From *Notre-Dame* to Norton Dome: Destruction and Reconstruction of a Cathedral of Newtonianism.

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Abstract

The Norton Dome is a beautiful problem in theoretical physics that is supposed to challenge at the same time the principles of causality, inertia and determinism in Newtonian mechanics. A static undeformable ball at the top of a dome of a given shape seems to move spontaneously at a random moment, without the help of any external net force. We try to show here that the perfect rotational symmetry of the problem has not been taken into account as it should be in its solving. In this approach, we distinguish between trajectory study plan and real trajectory plan: the section of the dome in which the object will evolve or not isn't the result of a free choice or a probability but the pure consequence of physics. The differential equations of motion integrated over the entire dome precisely tell us that, if it moves, the ball should take all directions, which brings us back to a basic contradiction, not with determinism or completeness of Newtonian theory, but between the indeterministic solutions themselves: then, under penalty of ubiquity of the ball, its permanent immobility at the dome's summit remains, in accordance with the principle of inertia, the unique physical solution to the Norton's paradox. That will be confirmed by the analysis of six historical cases, followed by a version of Cauchy-Lipschitz's uniqueness theorem for non-Lipschizian systems. Finally, the inertia principle of Newtonian physics will become a mathematical theorem.

[Note to reader:]

This is the 3rd version of the original article. The original text has been globally preserved. The main additions are:

1. *A new Section 4 for extended study: "Historical Cases Revisited" analyzing classical examples (Poisson, Duhamel, Boussinesq, Bertrand) through the lens of the method proposed in this work;*
2. *A mathematical interlude on the Cauchy-Lipschitz/Picard–Lindelöf's theorem.*
3. *The mathematical formulation of an equivalent of the latter theorem for non-Lipschitzian equations in Section 5;*
4. *An updated bibliography.*

Plan

1. Introduction to the Norton's Paradox.

As early as the 19th century, scientists discussed the validity of Newtonian determinism, which had been elevated to sacred dogma a century earlier by Laplacianism (van Strian, 2014). They revealed multiple solutions to certain differential equations arising from the fundamental principle of dynamics, whereas determinism dictated one and only one behavior of a moving body in a force field based on given initial conditions. In the midst of the rise of spiritualism, mathematical objects in turn began to levitate or slide on their own, free wills awoke in matter, and 'phantom actions' were reported at the very heart of the austere rationalism of classical physics. Even the traditional distinction between cause and effect was no longer a given.

However, the fires and blows struck against the cathedral of determinism by these few poltergeists of science were considered anecdotal. "Abnormal" solutions only appeared in situations that are themselves "exotic", imaginary forces or infinite systems of masses pushed to the extreme...until an article by John Norton (2003) where he presents the entirely credible case of indeterminism of a ball in equilibrium placed at the top of a dome of well-defined shape in a most banal gravity field:

> ***3. Acausality in Classical Physics***
>
> While exotic theories like quantum mechanics and general relativity violate our common expectations of causation and determinism, one routinely assumes that ordinary Newtonian mechanics will violate these expectations only in extreme circumstances if at all. That is not so. Even quite simple Newtonian systems can harbor uncaused events and ones for which the theory cannot even supply probabilities. Because of such systems, ordinary Newtonian mechanics cannot license a principle or law of causality. Here is an example of such a system fully in accord with Newtonian mechanics. It is a mass that remains at rest in a physical environment that is completely unchanging for an arbitrary amount of time—a day, a month, an eon. Then, without any external intervention or any change in the physical environment, the mass spontaneously moves off in an arbitrary direction, with the theory supplying no probabilities for the time or direction of the motion.

In the following, we will say indistinctly particle, mass, ball, object…to speak about the unit mass point. First, J. Norton classifies the notion of causality into what he calls "folk science". He supports his thesis with this dome (Norton, 2003):

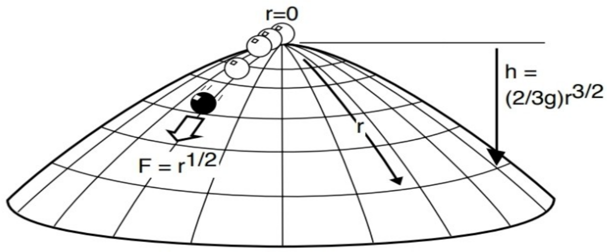


Figure 1a. Mass sliding on a dome

A point-like unit mass slides frictionlessly over the surface under the action of gravity. The gravitational force can only accelerate the mass along the surface. At any point, the magnitude of the gravitational force tangential to the surface is $F = d(gh)/dr = r^{1/2}$ and is directed radially outward. There is no tangential force at r=0. That is, on the surface the mass experiences a net outward directed force field of magnitude $r^{1/2}$. Newton's second law, F=ma, applied to the mass on the surface, sets the radial acceleration d^2r/dt^2 equal to the magnitude of the force field:

(1) $d^2r/dt^2 = r^{1/2}$

It turns out that the dome problem does not satisfy at its summit the conditions of the famous theorem of Cauchy-Lipschitz of 1868 (also named the Picard-Lindelöf theorem: we will use both appellations interchangeably) on the uniqueness of solutions to differential equations of the type: $\ddot{r} = f(r(t))$ with initial conditions $r(0) = r_0$ and $\dot{r}(0) = v_0$.

Mathematical Interlude: The Picard–Lindelöf Theorem

a) Picard–Lindelöf Theorem (Cauchy–Lipschitz) Statement (first-order ODE)

Let us consider the differential equation:

$$\frac{dy}{dt} = f(t, y)$$

with the initial condition $\mathbf{y(t_0) = y_0}$, where $\mathbf{f}$ is defined on a domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ and takes values in $\mathbb{R}^n$.

Assumptions:

There exists a neighborhood of (t_0, y_0) such that:

1. **f(t, y)** is continuous;
2. **f(t, y)** is Lipschitz continuous with respect to **y**, that is:

$$\| f(t, y_1) - f(t, y_2) \| \leq L \cdot \| y_1 - y_2 \|,$$

for all $\mathbf{y_1, y_2}$ in a neighborhood of $\mathbf{y_0}$, and for some constant $\mathbf{L > 0}$.

