B)$ (MP, 4, 11)
13. $\Box(A \wedge \neg B) \rightarrow \Box\neg(A \rightarrow B)$ (tautology: $p \wedge \neg q \equiv \neg(p \rightarrow q)$ under $\Box$)
14. $\Box\neg(A \rightarrow B)$ (MP, 12, 13)

Therefore, $\Box\neg(A \to B) \in T$. Since $\neg(A \to B) \in T$, $|A \to B|\_T = fn$.

- If $|A|\_T = tn$ and $|B|\_T = fc$, then $|A \to B|\_T = fc$

$|A|\_T = tn \Rightarrow A \in T, \Box A \in T$, and $|B|\_T = fc \Rightarrow \neg B \in T, \neg\Box\neg B \in T$.
Since $A \in T$ and $\neg B \in T$, then $\neg(A \to B) \in T$. Now assume $\Box\neg(A \to B) \in T$. Then $\Box(A \land \neg B) \in T$.
We prove auxiliary theorem (**):

1.$(A \land B) \to A$ (Axiom 1: tautology)
2. $\Box((A \land B) \to A)$ (Nec, 1)
3. $\Box((A \land B) \to A) \to (\Box(A \land B) \to \Box A)$ (Axiom 2: K)
4. $\Box(A \land B) \to \Box A$ (MP, 2, 3)
5. $(A \land B) \to B$ (Axiom 1: tautology)
6. $\Box((A \land B) \to B)$ (Nec, 5)
7. $\Box((A \land B) \to B) \to (\Box(A \land B) \to \Box B)$ (Axiom 2: K)
8. $\Box(A \land B) \to \Box B$ (MP, 6, 7)
9. $((\Box(A \land B) \to \Box A) \land (\Box(A \land B) \to \Box B)) \to (\Box(A \land B) \to (\Box A \land \Box B))$ (Axiom 1: tautology)
10. $(\Box(A \land B) \to \Box A) \land (\Box(A \land B) \to \Box B)$ (from 4, 8)
11. $\Box(A \land B) \to (\Box A \land \Box B)$ (MP, 9, 10)

Since we have $\Box(A \land \neg B) \in T$, $(\Box A \land \Box\neg B) \in T$, therefore $\Box\neg B \in T$, which contradicts to the assumption.
Therefore, $\neg(A \to B) \in T$ and $\neg\Box\neg(A \to B) \in T$, so $|A \to B|\_T = fc$.

- If $|A|\_T = tc$ and $|B|\_T = tc$, then $|A \to B|\_T = t$

$A \in T, \neg\Box A \in T$ and $B \in T, \neg\Box B \in T$. Assume $|A \to B|\_T = f$, then $\neg(A \to B) \in T$, so $A \in T, \neg B \in T$, contradiction. So, $|A \to B|\_T = t$.

- If $|A|\_T = fc$ and $|B|\_T = tc$, then $|A \to B|\_T = t$

$\neg A \in T, \neg\Box\neg A \in T$ and $B \in T, \neg\Box B \in T$. Assume $|A \to B|\_T = f$, then $\neg(A \to B) \in T$, so $A \in T, \neg B \in T$, contradiction. So,$|A \to B|\_T = t$.

- If |A|_T = fc and |B|_T = fc, then |A → B|_T = t

¬A ∈ T, ¬□¬A ∈ T and ¬B ∈ T, ¬□¬B ∈ T. Assume |A → B|_T = f, then ¬(A → B)∈ T, so A ∈ T, ¬B ∈ T, contradiction. So,|A → B|_T = t.

- If |A|_T = fc and |B|_T = fn, then |A → B|_T = tc

¬A ∈ T, ¬□¬A ∈ T and ¬B ∈ T, □¬B ∈ T. ¬A → (A → B)∈ T (tautology), so (A → B)∈ T. Assume □(A → B)∈ T

We use the following auxiliary proof:

1. □¬B premise
2. □(A → B)
3. □(¬B → ¬A) (2, tautologies, N)
4. □(¬B → ¬A)→(□¬B → □¬A) (K)
5. (□¬B → □¬A), (4, MP)
6. □¬A (1,5,MP)

This contradicts the assumption. Therefore, (A → B) ∈ T and ¬□(A → B)∈ T, so |A → B|_T = tc.

- If |A|_T = tc and |B|_T = fn, then |A → B|_T = fc

A ∈ T, ¬□A ∈ T and ¬B ∈ T, □¬B ∈ T, so A ∧ ¬B ∈ T, or ¬(A → B)∈ T. Now, assume □(A ∧ ¬B)∈ T, then □A ∧ □¬B ∈ T by theorem (**), so □A ∈ T, contradiction. Therefore, ¬(A → B)∈ T and ¬□¬(A → B))∈ T, |A → B|_T = fc.

- If |A|_T = tc and |B|_T = fc, then |A → B|_T = fc

A ∈ T, ¬□A ∈ T and ¬B ∈ T, ¬□¬B ∈ T, so A ∧ ¬B ∈ T, or ¬(A → B)∈ T. Now, assume □(A ∧ ¬B)∈ T, then □A ∧ □¬B ∈ T by theorem (**), so □A ∈ T, contradiction. Therefore, ¬(A → B)∈ T and ¬□¬(A → B))∈ T, |A → B|_T = fc.

The proof in the opposite direction:

- If $|A \to B|\_T = fn$, then $|A|\_T = tn$ and $|B|\_T = fn$

$\neg(A \to B) \in T$, so $A \in T$, $\neg B \in T$. Also, we have $\Box\neg(A \to B) \in T$. Therefore, $\Box(A \wedge \neg B) \in T$, $\Box A \wedge \Box\neg B \in T$ by theorem (**). So we have $A \in T$, $\neg B \in T$, $\Box A \in T$, $\Box\neg B \in T$, therefore, $|A|\_T = tn$ and $|B|\_T = fn$.