Conclusion:

Then there exists a time interval **I = [t_0 − ε, t_0 + ε]**, with **ε > 0**, and a unique solution **y(t)** defined on **I**, of class **C^1**, such that:

$$\frac{dy}{dt} = f\left(t, y(t)\right) \quad \text{, with } \mathbf{y(t_0) = y_0}.$$

b) Extension to Second-Order ODEs

Consider the second-order equation:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Let us define a new variable **v(t) = dy/dt**. Then we rewrite the system as:

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = f\left(t, y, v\right)$$

This is a first-order system in two variables:

$$\frac{dY}{dt} = F\left(t, Y\right) \quad \text{, where} \quad Y = \left(y, v\right)$$

Assumptions:

If $f\left(t, y, v\right)$ is continuous and Lipschitz continuous in **(y, v)**, then the system has a **unique solution** near the initial condition.

c) Special case: autonomous second-order ODE

Suppose the equation is autonomous, i.e time-independent:

$$\frac{d^2y}{dt^2} = f\left(y\right)$$

Then define $v = \frac{dy}{dt}$. The system becomes:

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = f\left(y\right)$$

If $f\left(y\right)$ is Lipschitz near the initial condition **y_0**, then the system has a **unique solution**.

On Norton’s dome, the governing equation is:

$$\frac{d^2r}{dt^2}=\sqrt{r}$$

Given that the net force on the right-hand side of the dynamics differential equation is the square root of the curvilinear abscissa **r**, therefore not differentiable at zero with respect to **r**, the Lipschitz condition is not satisfied.

In other words, $f(r)=\sqrt{r}$ is not Lipschitzian in **r = 0** since:

$$\lim_{r\to 0}\frac{(\sqrt{r})}{r}=\infty$$

⇒ the theorem does not apply.

Multiple solutions of the movement are then to be expected. And this is what happens...Norton deduces an infinity of possible solutions: the mass seems capable of moving without cause in any direction and at an arbitrary instant. More precisely, a unitary point mass, initially at perfect rest, will slide without friction, delivered to the sole force of its tangential weight, along the wall of a dome of equation :

$$h=\left(\frac{2}{3g}\right)r^{3/2}$$

In the polar coordinate system attached to the point, the weight vector **P** has the following components:

$$P_r=g\sin\Theta$$

$$P_\Theta=g\cos\Theta$$

where **θ** is the angle between the tangent to the dome at a given point and the horizontal **x**. We then obtain the following relations:

$$\sin\Theta=\frac{dh}{dr} \qquad P_r=\sqrt{r}$$

$$\frac{dh}{dr}=\sqrt{r}/g \qquad P_\Theta=\sqrt{g^2-r}=-R$$

from which we deduce the dynamic equation of the point identified by its curvilinear coordinate **r**:

$$\frac{d^2r}{dt^2} = \sqrt{r}$$

The reaction **R** of the support, directed along the normal to the tangent vector, in turn verifies the equation and the inequality :

$$\frac{1}{\sqrt{r(g^2-r)}}\left(\frac{dr}{dt}\right)^2 = \sqrt{g^2-r} - R$$

$$R \geqslant 0$$

The 2nd condition allows the mass to remain in contact with its support. It is clear that **r** is positive and must remain less than $\mathbf{g^2}$, but the mass takes off as soon as its speed exceeds a certain critical value depending on **r** :

$$v_{crit}(r=0) = 0$$

$$v_{crit}(r>0) = \sqrt[4]{r}\sqrt{g^2-r}$$

We then obtain two types of solutions to our differential equations. One is the classical solution of rest for all **t** of the mass at the top :

$$\forall t,\ r(t) = 0$$

The other new family of solutions that Norton derives is the following :

$$\forall T \geqslant 0,\quad \begin{cases} t < T : & r(t) = 0 \\ t \geq T : & r(t) = \frac{1}{144}(t-T)^4 \end{cases}$$

In other words, at any instant **T**, the ball at rest, in perfect equilibrium between its weight and the reaction of the support (therefore a zero net force), leaves its summit and begins to slide without any added physical intervention. There is an apparent violation of causality (no reason for the movement) and of the principle of inertia according to which any mass at rest or in uniform rectilinear translation perseveres in its state as long as no external net force acts on it.

Another issue is how such a breaking of symmetry (a random trajectory starting from the top) can occur in such a perfectly symmetrical problem ? Newtonian mechanics should respect the famous principle of symmetry…In reality, as we will see, the latter also applies to problems with multiple solutions when these are superimposed. This is the case for Norton's possible dynamical solutions around the axis.

But the fact that **T** is arbitrary also implies a contradiction with determinism: the same initial state seems to lead to an infinity of possible trajectories. According to Norton, indeterminism is declared but the principle of inertia would be safe because no force is exerted on the ball at the « excitation time » **t=T** and outside there is no first instant where the movement would not be accompanied by a force.

This idea would be questionable in itself if we consider that the force (collinear to acceleration) "precedes" the velocity and position of the movement. Indeed, by deriving the position **r(t)** repeatedly with respect to time, a constant appears at the *4th* derivative:

$$\frac{dr}{dt} = \frac{1}{36}(t-T)^3 \qquad \frac{d^3r}{dt^3} = \frac{1}{6}(t-T) \qquad \frac{d^4r}{dt^4} = \frac{1}{6}$$

$$\frac{d^2r}{dt^2} = \frac{1}{12}(t-T)^2$$

While everything else is at rest, something seems to be brewing at the level of the "acceleration" of the net force at **t=T** (called the *jounce*), which will "then" impact (in the reverse order of successive integrations) the force itself, then the speed and finally the position from the following instant $\mathbf{T^+ = T + dT}$.

We find ourselves in a weird situation where the principle of inertia would be never violated "punctually" (at any instant **t**) but always "globally" (between two instants **T** and $\mathbf{T^+ > T}$), since no external net force, apart from the two forces in equilibrium at the initial time, acts on the system at rest, nor later when it starts. It is not certain that this last formulation is not in real contradiction with the definition of the inertia principle or one of its consequences.

In fact, the principle of inertia considers in a sense as « internal » the forces exerted on the initial system in equilibrium (rest or pseudo-equilibrium), to be distinguished from the « external » forces of which it speaks that would disturb this system at a later time.

In the case of the dome, knowing that no force other than the actions of the weight and the support on the object intervenes at any moment of the experiment, the movement is only a result of the « internal » forces of the initial moment, without any external disturbance, hence its spontaneous nature by definition.

In this sense, there is indeed a contradiction between the spontaneous solutions of the Norton dome and the principle of inertia - whose corresponding solution is ***r(t)=0*** for all **t**. But knowing that both come from the resolution of Newtonian equations, it then becomes difficult to say which one should be dismissed as unphysical, or at least contrary to the physical formalism used.