- If $|A \to B|\_T = fc$, then either $|A|\_T = tn$ and $|B|\_T = fc$ or $|A|\_T = tc$ and $|B|\_T = fc$ or $|A|\_T = tc$ and $|B|\_T = fn$

Since $|A \to B|\_T = fc$, $\neg(A \to B) \in T$, $A \in T$, $\neg B \in T$. Also, $\neg\Box A \vee \Box A \in T$, $\neg\Box\neg B \vee \Box\neg B \in T$. Therefore, ($|A|\_T = tn$ or $|A|\_T = tc$) and ($|B|\_T = fn$ or $|B|\_T = fc$). But $|A|\_T = tn$ and $|B|\_T = fn$ if and only if $|A \to B|\_T = fn$. Therefore, $|A|\_T = tn$ and $|B|\_T = fc$ or $|A|\_T = tc$ and $|B|\_T = fc$ or $|A|\_T = tc$ and $|B|\_T = fn$.

- If $|A \to B|\_T = tn$, then either $|A|\_T = fn$ or $|B|\_T = tn$ or $|A|\_T = tc$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = fc$

Assume $|A|\_T = tn$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = fn$. Let's $|A|\_T = tn$ and $|B|\_T = tc$, so $A \in T$, $\Box A \in T$ and $B \in T$, $\neg\Box B \in T$.Since $|A \to B|\_T = tn$, $\Box(A \to B)) \in T$, $(\Box A \to \Box B) \in T$ (Distribution axiom), $\Box B \in T$ ($\Box A \in T$, MP). This contradicts to the assumption.
Now assume $|A|\_T = fc$ and $|B|\_T = fn$, so $\neg A \in T$, $\neg\Box\neg A \in T$ and $\neg B \in T$, $\Box\neg B \in T$. Since $|A \to B|\_T = tn$, $(A \to B)) \in T$, $\Box(A \to B)) \in T$, $(A \to B) \to (\neg B \to \neg A)) \in T$, $\Box((A \to B) \to (\neg B \to \neg A)) \in T$, $\Box(A \to B) \to \Box(\neg B \to \neg A)) \in T$, $\Box(\neg B \to \neg A)) \in T$, $\Box\neg B \to \Box\neg A \in T$, $\Box\neg A \in T$, contradiction.

Therefore, $|A|\_T = fn$ or $|B|\_T = tn$ or $|A|\_T = tc$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = fc$.

- If $|A \to B|\_T = tc$, then either $|A|\_T = tn$ and $|B|\_T = tc$ or $|A|\_T = tc$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = tc$ or $|A|\_T = fc$ and $|B|\_T = fc$ or $|A|\_T = fc$ and $|B|\_T = fn$

Assume $|A \to B|\_T = tc$ but $|A|\_T = fn$ or $|B|\_T = tn$ or $|A|\_T = tc$ (tn or tc) and $|B|\_T = f$ (fn or fc).
If $|A|\_T = fn$ or $|B|\_T = tn$, then $|A \to B|\_T = tn$ (proved before). If $|A|\_T = t$ and $|B|\_T = f$, then $\neg(A \to B) \in T$, contradiction.

Therefore, $|A|_T = tn$ and $|B|_T = tc$ or $|A|_T = tc$ and $|B|_T = tc$ or $|A|_T = fc$ and $|B|_T = tc$ or $|A|_T = fc$ and $|B|_T = fc$ or $|A|_T = fc$ and $|B|_T = fn$.

Verify Modal Operator (□)

KTm: $|\Box A|_T$: $\Box tn = t$, $\Box tc = f$, $\Box fc = f$, $\Box fn = f$. $|A|_T = tn$: $A \in T$, $\Box A \in T$, $|\Box A|_T = t$. $|A|_T = tc$: $A \in T$, $\neg\Box A \in T$, $|\Box A|_T = f$. $|A|_T = fc$: $\neg A \in T$, $\neg\Box\neg A \in T$, $\neg A \to \neg\Box A \in T$ (contraposition to T), $\neg\Box A \in T$, $|\Box A|_T = f$. $|A|_T = fn$: $\neg A \in T$, $\Box\neg A \in T$, $\neg A \to \neg\Box A \in T$ (contraposition to T), $\neg\Box A \in T$, $|\Box A|_T = f$.

Axioms in $|\cdot|_T$

As the interpretation $|\cdot|_T$ preserves the table definitions for all logical connectives and the modal operator, every axiom of the KTm system is a valid formula, and the inference rules maintain this validity.

Conclusion

$|\cdot|_T$ matches KTm semantics (¬, →, □). All axioms hold in T, and validity (tn or tc) is preserved.

Assume formula E is valid but is not provable in KTm. Then ¬¬E is also not provable in KTm. Therefore the set {¬E} is consistent with KTm, it can be expanded to maximal set T, by lemma 1. All formulas from T are valid in the interpretation given in lemma 2, therefore ¬E is also valid in this interpretation. Thus formula E is not valid in this interpretation, which contradicts to the assumption. The completeness theorem is proved.

4.4 System KT4m: Reflexive and Transitive Logic

KT4m aligns with Kripke's S4 (reflexive and transitive accessibility).

In Kripke semantics, KT4 (akin to S4) has a reflexive and transitive accessibility relation. Reflexivity ensures □p implies p in the actual world, and transitivity (if w accesses v and v accesses u, then w accesses u) strengthens necessity. In the sequence model, $tn = \langle t, t, t, ...\rangle$ fully satisfies transitivity, so $\Box tn = tn$ as necessity propagates indefinitely. For $tc = \langle t, f, f, ...\rangle$, $\Box tc = f$ (fc or fn) since truth doesn't persist transitively. For $fc = \langle f, t, t, ...\rangle$ and $fn = \langle f, f, f, ...\rangle$, $\Box fc = f$ and $\Box fn = f$ because neither ensures truth in w_0 (reflexivity) nor across all transitively reachable worlds, restricting necessity to tn alone.

- **Modal Operator (□)**:

p	□p
tn	tn
tc	f
fc	f
fn	f

- **Axioms**:

1. Propositional Tautologies
2. K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
3. T: $\Box p \rightarrow p$
4. 4: $\Box p \rightarrow \Box\Box p$

- **Rules**: Modus Ponens, Necessitation.