One would need here a sort of impartial arbiter, outside of strict Newtonian physics, to decide between them – a referee who will be sought further in some logical consistency of physical movement. In our view, the Norton's dome is definetely more remarkable for its spectacular and rather unprecedented violation of the inertia principle than for its indeterminism (the latter not being rare in problems like those of the three-body type).

Besides, Norton's approach has also be criticized for forcibly 'agglutinating' heterogeneous solutions with different initial conditions (the lasting rest of the ball where all the quantities are zero up to time **T**, and its movement from a pseudo-rest at **T** where the acceleration of the force would be equal to *1/6*), which would be contrary to good practice in physics (G. Davies, 2017).

However, this counter-argument does not quite hold up if we limit ourselves to the case **T=0**, with then only one type of solutions and only one set of initial conditions, although the paradox persists. Furthermore, laws of motion have been perfectly applied on the half-profile: nothing indicated the subsequent emergence of these position derivatives of order higher than two in the initial state.

Then, we will see that the truth may lie elsewhere...

2. Taking into account the rotational symmetry.

The crucial moment when geometry is mentioned in Norton's article is in the following passage:

> *Properties*
>
> Two distinct features of this spontaneous excitation require mention.
>
> *No cause.* No cause determines when the mass will spontaneously accelerate or the direction of its motion. The physical conditions on the dome are the same for all times t prior to the moment of excitation, t=T, and are the same in all directions on the surface.
>
> *No probabilities.* One might think that at least some probabilistic notion of causation can be preserved in so far as we can assign probabilities to the various possible outcomes. Nothing in the Newtonian physics requires us to assign the probabilities, but we might choose to try to add them for our own conceptual comfort. It can be done as far as the *direction* of the spontaneous motion is concerned. The symmetry of the surface about the apex makes it quite natural for us to add a probability distribution that assigns equal probability to all directions. The complication is that there is no comparable way for us to assign probabilities for the *time* of the spontaneous excitation that respect the physical symmetries of solutions (3). Those solutions treat all candidate excitation times T equally. A probability distribution that tries to make each candidate time equally likely cannot be proper—that is, it cannot assign unit probability to the union of all disjoint outcomes.[7] Or one that is proper can only be defined by inventing extra physical properties, not given by the physical description of the dome and mass, Newton's laws and the laws of gravitation, and grafting them unnaturally onto the physical system.[8]

We will discuss this postulate according to which the physical direction of the mobile's trajectory could be modeled by a uniform probability law of the type $\mathbf{dP = d\varphi/360°}$, with $\boldsymbol{\varphi}$ the angle of rotation around the vertical axis $\mathbf{h}$.

Of course, nothing prevents choosing a study section in the sense of a work plan (e.g. a profile view of the dome) to apply the laws of physics and predict the direction that the ball will follow at the top, hence the real section of its evolution. But these two types of direction, one (free) for the study of the problem, the other (imposed) that the laws of motion dictate to us, must not be confused: it is unjustified here to freely assign a direction (certain or probable) to the mobile since it is up to physics to say so. The latter is full of examples (electromagnetism, inertial forces in an accelerated frame, Coriolis forces, etc.) where the direction followed by the object does not belong to the work plan.

Yet we find this confusion between physical direction and study direction recurrently in the literature related to the dome by reading that the mass at rest begins to slide spontaneously from the summit in an "any direction". In the problem that concerns us, Norton himself obtained two types of possible physical solutions for each study section/plan/direction:

1. Constant rest for all **t**,

2. Rest until **t < T**, then “acausal” movement (at time **T**) *in the same direction*.

It is to this "binary" rule of the game that we will have to limit ourselves. For each of the directions around the vertical axis, nothing in the Newtonian physics of the dome indicates the possibility for the object to behave differently, let alone follow the direction of our wishes. For the physical analysis, intuition guides us to a section of the dome.

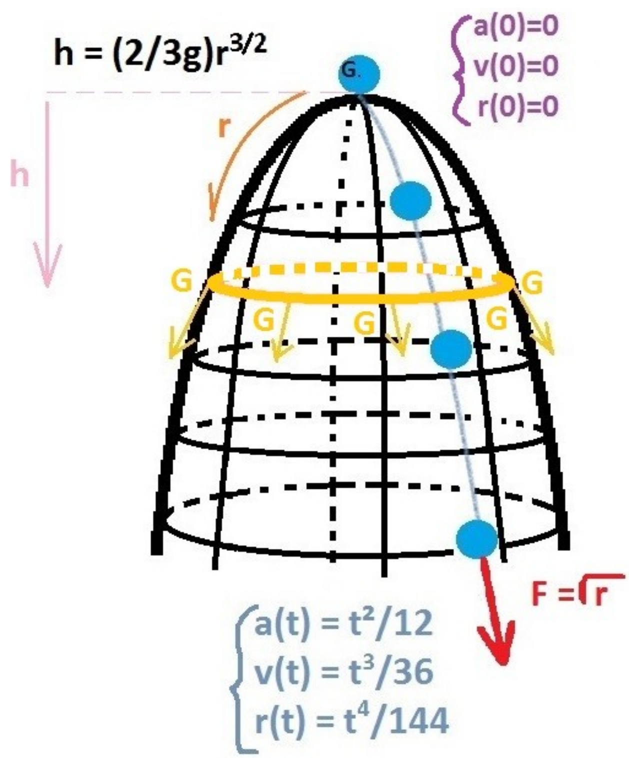
On the chosen half-profile of study, all the forces in play – including the zero initial conditions – are coplanar: the fundamental principle of dynamics then implies that any possible movement of the mass will take place exclusively on this common plane, that of the study. The same reasoning being valid for any section around the axis of rotation of the dome, physics leads to a single possible conclusion: the ubiquity of the mobile particle on the dome.

Now, if we crudely count all the study directions to reconstruct the dome by revolution around the **h** axis, what do we obtain in terms of the kinematics of the mobile? An infinity of trajectories covering more or less the dome, some always remaining at the top, others starting at distinct or non-distinct times **T** (an infinite "excitation time" **T** being equivalent to the resting state of the particle). From the point of view of the cylindrical or rotational geometry of the dome, all these trajectories or states of rest are carried out simultaneously by the mass.

This panorama offers us a space of extremely heterogeneous kinematic possibilities, unless we accept the principle of symmetry, known as Neumann-Curie's (1894): *“lorsque certaines causes produisent certains effets, les éléments de symétrie des causes doivent se retrouver dans les effets produits. Lorsque certains effets révèlent une certaine dissymétrie, cette dissymétrie doit se retrouver dans les causes qui leur ont donné naissance”*, in other words: *when certain causes produce certain effects, the effects have at least the symmetry of the causes.* In the case of multiple solutions to the problem, the « effects » are to be taken in the sense of superposition of all possible solutions.

Here – which avoids entering into the debate on the relevance of the concept of causality – the causes are to be understood simply as a combination of the geometry of the problem and forces in play at the initial moment (in this case, a dome, gravity and the reaction of the support), and the effects as the future evolution of the system. Their perfect symmetry of rotation implies the perfect symmetry of the trajectories of the mobile around **h**.

At this point, there are only two main solutions on the whole dome: either the mass at the top remains at rest indefinitely, or it takes all directions at once to slide spontaneously along the wall at the same time **T** following the same law of motion according to a perfect choreography (the centers of gravity **G** forming a uniform ring descending the dome at the same speed) :



Yet, on the one hand, Norton's set of possible solutions around the rotation axis (whose juxtaposition covers exactly the entire dome) respect the principle of symmetry as much as the set of contradictory solutions – this is an accepted extension of the Curie's principle. On the other hand, invoking the principle of symmetry is not necessary to reveal the whole contradiction above of the evolutions of a mass supposed to move without cause.

To summarize all the cases, the particle does not suffer from manifest indeterminism in time but from hidden ubiquity in space: if it does not go "nowhere", then it goes "everywhere" - and vice versa. We have every good reason to eliminate the last solution, at least out of respect for the classical principle of non-contradiction, valid even in quantum mechanics, which prohibits the same point from following several simultaneous trajectories (there is also a violation of the principle of conservation of total energy which becomes infinite with an infinity of masses in motion, etc. but we will not discuss it).

No conflict with Newtonian formalism, the principle of inertia, or that of sufficient reason, no incompleteness of physics, are necessary here: the particle must remain at rest, unless we endow it with a mystical or paranormal property of ubiquity, where its localizations contradict each other.

Let's test mathematically this view on a particular section, namely a complete profile of the dome. In order not to impose the movement of the particle on the left or right side, we'll let the curvilinear coordinate take negative values, calling it **s** (zero-valued variable at the top). A polar coordinate system $(\mathbf{u_r}, \mathbf{u_\theta})$ adapted to this relative coordinate **s** is chosen. The dome's curve will have equation:

$$h = (2/3g)|S|^{3/2}$$

We can then easily verify – considering each half-profile – the new equation of motion on this complete dome profile:

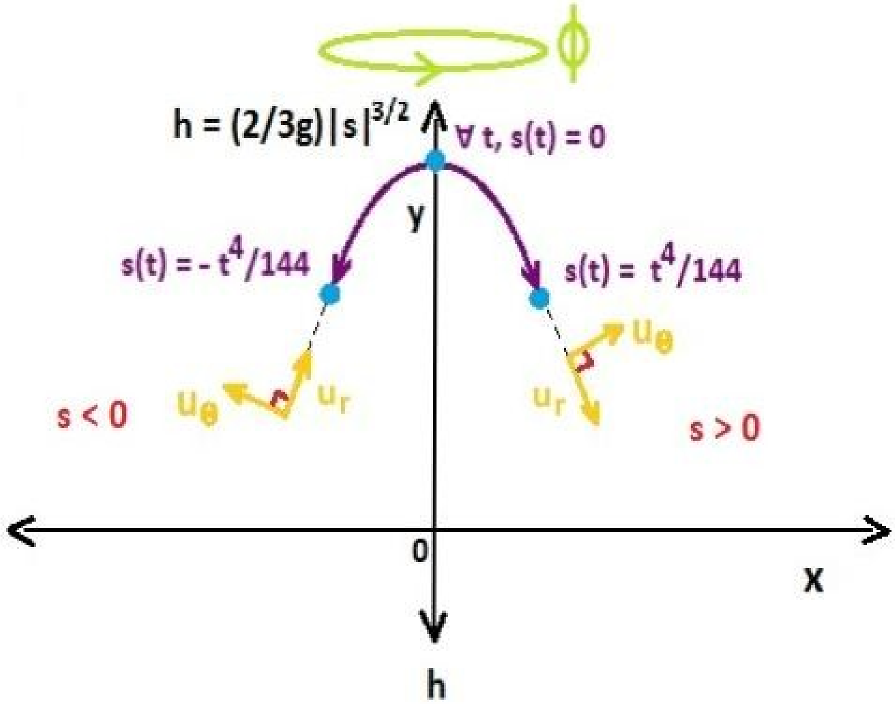
$$\frac{d^2|S|}{dt^2} = \sqrt{|S|}$$

which amounts of finding a C^2 positive function of time. The following solutions are derived from Norton's:

$$\forall T \geqslant 0,\quad \begin{cases} t < T: & S(t) = 0 \\ t \geqslant T: & |S(t)| = \frac{1}{144}(t-T)^4 \end{cases}$$

$$\Rightarrow \begin{cases} S(t) = \frac{1}{144}(t-T)^4 \\ S(t) = -\frac{1}{144}(t-T)^4 \end{cases}$$

Now this different approach involves the absolute value of **s**. This is convenient so as not to prejudge the physical direction that the mobile will take. The parametric representation in time of these solutions clearly shows us the displacement of mass on both sides of the dome at once:



Over this entire dome study section, it is confirmed that Newtonian physics obeys the principle of symmetry in the strong sense of a ubiquity of the particle and not of the multiplicity of solutions. Norton's solution for **s ≥ 0** is only a window that hides the global view of the entire solution **s** and its contradictions over the dome profile. In the current solutions for **|s|**, there is no probability or arbitrary choice between the two directions: it is merely a contradiction.

Furthermore, this contradiction is repeated all around the axis making all Norton solutions contradictory: we see that clearly by posing our curvilinear abscissa as a function of both time and rotation angle **φ**, i.e. ***s = s(t, φ)*** – solving the differential equation above gives identical results.

At no time is it a question of the particle moving towards a section other than the study section, no transverse force appears in the dynamic balance: the differential equations do not describe a possible trajectory of the ball on the study plane but its only possible trajectory on the dome. However, all study planes say paradoxically the same thing.