Soundness Theorem: All theorems of KT4m are valid.

Proof:

- **Axiom 1**: Tautologies valid (see KTm).
- **Axiom 2**: **□(p → q) → (□p → □q)**

Assume there exists an interpretation where □(p → q) → (□p → □q) takes fn or fc. Then □(p → q) is t (tn or tc) and (□p → □q) is f(fn or fc). Then □p is t, and □q is f. By definition of modal operator, p is tn, and q is tc, fc, or fn. Then, from definition of implication, p → q takes tc, fc or fn, so □(p → q) takes f, contradiction to the assumption. Formula is valid.

- **Axiom 3: □p → p**

Assume □p → p takes f for some p. Then □p is t, and p is f. But since □p is t, then p is tn, and we get contradiction. Axiom 3 is valid.

- **Axiom 4: □p → □□p**

□	p	→	□	□	p

tn	tn	tn	tn	tn	tn

If p = tn then Axiom 4 takes tn. If p is tc, fc or fn, then □p takes f, and Axiom 4 takes t. Therefore, Axiom 4 is valid.

- **Rules**: Same as KTm.

Completeness Theorem: Every valid formula in RMS for KT4m is provable.

- Proof for **Lemma 1** and **Lemma 2**: the same as for KTm.
- **Conclusion**: Same as KTm.

4.5 System S5m: Reflexive, Transitive, and Symmetric Logic

S5m matches Kripke's S5 (full accessibility).

In Kripke semantics, S5 is characterized by a reflexive, transitive, and symmetric (equivalence) relation, ensuring all worlds are mutually accessible. Thus, □p is true at a world if and only if p is true in all worlds, with no room for contingency under this universal accessibility. In the RMS sequence model, where truth values are represented as sequences over worlds with w_0 as the actual world, if p = tn = <t, t, t, ...>, then by transitivity, □p = tn, as p's truth persists across all accessible worlds, securing absolute necessity. However, if p ≠ tn—taking values tc = <t, f, f, ...>, fc = <f, t, t, ...>, or fn = <f, f, f, ...>—then □p cannot be true (i.e., neither tn nor tc), since p fails to hold universally; by Axiom 5 (◇p → □◇p, or equivalently ¬□¬p → □¬□¬p), it follows that ¬□p must be necessary (¬□p→ □¬□p), implying □p = fn, as necessary falsity is the only consistent value in S5m for such cases. Thus, for p = tc, fc, or fn, □p = fn, reflecting that any deviation from necessary truth collapses to necessary falsity under S5's equivalence structure.

- **Modal Operator (□)**:

p	□p
tn	tn
tc	fn
fc	fn

fn	fn

- **Axioms**:

1. Propositional Tautologies
2. K: □(p → q) → (□p → □q)
3. T: □p → p
4. 4: □p → □□p
5. 5: ◇p → □◇p

- **Rules**: Modus Ponens, Necessitation.

Soundness Theorem: All theorems of S5m are valid.

Proof:

- **Axiom 1**: Tautologies valid (see KTm).
- **Axiom 2**: Proof similar to previous systems. Valid.
- **Axiom 3: □p → p** Proof similar to previous systems. Valid.
- **Axiom 4: □p → □□p** Proof similar to previous systems. Valid.
- **Axiom 5: ◇p → □◇p**

¬	□	¬	p	→	□	¬	□	¬	p
tn	fn	fn	tn	tn	tn	tn	fn	fn	tn
tn	fn	fc	tc	tn	tn	tn	fn	fc	tc
tn	fn	tc	fc	tn	tn	tn	fn	tc	fc
fn	tn	tn	fn	tn	fn	fn	tn	tn	fn

Axiom 5 is a valid formula.

- **Rules**: Same as KTm.

Completeness Theorem: Every valid formula in RMS for S5m is provable.

Proof for **Lemma 1** and **Lemma 2**: the same as for KTm.

- **Conclusion**: Same as KTm.

4.6 RMS Semantics for Non-Reflexive Modal Logics

Before delving into the specifics of the KDm system, it's worth considering how Resolution Matrix Semantics (RMS) can be applied in the opposite direction from our previous explorations—toward modal systems weaker than KTm, where the Kripkean framework lacks reflexivity. In reflexive systems like KTm, the accessibility relation ensures that every world accesses itself, constraining the behavior of the modal operator □ and its truth values.

Our intuition suggests that relaxing this condition—moving to systems without reflexivity—should result in even broader, less determinate truth values for □, reflecting greater semantic flexibility due to fewer structural restrictions. KDm, corresponding to Kripke's KD system with seriality but no reflexivity, serves as a confirmation of this intuition.

Here, we apply RMS semantics to analyze a non-reflexive logic, observing how the definition of the modal operator adapts to this weaker framework. KDm's modal operator allows a broader range of indeterminate values than KTm. One sub-interpretation subtly restricts the implication matrix to align precisely with the system's properties. For instance, when p = fc and q = tc, p → q yields tc instead of t (tn or tc). Similarly, when p = q = tc or p = q = fc, p → q results in tn rather than t (tn or tc).

4.6.1 System KDm:

System KDm corresponds to Kripke's KD system, featuring a serial accessibility relation (for every world w, there exists some v such that wRv). In RMS, KDm uses the same language and foundational semantics as KTm, with adjusted axioms and modal operator definitions.

In the development of RMS semantics, system KDm stands out as the first system in this paper to incorporate the indeterminate truth value "t/f," which can resolve to any of the four values: tn, tc, fc, or fn (or their combinations). This indeterminacy arises specifically when p takes the value fc (contingent false), where there exists a possibility that □p is true—meaning p could be false in the actual world (w_0) but true in all accessible worlds in the sequence, such as <f, t, t, ...>.