Besides, by deriving the double-direction solutions **|s|**, we find all the quantities zero at **t=T**, except both (now the « dissociative » nature of the ball appears from its very initial state):

$$\boldsymbol{d^4 s/dt^4 = 1/6}$$

$$\boldsymbol{d^4 s/dt^4 = -1/6}$$

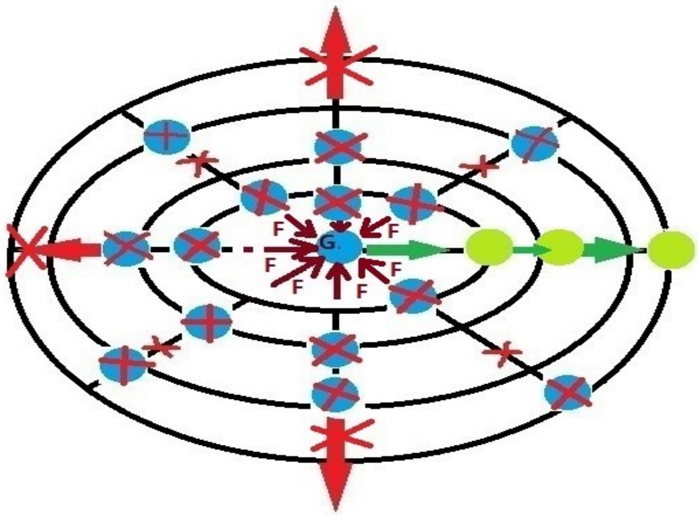
Finally, the icing on the cake: the above demonstration itself can be transposed to any other non-zero mathematical solutions of motion $\boldsymbol{r^*(t)}$ *defined* on the positive half-profile of the dome – other than the Norton solutions – and some interval **I** of time to show that these solutions are physically impossible. It suffices to solve the fundamental equation of dynamics with $\mathbf{|s| = r^*}$ on a complete profile, hence respectively $\boldsymbol{s(t) = r^*(t)}$ and $\boldsymbol{s(t) = - r^*(t)}$ for all **t** in **I**, on each side of the profile. It even works for a sign-changing trajectory of $\mathbf{r^*}$ by isolating the purely positive or negative parts.

Similarly for a half-profile, we will find the same paradox between rest on one side and motion $\boldsymbol{r^*(t)}$ of the mass on the other. We can conclude that the continuous rest solution is the only well-defined and physically valid mathematical solution to the dome problem. Such a result will be generalized in Section 5.

3. Physical solution of motion.

Can we remedy this inconsistency of the motion solutions to the dome problem? Yes, provided at least that we destroy the initial symmetry of the problem, since this rotating geometry (the shape of the dome and the state of rest of the mass) itself creates the paradox.

By applying symmetrical forces of the same intensity all around the particle, except in the desired direction of motion, all the forces cancel each other out in pairs except for one: the particle is allowed to move physically in a precise direction and no longer spontaneously, but with the help of an initial net force. All other contradictory trajectories should thus be eliminated:



One could also wonder if it would not be enough to cut the dome like a cherry cake, replacing the additional forces with "vacuum" to stop the particle... Except that then nothing would prevent Norton-type acausal solutions (with non-zero initial "reactivity" but undetectable in acceleration, speed and position) from "making the latter move by itself" in other directions to regain its magical ubiquity.

Here, we should perhaps clarify a little more the case of a simple half-profile: one can always see it as a complete asymmetrical profile, with a half-profile on the right (the dome) and a half-profile on the left, for example the ball at rest at **t=0** (**x_0=0**, **h_0=0**) on a platform overlooking a vertical precipice in a gravity field – which is tangentially zero therefore Lipschitzian. We then solve the fundamental principle of dynamics on each side, wondering what global physical movement the ball would follow on this half-plane of study.

On the right, solutions would be the ball starting to spontaneously descend the wall of the dome at any time (Norton's solutions), or the ball staying continuously at rest. *On the left,* even assuming that there is no acausal motion towards the precipice, the fundamental principle gives:

- On the **x** axis : $d^2x(t)/dt^2 = 0$, then after integration : $x(t) = 0$
- On the **h** axis : $d^2h(t)/dt^2 = g$, then after integration : $h(t) = gt^2/2$

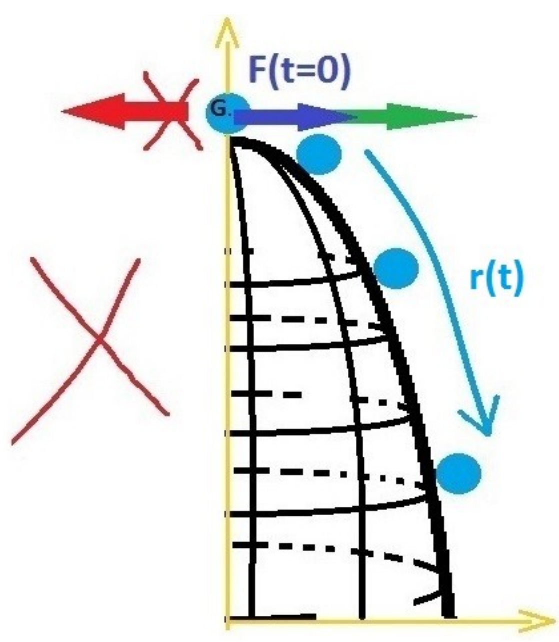
It appears that on the right side the only solution compatible with our initial rest conditions would be : $x(t) = 0$ and $h(t) = 0$ for all **t** (standing rest at the apex because the mass cannot fall into the precipice).

Now, by bringing together these two behaviors on both sides for the same ball, the paradox still arises that the object would start towards the right but would remain at rest at the same time. Thus it seems that even by eliminating the possibility of a solution of acausal motion to the left, even without any mention of the Curie symmetric principle, the fact of successively considering the half profile seen from the right, then seen from the left would still give rise to contradictory displacement solutions.

Moreover this approach doesn't just suppose the validity of the principle of symmetry for the dome but it demonstrates it as a consequence of the Newtonian formalism (see *section 2*). Taking into account the symmetry of the dome teaches us that for a particle in acausal motion everything can change according to the space to be studied.

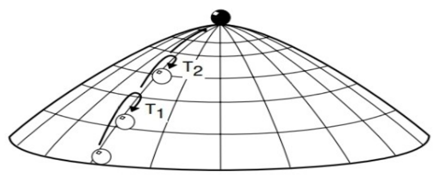
Considering only a half-profile of the dome would not allow those contradictions inherent in spontaneous physical behavior to disappear. The possibility of a precipice on the other side of the half-dome must be eliminated and replaced by a directed force to prevent the ball from remaining at rest or falling into the void.

On a half-profile of the dome (which is the pattern to rotate to restore the complete dome), we get for **T=0**:



This initial force **F(t=0)** could be for example the reaction of a wall against which the mass would be placed. It acts as a non-zero jerk force. This time, the symmetry is broken, the ball will have only one direction to follow.

Let's mention that Norton proposes another way to obtain his "acausal solutions": he asks to consider a mobile starting from the bottom of the dome to which we would impart an energy or initial speed sufficiently calibrated to hoist it exactly to the top. If we reverse the movement we would find, by the well-known principle of time invariance of Newtonian differential equations, the spontaneous sliding movement of the mass in question (Norton, 2003):



However, as we saw above, this would be forgetting that the solution obtained by time inversion is not the only trajectory starting from static conditions but, after analysis of the rotational symmetry of the problem, one among an infinity of simultaneous trajectories covering the surface of the dome.

Certainly, only one trajectory starting from the top will arrive at the bottom with the velocity vector in the exact opposite direction to that of the initial projection experiment but, without this arbitrary "final condition", nothing will forces the static particle at the top to take this one direction rather than another (an infinity of others...).

Finally, we would be curious to have an idea of the physical solution with non-zero initial force $\mathbf{F_0}$ in a certain direction. Here we set **T=0**. A rich study of the Norton dome problem (D. Malament, 2007) shows that if the mass is not at zero speed at the top for **t=0**, it will detach from the wall at the slightest movement.

Then applying $\mathbf{F_0}$, what will happen at time $\mathbf{t_1 = \Delta t}$ close to **t=0**? To ensure the adhesion of the mass, the following inequality must be verified (see section 1):

Limited developments in the neighborhood of zero give us:

$$a(0) = \left.\frac{d^2r}{dt^2}\right)_{t=0} = F_0$$

$$t \to 0 :$$

$$v(t_1) = \left(\frac{dr}{dt}\right)_0 + \Delta t \left.\frac{d^2r}{dt^2}\right)_0$$

Hence :

$$a(t_1) = \left.\frac{d^2r}{dt^2}\right)_{t_1} = \sqrt{r_1} = \Delta t \sqrt{\frac{F_0}{2}}$$

$$v(t_1) = \left.\frac{dr}{dt}\right)_{t_1} = F_0 \, \Delta t$$

$$r(t_1) = r(0) + \Delta t \left.\frac{dr}{dt}\right)_0 + \frac{1}{2}\Delta t^2 \left.\frac{d^2r}{dt^2}\right)_0$$

$$r(t_1) = \frac{1}{2} F_0 \, \Delta t^2$$

For **T1 = Δt** sufficiently small, we then observe that $\mathbf{(dr/dt)^2/\sqrt{r}}$ is indeed bounded above by $\mathbf{g^2 - r}$, which verifies the sliding condition at least up to **t1 > 0**.

4. Historical Cases Revisited.

We are going to analyse in depth those six cases (extracted in particular from the work of M. van Strien, 2014) through the prisme of the ubiquitous method proposed in this work.

CASE 1 - Poisson (1806) — Force $F(r) = c.r^a$ with $0 < a < 1$

Poisson seems to be historically the first scholar to study multiples solutions of newtonian differential equations, leaving open the possibility that they could be encountered in the physical world. He studied non-Lipschitzian forces before its time (the Cauchy-Lipschitz theorem would only begin to emerge decades later...).

Starting from the differential equation of a classical unit mass particle in a rectilinear motion subject to an accelerating force $F(r)=c.r^{a}$, with **r** the distance from the origin, **a** and **c** constants, $0< a< 1$, he gets:

$$d^2r/dt^2=c.r^{a}$$

at **zero initial conditions** for position and velocity: $r(0)=0, r'(0)=0.$

This equation admits both the trivial static solution :

$$r(t)=0 \quad , \quad \forall t$$

and a family of spontaneous motion solutions for arbitrary time **T**:

$$r(t)=A(t-T)^{2/(1-a)} \quad \text{for} \quad t\geq T \quad \text{, with } \mathbf{A} \text{ constant.}$$

First, one could consider a circular symmetry, as for the Norton's dome, or just a mirror symmetry to complete the real axis for the coordinate **r** in one dimension.

Then, by extending to a signed coordinate **s**, the equation becomes:

- For $s\geqslant 0$:

$$\frac{d^2 s(t)}{dt^2}=c\cdot s^{a}(t)$$

- For $s\leqslant 0$:

$$\frac{d^2 s(t)}{dt^2}=-c\cdot(-s(t))^{a}$$

thus, after symmetrization using $s\rightarrow|s|$:

$$|\ddot{s}|(t)=c\cdot|s(t)|^{a}$$

from which we obtain both solutions $s(t)$ and $-s(t)$ as valid, leading to spatial ubiquity, i.e. simultaneous departures in opposite directions.

But, even without symmetrization, we show a half-profile contradiction: applying Newton’s laws on a left domain with purely vertical force (no tangential field ⇒ Lipschitzian force with respect to the variable $s\leqslant 0$) yields persistence at rest as the *unique solution* at left side – which is nothing other than the inertia principle:

$$d^2s/dt^2=0 \quad \Rightarrow \quad s(t)=0$$

while the right-side solution permits spontaneous departure. This contradiction cannot be resolved without imposing a directional force.

A single rigid object governed by Newtonian laws cannot behave differently across adjacent domains: the model is thus physically inconsistent, but the hidden contradiction only appears by geometrically extending the space of the system.

CASE 2 : Poisson (1806) : friction-like Force $F(u)=-d\sqrt{u}$

A unit-mass particle is subject to a non-conservative force that depends here on speed **u**, not position:

$$F(u)=-d\sqrt{u}\,,$$
$$\textit{with } u\geq 0,\quad \mathbf{d}\ \text{ a constant}$$

which leads to the differential equation of movement:

$$\ddot{r}(t)=-d\sqrt{\dot{r}(t)}$$

$$r(0)=0,\ \dot{r}(0)=0.$$

Depending on the sign of **d**, this applied force is oriented along the velocity vector (driving force) or opposite (braking force).