Suppose p = fc = <f, t, t, t, ...>, meaning p is false in the actual world (w_0) but true in all accessible worlds (w_1, w_2, ...), and q = tc = <t, f, f, f, ...>, meaning q is true in w_0 but false in all or at least in some accessible worlds. In Kripkean terms, fc → tc would evaluate p → q across worlds: at w_0, <f → t> = t, but in accessible worlds, for some number i, (e.g., w_i), <t → f> = f, since t at w_i (for p) implies f at w_i (for q), yielding falsehood. Thus, in RMS for KDm, fc → tc = tc in this sub-interpretation, reflecting contingency rather than necessity, which we refine in the implication matrix below to maintain consistency with KD's weaker structure.

Since KDm corresponds to Kripke's KD system with a serial but non-reflexive accessibility relation, the lack of reflexivity allows p to be false in the actual world while still permitting □p to hold if all subsequent worlds are true, rendering □p's truth value undetermined in the general case and thus spanning the range {t, f}. This marks a significant departure from the more constrained modal operator definitions in reflexive systems like KTm, highlighting KDm's greater semantic flexibility.

4.6.2 RMS Semantics for KDm

Implication (→):

p → q	tn	tc	fc	fn
tn	tn	tc	fc	fn
tc	tn	t/tn*	fc	fc
fc	tn	t/tc*	t/tn*	tc
fn	tn	tn	fn	fn

*) sub-interpretation for p = fc, □p = t

- **Modal Operator (□)**:

p	□p
tn	t
tc	f
fc	f/t*
fn	f

*) sub-interpretation for p = fc, □p = t

Axioms and Rules

- **Axioms**:
    1. Propositional Tautologies (e.g., p ∨ ¬p).
    2. Distribution Axiom (K): □(p → q) → (□p → □q).
    3. Seriality Axiom (D): □p → ¬□¬p.
- **Inference Rules**:
    - Modus Ponens (MP): From p and p → q, infer q.
    - Necessitation (Nec): From p, infer □p (if KDm ⊢ p).

Soundness Theorem

Theorem: KDm is sound—for every theorem A in KDm, A is valid in RMS (i.e., |A|_I ∈ {tn, tc} in every sub-interpretation I' of every interpretation I).

Proof: We must show that all axioms are valid and the inference rules preserve validity.

- **Axiom 1: Propositional Tautologies**
  Consider p ∨ ¬p.

p	¬p	p v ¬p (p → ¬p)
tn	fn	tn
tc	fc	tn/tc
fc	tc	tn/tc
fn	tn	tn

Axiom 1 is a valid formula.

- **Axiom 2: Distribution Axiom (□(p → q) → (□p → □q))**

□	(p	→	q)	→	(□p	→	□q)
f	tn	fn	fn	t	t	f	f
f	fc*	tc	fn	t	t	f	f
f	tn	fc**	fc	t	t	f	f
f	tn	tc	tc	t	t	f	f

*) 1^{st} sub-interpretation □fc = t;

**) 2^{nd} sub-interpretation □fc = f.

- **Axiom 3: Seriality Axiom (□p → ¬□¬p)**

□	p	→	¬	□	¬	p
t	tn	t	t	f	fn	tn
f	tc	t	t/f	f/t	fc	tc
t/f	fc	t	t	f	tc	fc
f	fn	t	f	t	tn	fn

Axiom 3 is a valid formula.

- **Inference Rule 1: Modus Ponens**
  If |p|_I = tn or tc and |p → q|_I = tn or tc, then |q|_I = tn or tc (KDm implication table applies).
- **Inference Rule 2: Necessitation**
  Same proof as in KTm.

Conclusion: All axioms and rules hold; KDm is sound.

Completeness Theorem

Theorem: KDm is complete—for every valid formula A in RMS, KDm ⊢ A.

Proof: We use maximal consistent sets, adapting the KTm approach.

- **Lemma 1**: A consistent set Γ can be extended to a maximal consistent set T with:
  - For all p, exactly one of (p ∧ □p), (p ∧ ¬□p), (¬p ∧ ¬□¬p ∧ ¬□p), (¬p ∧ ¬□¬p ∧ □p) or (¬p ∧ □¬p) ∈ T.
  - If Γ ⊢ B and Γ ⊆ T, then B ∈ T.
  - B ∨ C ∈ T iff B ∈ T or C ∈ T.
  - B ∧ C ∈ T iff B ∈ T and C ∈ T.
    Proof: Enumerate formulas, add each if consistent with KDm axioms (K, D), ensuring maximality via closure and contradiction avoidance (as in KTm).
- **Lemma 2**: There exists a valuation |·|_T such that:
  - |A|_T = tn iff A ∈ T and □A ∈ T.
  - |A|_T = tc iff A ∈ T and ¬□A ∈ T.
  - |A|_T = fc iff ¬A ∈ T and ¬□¬A ∈ T and ¬□A ∈ T in sub-interpretation □fc=f;
  - |A|_T = fc iff ¬A ∈ T and ¬□¬A ∈ T and □A ∈ T in sub-interpretation □fc=t.

- |A|_T = fn iff ¬A ∈ T and □¬A ∈ T.

Proof of Lemma 2:

- **Define the Valuation |·|_T**:
  By Lemma 1, T assigns each p exactly one of:
  - p ∧ □p (tn: true and necessary).
  - p ∧ ¬□p (tc: true but not necessary).
  - ¬p ∧ ¬□¬p ∧ ¬□p (fc: false but not necessarily false and not necessarily true).
  - ¬p ∧ ¬□¬p ∧ □p (fc: false and necessarily true and therefore not necessarily false).
  - ¬p ∧ □¬p (fn: false and necessarily false).

  This ensures |p|_T ∈ {tn, tc, fc_1, fc_2, fn} uniquely for propositional variables.
- **Verify Negation (¬)**:
  - |A|_T = tn: A ∈ T, □A ∈ T. ¬¬A ∈ T, □¬¬A ∈ T (tautologies, MP), so |¬A|_T = fn.
  - |¬A|_T = fn: ¬A ∈ T, □¬A ∈ T. ¬¬A ∈ T, □A ∈ T, so |A|_T = tn.
  - |A|_T = tc: A ∈ T, ¬□A ∈ T. ¬¬A ∈ T, ¬□¬(¬A) ∈ T, so (¬(¬A) ∈ T, ¬□¬(¬A) ∈ T and ¬□(¬A) ∈ T) or (¬(¬A) ∈ T, ¬□¬(¬A) ∈ T and □(¬A) ∈ T) so |¬A|_T = fc.
  - |¬A|_T = fc: (¬(¬A) ∈ T, ¬□¬(¬A) ∈ T and ¬□(¬A) ∈ T) or (¬(¬A) ∈ T, ¬□¬(¬A) ∈ T and □(¬A) ∈ T), therefore (¬A) ∈ T, ¬□(¬A) ∈ T, so |¬A|_T = tc.
  - |A|_T = fn: ¬A ∈ T, □¬A ∈ T. |¬A|_T = tn.
  - |¬A|_T = tn: (¬A) ∈ T, □(¬A) ∈ T, |A|_T = fn.
- **Verify Implication (→)**:
  - The proof is similar to the same proof in the system KTm.
  - Let's show the case: if p = fc, q = tc, then p → q = t/tc.
  - **1^{st} sub-interpretation (for p = fc, □p = f):**
  - Let's prove that if p = fc, q = tc, then p → q = t
  - We have ¬p ∈ T and ¬□¬p ∈ T and ¬□p ∈ T, q ∈ T, ¬□q ∈ T. Assume p → q = f, then p ∈ T and ¬q ∈ T, contradiction
  - **2^{nd} sub-interpretation (for p = fc, □p = t):**
  - We prove that if p = fc, q = tc, then p → q = tc
  - We have ¬p ∈ T and ¬□¬p ∈ T and □p ∈ T, q ∈ T, ¬□q ∈ T. Assume p → q = f, then p ∈ T and ¬q ∈ T, contradiction. Now assume p → q = tn, therefore □(p → q) ∈ T, (□p → □q) ∈ T (by axiom K), and since □p ∈ T, we get □q ∈ T, contradiction.
- **Verify Modal Operator (□)**:

- $|A|\_T = tn$: $A \in T$, $\Box A \in T$, $|\Box A|\_T = t$.
- $|A|\_T = tc$: $A \in T$, $\neg\Box A \in T$, $|\Box A|\_T = f$.
- $|A|\_T = fc$: $\neg A \in T$, $\neg\Box\neg A \in T$, $\neg\Box A \in T$, $|\Box A|\_T = f$ – 1st sub-interpretation (for p = fc, $\Box p = f$)
- $|A|\_T = fc$: $\neg A \in T$, $\neg\Box\neg A \in T$, $\Box A \in T$, $|\Box A|\_T = t$ – 2nd sub-interpretation (for p = fc, $\Box p = t$)
- $|A|\_T = fn$: $\neg A \in T$, $\Box\neg A \in T$, $\Box\neg A \rightarrow \neg\Box A \in T$ (D), $\neg\Box A \in T$, $|\Box A|\_T = f$.

- **Axioms in $|\cdot|\_T$**:
  - Tautologies, K, D hold (as in soundness).
  - $|\cdot|\_T$ matches KDm semantics.
- **Conclusion**: If A is valid but not provable, ¬A is consistent, extends to T, $|\neg A|\_T$ = tn or tc, $|A|\_T$ = fn or fc, contradicting validity. KDm is complete.

4.6.3 System Km (Minimal Logic)

System Km corresponds to Kripke's minimal modal system K, which imposes no constraints on the accessibility relation—no requirements such as seriality, reflexivity, transitivity, or symmetry apply. In Resolution Matrix Semantics (RMS), Km adopts the same language and foundational semantics as other systems (e.g., KTm, KDm), but its lack of relational restrictions results in a highly flexible modal operator definition. Unlike KDm, which introduces seriality and constrains □fn to f, Km permits □fn to take t/f - fully indeterminate values in certain cases, reflecting the absence of structural assumptions about accessible worlds.

In the context of a single, non-reflexive world with no accessible worlds within System Km, any proposition p's truth value is determined solely by its value at w_0. If p is false at w_0, its truth value is fn (necessary false). However, the necessity operator □p is true at w_0 because p is considered true in all accessible worlds—vacuously true since there are none. Thus, the truth value of □p is tn (necessary truth). This scenario illustrates that in System Km, for p with truth value fn, □p can take the value t (either tn or tc), specifically tn in this case, aligning with the system's definition where □p = t/f for p = fn. This highlights the indeterminacy inherent in System Km, where the absence of constraints on the accessibility relation allows for such cases, and notably, both □p and □¬p can be tn, reflecting the vacuous truth of necessity in isolated worlds.

RMS Semantics for Km

Km employs the same language as KDm.

Negation (¬)

The Negation matrix definition is the same as in KDm.

Implication (→)

p → q*	tn	tc	fc	fn
tn	tn	tc	fc	fn
tc	tn	t/tn	fc	fc
fc	tn	t/tc	t/tn	tc
fn	tn	tn	fn	fn

*) 1st sub-interpretation: for p = fn, □p = f

p → q**	tn	fn
tn	tn	tn
fn	tn	tn

**) 2nd sub-interpretation: for p = fn, □p = t; no tc or fc

Modal Operator (□)

In Kripke's K, □p is true at w_0 if p is true in all worlds accessible from w_0, but with no constraints on R, □p's value depends entirely on the interpretation. In RMS:

p	□p
tn	t
tc	f
fc	f/t*
fn	f/t**

*) for p = fc, □p = f, this value is f; for p = fc, □p = t, this value is t.

**) for p = fn, □p = f, this value is f; for p = fn, □p = t, this value is t.

For the modal operator definition, if p = fn and □p = t, in fact, there are no contingent truth values, as it was mentioned before. So, for this case, the definition for modal operator is as follows:

p	□p
tn	t
fn	t

Axioms and Rules

Axioms:

1. Propositional Tautologies (e.g., p ∨ ¬p).
2. Distribution Axiom (K): □(p → q) → (□p → □q).

Inference Rules:

1. Modus Ponens (MP)
2. Necessitation (Nec)

Km omits additional axioms like D, T, 4, or 5, making it the baseline modal system.