Again, a static solution:

$$r(t)=0,\ \forall t.$$

By posing:

$$u(t)=\dot{r}(t)$$

This equation becomes:

$$\dot{u}(t)=-d\sqrt{u(t)}$$

If $d\leqslant 0$ and $t\geqslant T$:

$$\int\frac{du}{\sqrt{u}}=-d\int dt$$

$$\Rightarrow\quad 2\sqrt{u}=-d(t-T)$$

$$\Rightarrow\quad u(t)=\frac{d^2}{4}(t-T)^2$$

We get non-trivial solutions for $t\geq T$, with arbitrary time departure **T**:

$$\dot{r}(t)=\frac{d^2}{4}(T-t)^2$$
$$r(t)=\frac{d^2}{12}(T-t)^3$$

Velocity is increasing from $u(T)=0$ to $u(t)>0$.

But for $d \geqslant 0$, we get a field of frictional (decelerating) forces. If $u(T)=K>0,\ t \geqslant T$ and $t \leqslant T+\frac{2\sqrt{K}}{d}$:

$$\int \frac{du}{\sqrt{u}}=-d\int dt$$

$$\Longrightarrow \quad 2\sqrt{u}-2\sqrt{K}=-d(t-T)$$

$$\Longrightarrow \quad u(t)=\left(\sqrt{K}-\frac{d}{2}(t-T)\right)^2$$

Non-trivial solutions are then decelerating movements from $r(T)=R$ to $r=R+\frac{2}{3\mathrm{d}}\sqrt{K}$:

$$\dot{r}(t)=\left[\sqrt{K}-\frac{d}{2}(t-T)\right]^2$$

$$\Longrightarrow \quad r(t)=R+\frac{2}{3\mathrm{d}}K\sqrt{K}-\frac{2}{3\mathrm{d}}\left[\sqrt{K}-\frac{d}{2}(t-T)\right]^3$$

By modifying **d**, one obtains trajectories with spontaneous motions, braking, stops or restartings.

As force is only studied by Poisson for $u(t)=\dot{r}(t)\geq 0$, one can extend it to the whole real axis of speeds by considering (with variable **v** the signed version of velocity **u**):

- **v** $\geqslant$ **0** (the particle moves towards the $s>0$) :

$$\ddot{v}(t)=-d\cdot\sqrt{v(t)}$$

- **v** $\leqslant$ **0** (the particle moves towards the $s<0$) :

$$\ddot{v}(t)=+d\cdot\sqrt{-v(t)}$$

Then, joining both cases:

$$|\ddot{v}|(t)=-d\cdot\sqrt{|v(t)|}$$

If **d** $\leqslant$ **0** for example (driving force), solutions become:

$$v_+(t)=+\frac{d^2}{4}(T-t)^2$$

$$v_-(t)=-\frac{d^2}{4}(T-t)^2$$

Trajectories are then:

$$s_+(t) = + \frac{d^2}{12}(T-t)^3$$

$$s_-(t) = - \frac{d^2}{12}(T-t)^3$$

Here appears the same contradictions as above: the system allows arbitrary redirections in velocity space, then in positions, creating inconsistency if symmetrized.

Same thing on a half-profile: right-side motion coexists with inert left-side rest, violating object uniqueness. Again, ubiquity of motion appears, in contradiction with Newtonian realism.

Poisson didn't push his analysis to a larger physical space, however he discusses at length the relevance of singular solutions. For him, as for Duhamel later (see below), it is clear that if one has to choose, then one must eliminate the solutions that do not respect the principle of inertia. Poisson seems to interpret the latter as the attribution of causality to the notion of force.

Yet, as we will see with Duhamel, nothing in the "indeterministic" dynamical equations in themselves allows us to decide between the constant rest and the regular "acausal" solutions. The preference given to the solution of rest by the principle of inertia looks more like a metaphysical than a mathematical choice, strictly speaking...

CASE 3 : Duhamel (1845) — Philosophical objection to non-uniqueness

This case is more modest in its mathematical scope, but interesting from a doctrinal point of view, showing how some physicists recognized the existence of multiple solutions, while explicitly choosing to reject them on the basis of physical, metaphysical or moral reasoning.

Duhamel revisits the examples studied by Poisson, notably those where the force is:

$$F(v) = -c \cdot v^a, 0 < a < 1$$

He does not present a new system, but generalizes those of Poisson (where $a = 1/2$). Considering the equation of motion:

$$\ddot{r}(t) = -c \cdot \dot{r}^a(t)$$

Duhamel mathematically admits that there are an infinite number of solutions, as seen previously, but he maintains on physical grounds that only the rest solution is acceptable:

$$r(t)=0,\ \forall t$$

Although he does not formalize this idea, Duhamel seems to invoke the principles of inertia and causality to reject indeterminism and disqualify uncaused solutions. However, as already seen earlier with the Norton's dome and Poisson, the principle of inertia is not independent from the dynamics equation: in non-Lipschitzian cases, rest and delayed starts are both exact mathematical solutions of the equation of motion. Inertia is just a particular one, unable *per se* to discriminate among others or declare any non-static solution unphysical.

Since there is no internal criterion in the three classical laws of motion to choose between indeterministic solutions, we extended the differential equation in **r** to:

$$|\ddot{s}|(t)=c\cdot|s(t)|^{a}$$

This equation again allows for dynamical ubiquity which:

- ✔ Violates the physical identity of the particle,
- ✔ And logically invalidates solutions other than rest.

Then, our method provides a logical formalization of what Duhamel intuitively asserts: multiple solutions lead to a global spatial contradiction. We preserve the entire Newtonian framework: no additive principle, but a global logical test: if the set of solutions creates a spatial contradiction (ubiquity), then they must be eliminated. This principle is not dynamic, but logico-geometric. It allows us to decide between solutions without betraying the initial equations or importing an external axiom (such as a minimization or stability principle).

Finally, to answer to Duhamel, inertia in itself cannot rule out others solutions by fiat. Perhaps this is why, aware of its weaknesses, scientists like Newton elevated it to the rank of principle. Instead, ubiquity of undeformable objects can more rigorously justify the rejection of spontaneous solutions without overdetermining classical physics. It makes uniqueness not arbitrary, but necessary for the coherence of the physical world: one object equals one position at each instant.

In the final section, we will propose abandoning inertia as a simple principle, subject to the arbitrariness of physical systems, and making it a theorem. Let's move on to the next historical cases that will only confirm this need to evolve the scientific paradigms.

CASE 4 : Boussinesq (1879) — Generalized dome.

Boussinesq discusses the same kind of bifurcation in mechanical systems and introduces the idea of spontaneous rupture beyond deterministic prediction. He writes that:

“Lorsque les équations de la mécanique ne suffisent plus à déterminer le mouvement, il faut invoquer une cause étrangère, que je nomme agent directeur.”