Soundness Theorem

Theorem: Km is sound—for every theorem A in Km, A is valid in RMS (i.e., |A|_I ∈ {tn, tc} in every sub-interpretation I' of every interpretation I).

Proof: Show that all axioms are valid and inference rules preserve validity.

Axiom 1: Propositional Tautologies

The same way as in previous cases; tautologies take only tn or tc.

Axiom 2: Distribution Axiom (□(p → q) → (□p → □q))

The proof for for p = fn, □p = f, 1st sub-interpretation, is the same as for system KDm. Now consider 2nd sub-interpretation, p = fn, □p = t.

Since □q is a valid formula in this sub-interpretation, then (□(p → q) → (□p → □q)) is valid.

The proof for inference rules is the same as in previous systems.

Conclusion: All axioms and rules hold; Km is sound.

Completeness Theorem

Theorem: Km is complete—for every valid formula A in RMS, Km ⊢ A.

Proof: Use maximal consistent sets, adapting the KDm approach.

Lemma 1

A consistent set Γ can be extended to a maximal consistent set T with:

- For all p, exactly one of (p ∧ □p), (p ∧ ¬□p), (¬p ∧ ¬□¬p), or (¬p ∧ □¬p) ∈ T.
- If Γ ⊢ B and Γ ⊆ T, then B ∈ T.
- B ∨ C ∈ T iff B ∈ T or C ∈ T.
- B ∧ C ∈ T iff B ∈ T and C ∈ T.

Proof: Enumerate formulas, add each if consistent with Km axioms (K), ensuring maximality via closure and contradiction avoidance (as in KDm).

Lemma 2

There exists a valuation |·|_T such that:

- |A|_T = tn iff A ∈ T and □A ∈ T.
- |A|_T = tc iff A ∈ T and ¬□A ∈ T.
- |A|_T = fc iff ¬A ∈ T and ¬□¬A ∈ T and ¬□A ∈ T in sub-interpretation □fc=f;
- |A|_T = fc iff ¬A ∈ T and ¬□¬A ∈ T and □A ∈ T in sub-interpretation □fc=t.
- |A|_T = fn iff ¬A ∈ T and □¬A ∈ T and ¬□A ∈ T in sub-interpretation □fn=f;
- |A|_T = fn iff ¬A ∈ T and □¬A ∈ T and □A ∈ T in sub-interpretation □fn=t.

Proof of Lemma 2:

- **Define the Valuation |·|_T:**
- p ∧ □p (tn: true and necessary).
- p ∧ ¬□p (tc: true but not necessary).
- ¬p ∧ ¬□¬p ∧ ¬□p (fc: false but not necessarily false and not necessarily true, sub interpretation).
- ¬p ∧ ¬□¬p ∧ □p (fc: false but not necessarily false and necessarily true, sub interpretation).
- ¬p ∧ □¬p∧ ¬□p (fn: false and necessarily false and not necessarily true, sub interpretation).

- ¬p ∧ □¬p∧ □p (fn: false and necessarily false and necessarily true, sub interpretation).

  This ensures |p|_T ∈ {tn, tc, fc_1, fc_2, fn_1, fn_2} uniquely for propositional variables.

- **Verify Negation (¬):** Proof similar to KDm.
- **Verify Implication (→):** Similar to KDm, adjusted for Km's implication table
- **Verify Modal Operator (□):** Similar to KDm, adjusted for Km\s modal operator table

|·|_T matches Km semantics.

- **Conclusion**: If A is valid but not provable, ¬A is consistent, extends to T, |¬A|_T = tn or tc, |A|_T = fn or fc, contradicting validity. Km is complete.

5. Tableau Method for Resolution Matrix Semantics

The tableau method, a classical tool in propositional logic for testing validity, systematically decomposes formulas into their subformulas to determine if a contradiction arises under all possible truth assignments [2, 10]. In Resolution Matrix Semantics (RMS), we adapt this approach to modal contexts by leveraging the finite set of truth values—necessary truth (tn), contingent truth (tc), contingent false (fc), and necessary false (fn)—alongside indeterminate values (t, f, t/f). A formula A is valid in an RMS system if its negation ¬A cannot be assigned a "true" value (tn or tc) in any interpretation, meaning every branch of the tableau closes (i.e., contains a contradiction). This section outlines the tableau method for RMS and provides examples for the system KTm, illustrating cases where a formula is a theorem (valid) and where it is not.

Tableau Rules in RMS

The tableau begins with the negation of the formula to be tested (¬A), assuming it takes a true value (tn or tc). We then apply decomposition rules based on the RMS truth tables for connectives (¬, →, □) and modal operators, splitting branches for indeterminate values (t, f, t/f) into sub-interpretations. A branch closes if it assigns contradictory values to a subformula (e.g., p = tn and p = fn).

For instance, the following is the complete set of tableau rules for the KTm system. Each application of the tableau rules begins with the **FA** rule, followed by the application of all relevant rules to decompose the initial formula, continuing until either all branches close or at least one branch remains open.

Tableau Rules – KTm:

TA

TA	
TnA	TcA

FA

FA	
FnA	FcA

Tn¬

Tn(¬A)
FnA

Tc¬

Tc(¬A)
FcA

Fc¬

Fc(¬A)
TcA

Fn¬

Fn(¬A)
TnA

Tn→

Tn(A → B)				
FnA	TnB	TcA, TcB	FcA, TcB	FcA, FcB

Tc→

Tc(A → B)				
TnA, TcB	TcA,TcB	FcA, TcB	FcA, FcB	FcA, FnB

Fc→

Fc(A → B)		
TnA, FcB	TcA, FcB	TcA, FnB

Fn→

Fn(A → B)
TnA, FnB

Tn□

Tn(□A)
TnA

Tc□

Tc(□A)
TnA

Fc□

Fc□A		
TcA	FcA	FnA

Fn□

Fc□A		
TcA	FcA	FnA

Examples

1. Let's show □p → p is a theorem of KTm.

F(□p → p)			
Fc(□p → p)			Fn(□p → p)
Tn□p, Fcp	Tc□p, Fcp	Tc□p, Fnp	Tn□p, Fnp
Tnp, Fcp	Tnp, Fcp	Tnp, Fnp	Tnp, Fnp

Each branch is closed, as we have contradictions in each branch. Therefore, formula □p → p is a theorem of KTm.

2. Now we show that p → □p is not a theorem of KTm.

F(p → □p)											
Fc(p → □p)									Fn(p → □p)		
Tnp, Fc□p			Tcp, Fc□p			Tcp, Fn□p			Tnp, Fn□p		
Tnp, Tcp	Tnp, Fcp	Tnp, Fnp	Tcp, Tcp	Tcp, Fcp	Tcp, Fnp	Tcp, Tcp	Tcp, Fcp	Tcp, Fnp	Tnp, Tcp	Tnp, Fcp	Tnp, Fnp

In the selected cells, there is no contradiction; selected branches remain open, therefore, formula p → □p is not a theorem of KTm.

The tableau method in RMS efficiently tests validity by exploring truth value assignments, closing branches when contradictions arise. This adaptation enhances RMS's practicality, as noted in the Introduction, offering a systematic alternative to Kripkean model-checking.

6 Relationship Between Modal Operator Truth Values and System Generality

In this chapter, we examine the interplay between the axiomatic structure of modal systems Km, KDm, KTm, KT4m, and S5m and the truth values assigned to their modal operators (□) in Resolution Matrix Semantics (RMS). These systems constitute a hierarchy in which the axioms of Km form a subset of KDm, those of KDm a subset of KTm, those of KTm a subset of KT4m, and those of KT4m a subset of S5m, embodying a progressive refinement of their logical constraints.

The truth values assigned to □ exhibit a reverse pattern: the more general the system (with fewer axioms), the more indeterminate or vague its modal operator truth values, while the more specific systems (with more axioms) constrain these values progressively. This inverse relationship underscores a key insight of RMS: generality in axiomatic structure correlates with greater flexibility or ambiguity in semantic assignments. Here, we extend this analysis with philosophical reasoning on how the indeterminacy of □ in RMS mirrors the degree of restriction on the accessibility relation R in Kripkean semantics, particularly with the updated, stricter definition of □ for S5m, and explore its implications for philosophy and science.

6.1 Axiomatic Hierarchy

The systems Km, KDm, KTm, KT4m, and S5m correspond to the Kripkean systems K, KD, KT (T), KT4 (S4), and S5, respectively, with a clear progression in their axioms:

- **Km**: Includes only propositional tautologies and the Distribution Axiom (K: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$), making it the most general system with no restrictions on the accessibility relation.
- **KDm**: Adds the Seriality Axiom (D: $\Box p \rightarrow \neg\Box\neg p$) to Km, requiring at least one accessible world from each world.
- **KTm**: Extends KDm with the Reflexivity Axiom (T: $\Box p \rightarrow p$), ensuring every world accesses itself.
- **KT4m**: Further augments KTm with the Transitivity Axiom (4: $\Box p \rightarrow \Box\Box p$), enforcing transitive accessibility.
- **S5m**: Adds the Symmetry Axiom (5: $\neg\Box\neg p \rightarrow \Box\neg\Box\neg p$) to KT4m, creating a reflexive, transitive, and symmetric (equivalence) relation where all worlds are mutually accessible.

Thus, the set of axioms grows from Km to S5m ($K \subseteq KD \subseteq KT \subseteq KT4 \subseteq S5$), reflecting increasing constraints on the underlying Kripkean accessibility relation: none (Km), serial (KDm), reflexive (KTm), reflexive-transitive (KT4m), and reflexive-transitive-symmetric (S5m).

6.2 Modal Operator Truth Values. Inverse Relationship

Let's compare the truth value assignments for □ across the systems.

	□p				
p	Km	KDm	KTm	KT4m	S5m
tn	t	t	t	tn	tn
tc	f	f	f	f	fn
fc	t/f	t/f	f	f	fn

fn	t/f	f	f	f	fn

The progression of truth values reveals a clear pattern:

- Km has the most indeterminate values (t, f, t/f), with t/f for fc and fn, reflecting its lack of relational constraints.
- KDm narrows this slightly (t, f, t/f), with t/f only for fc, as seriality rules out □fn = t.
- KTm reduces indeterminacy further (t, f), eliminating t/f entirely, as reflexivity fixes □fc = f.
- KT4m constrains values to (tn, f), with □tn = tn and all others f, reflecting the strong reflexive-transitive structure.
- S5m achieves maximal determinacy, allowing only tn or fn, as symmetry, reflexivity, and transitivity enforce a binary necessity/falsity distinction across all mutually accessible worlds.

The moral holds: **the more general a system (fewer axioms), the vaguer its □ truth values in RMS, reflecting fewer restrictions on R in Kripkean semantics**. Km's maximal indeterminacy (t, f, t/f) mirrors an unrestricted R—a blank slate for interpretation. As axioms like seriality, reflexivity, transitivity, and symmetry accrue, □'s values narrow, with S5m's updated (tn, fn) matching KT4m's (tn, f) in determinacy but sharpening f to fn, reflecting the equivalence relation's total constraint. This evolution from Km's vagueness to S5m's precision captures a trade-off between generality and specificity, with RMS offering a truth-value lens that parallels Kripke's relational approach, revealing their deep structural kinship.

7. Applications and Insights

This chapter provides a brief sketch of possible applications of Resolution Matrix Semantics (RMS), exploring how RMS can be applied to fields such as deontic logic, artificial intelligence, and quantum computing, highlighting its ability to model complex, context-dependent reasoning with computational efficiency and philosophical depth. These applications not only demonstrate RMS's adaptability but also offer fresh insights into its potential to bridge theoretical logic with real-world challenges.

7.1 Promising Applications of RMS method in Deontic Logic and Other Modal Logic

RMS approach extends its truth-value-based framework beyond alethic modal systems (Km, KDm, KTm, KT4m, S5m) to deontic logic, which formalizes concepts of obligation (O), permission (P), and prohibition. Unlike Kripkean semantics, which relies on possible worlds and accessibility relations, RMS uses a finite set of truth values—necessary truth (tn), contingent truth (tc), contingent false (fc), and necessary false (fn)—augmented by indeterminate values (t, f, t/f). This approach offers a substantive alternative for modeling deontic operators by assigning truth values directly to normative statements, avoiding relational structures.

In deontic logic, the modal operator O (obligation) can be interpreted in RMS as follows: a proposition Op takes tn if p is obligatory in all contexts (e.g., a universal duty), tc if p is obligatory in the current context but not universally (e.g., a situational duty), fc if p is not obligatory but permissible in some contexts, and fn if p is forbidden (never obligatory). The indeterminate value t (tn or tc) can represent cases where p's obligation status is contextually ambiguous—obligatory in some sense but not precisely determined—while f (fc or fn) indicates a lack of obligation, possibly ranging from permissible to prohibited. This setup allows RMS to capture nuances like deontic dilemmas (e.g., $Op \wedge O\neg p$) by exploring sub-interpretations where conflicting obligations might resolve differently.

Permission (P) can be defined via negation and obligation, e.g., $Pp \equiv \neg O \neg p$, aligning with classical deontic relationships. In RMS, if $\neg p$ is fn (never obligatory to avoid), Pp could be tn (always permitted), while tc for $\neg p$ might yield t for Pp, reflecting contingent permission. The tableau method (Section 4.7) can test deontic formulas' validity, closing branches when contradictions arise (e.g., Op = tn and p = fn), ensuring obligations imply compliance in valid interpretations.

RMS's strength lies in its flexibility to handle deontic paradoxes without relational complexity. For instance, in addressing dilemmas or explosion (e.g., $Oa \wedge O\neg a \supset Ob$), RMS can assign indeterminate values to reflect uncertainty, then resolve them via sub-interpretations, preventing the classical explosion where contradictions imply arbitrary obligations, unlike world-based models. This approach draws inspiration from prior non-Kripkean matrix-based deontic systems [7, 8], adapting RMS's 4-valued structure to normative reasoning, with potential for further refinement in capturing strong versus weak obligations.

In a similar way, the RMS approach can be adapted to other branches of modal logic, such as epistemic (concerning knowledge and belief), doxastic (concerning belief and reasoning), temporal (concerning time and change), and dynamic (concerning actions and updates), among others.

7.2 Artificial Intelligence and Natural Language Processing

Resolution Matrix Semantics (RMS) brings a fresh perspective to artificial intelligence (AI) and natural language processing (NLP) by tackling the ambiguity and uncertainty that define human language and real-world reasoning. With its core set of truth values—necessary truth (tn), contingent truth (tc), contingent false (fc), and necessary false (fn)—supplemented by indeterminate options like t (tn or tc), f (fc or fn), and t/f (fully indeterminate), RMS moves beyond the rigid true/false binary of classical logic. This makes it an ideal tool for AI systems that grapple with imprecise data, offering a streamlined, truth-value-based alternative to the relational complexity of Kripkean semantics. Beyond language, RMS's flexibility extends to cutting-edge fields like quantum computing, where its handling of indeterminacy mirrors the probabilistic nature of quantum mechanics, enriching its utility across technology and science.

In NLP, RMS excels at capturing the graded truth of everyday language. A statement like "the weather is pleasant" isn't simply true or false—it's true in some contexts, false in others—fitting naturally as t, resolving to tn (universally pleasant) or tc (pleasant now) depending on the situation. This nuance boosts AI tasks like sentiment analysis: a review saying "the film was okay" might take t for "positive sentiment," reflecting its ambivalence rather than forcing a binary label. RMS's sub-interpretation mechanism resolves such indeterminacy systematically, enhancing chatbots, translations, and text understanding by aligning with the fuzzy, context-driven nature of human communication. The tailored tableau method further supports this by efficiently testing interpretations, ensuring practical applicability.

For AI decision-making, RMS shines in uncertain environments, such as autonomous vehicles or medical diagnostics. A vehicle sensor detecting "obstacle ahead" with partial confidence might warrant t—tn if backed by multiple readings, tc if tentative—allowing cautious navigation without overreaction. In medicine, "patient has condition X" could be t based on suggestive tests, guiding next steps without hasty judgments. This approach leverages RMS's focus on truth values over world-relations, simplifying computation for real-time systems. Epistemically, RMS models knowledge or belief—say, "the agent knows the path is clear"—with tn for certainty and f for doubt, supporting multi-agent coordination or belief updates with nuanced reasoning.

RMS's reach extends further into quantum mechanics, where its indeterminate values echo the superposition of quantum states. A particle's spin, indeterminate until measured, aligns with t/f—potentially resolving to tn (true across all contexts post-measurement) or fc (false here, true elsewhere)—mirroring quantum indeterminacy before collapse [11]. In quantum computing, RMS could represent qubit states as t pre-measurement, aiding algorithm design by mapping truth values to probabilities or amplitudes. This connection not only ties RMS to AI's theoretical underpinnings but also positions it as a bridge to quantum-enhanced NLP or decision systems, where probabilistic reasoning is key. By uniting language processing with quantum insights, RMS offers AI a versatile, philosophically rich framework that simplifies complexity while embracing the vagueness of reality.

8. Conclusion

This paper has presented Resolution Matrix Semantics (RMS) as an innovative, truth-value-based framework for modal logic, distinct from Kripkean relational semantics. By defining systems Km, KDm, KTm, KT4m, and S5m with an ambient 4-valued structure—necessary truth (tn), contingent truth (tc), contingent false (fc), and necessary false (fn)—supplemented by broad values (t, f, t/f), RMS offers a robust alternative that captures modal nuances without relying on accessibility relations. The proofs of soundness and completeness confirm the logical rigor of Resolution Matrix Semantics (RMS), further supported by a tailored tableau method that systematically validates formulas with practical efficiency, while its applications in deontic logic, artificial intelligence, and quantum computing domains illustrate its versatility.

RMS's emphasis on truth over relations simplifies computation and enhances its relevance to philosophy, technology, and science.

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