Boussinesq builds a mechanical system specifically intended to highlight non-unique solutions, linked to the failure of the Lipschitz condition. He designs a surface of revolution (dome) on which a particle of unit mass is deposited, subject only to gravity.

This dome is defined by a half-profile on the plane, with the height as a function of the path **r** from the apex. Boussinesq generalizes the classical form by proposing:

$$h(r)=\frac{K^2}{2\mathrm{g}}\left(\log(a/r)\right)^{2\mathrm{k}} r^{2\mathrm{m}}$$

One recovers the Norton's Dome by setting $a=e.r$ (where **e** is the Euler constant), $m=¾, K^2=1/m.$ A particle slides on the surface described by this height function which becomes non-Lipschitz at $r=0$ for $½< m< 1.$

The movement is governed by the law:

$$\ddot{r}(t)=g\,\frac{dh}{dr}$$

$$\frac{dh}{dr}=\frac{K^2}{g}\left[m\left(\log(a/r)\right)^{2\mathrm{k}}-k\left(\log(a/r)\right)^{2\mathrm{k}-1}\right]r^{2\mathrm{m}-1}$$

Both solutions of continuous rest and departure are mathematically permitted:

$$r(t)=0,\ \forall t$$

and an infinity of solutions of type:

$$r(t)=f(t-T),\ \forall t\geqslant T$$

with arbitrary **T** and: $f(0)=f\,'(0)=0$

Boussinesq deduces that, without external cause, the system is unable to choose “by itself” one direction of movement, which he philosophically interprets as the introduction of a non-material *guiding principle*. Applied to biology, he identifies this hidden variable with free will.

Now, let's extend symmetrically this half-profile around the dome axis by mean of the signed coordinate **s**. Like in Norton's dome, the Boussinesq profile and its associated differential equation become:

$$H(s)=h(|s|)=\frac{K^2}{2g}\left(\log\left(\frac{a}{|s|}\right)\right)^{2k}|s|^{2m}$$

- For **s** ⩾ **0**: $\ddot{s}(t)=g\cdot\frac{d}{ds}H(s)=g\cdot\frac{d}{ds}h(s)$
- For **s** ⩽ **0**: $\ddot{s}(t)=g\cdot\frac{d}{ds}H(s)=g\cdot\frac{d}{ds}h(-s)=-g\cdot\frac{d}{d(-s)}h(-s)$

Then, for all **s**:

$$|\ddot{s}|(t)=g\cdot\frac{d\,h(|s|)}{d\,|s|}$$

which releases the new simultaneous solutions for **t** ⩾ **T**:

$$|s(t)|=f(t-T)$$

then:

$$s_+(t)=f(t-T)$$

$$s_-(t)=-f(t-T)$$

Extension to **s** ∈ ℝ and transformation to $|s|$ allows incompatible mirrored solutions. At any time **T**, the object can go both in one direction and in the opposite (geometrical ubiquity). Only the standing rest state solution respects principles of material identity, inertia and spatial coherence.

Again, half-profile physical contradiction arises from:

1. Left side: vertical forces only (if no tangential force fields) → rest enforced on this side by lack of tangential field.
2. Right side: motion predicted → contradiction unless a directional trigger is introduced to break symmetry.

CASE 5 : Boussinesq (1879) — Two particles with indeterministic interaction (Boscovich atomic model)

Here, Boussinesq describes a two-body system (sometimes called "atoms") subject to a non-Newtonian central force.

- One of the examples considered seems to be inspired by the Boscovich atomistic theory of matter (1758, in *Theoria philosophiae naturalis*), where atoms are not considered as extended indivisible bodies but as points and centers of force, spinning around each other: Boscovich believed that the force between atoms was repulsive at very short distances, attractive at macroscopic distances - in accordance with the law of universal gravitation - and changed sign (alternately attractive and repulsive) in the intermediate zone.
- The interaction potential is not regular in **R**, but remains finite and continuous.
- An unstable circular orbit is possible, where the particles rotate at zero speed.

Indeed, by moving from its attractive nature to its opposite, this interaction should pass through a state of zero force. Boussinesq expects the singular orbits at such points. In this configuration, the two particles can remain in relative equilibrium at distance **R** in a circular orbit, without moving, or begin to move at an indeterminate time **T**, “without cause”.

The fundamental equation of dynamics for the distance **r** between the two particles is:

$$m\ddot{r}=F(r)$$

Boussinesq uses the following potential and derives its corresponding force:

$$V(r)=C\cdot|r-R|^{1+a},\ 0<a<1$$

$$\Longrightarrow\quad F(r)=-C(1+a)\cdot sgn(r-R)\cdot|r-R|^{a}$$

So (defining constant $k=C(1+a)$):

$$\ddot{r}=-k\cdot sgn(r-R)\cdot|r-R|^{a}$$

N.b. let's point out that this formulation of atomic interaction respects the Boscovich conditions ($F(r)>0\ if\ r<R$ and $F(r)<0\ if\ r>R$).

Thus, he finds:

- Trivial solution:

$$r(t)=R,\ \forall t$$

- Non-trivial solutions:

$$r(t)=R\pm A(t-T)^{\frac{2}{1-a}},\ \ for\ t\geq T$$

The two particles could either stay at rest, or spontaneously move closer, further apart, or even oscillate. These solutions appear for the same initial conditions and Boussinesq treats them as alternative and mutually exclusive solutions while in fact they are simultaneous and contradictory. It will be more obvious by introducing the signed variable $s=r-R$ which is equivalent at first to what Boussinesq did, but we will pay more attention in interpreting his solutions.

The equation becomes:

$$|\ddot{s}|(t)=-k\cdot|s(t)|^{a}$$

Then:

$$|s(t)|=A(t-T)^{\frac{2}{1-a}},\ \ for\ t\geq T$$

• It admits both:

➔ Inward solutions: $s_{-}(t)=-A(t-T)^{\frac{2}{1-a}},\ \ for\ t\geq T$

➔ And outward solutions: $s_{+}(t)=+A(t-T)^{\frac{2}{1-a}},\ \ for\ t\geq T$

System allows motion in opposing directions from identical initial conditions. Unless a symmetry-breaking directional force is used, this modelling of Boscovich atomic theory is then contradictory. As previously, it can be shown that the only consistent solution is perpetual rest of the two particles at distance **R**.

CASE 6 : Bertrand (1878-79) — Rejection of Boussinesq's framework and Philosophical Critique of Mechanical Indeterminism

Joseph Bertrand rejects any physical allowance of non-uniqueness, demanding a 'true' hidden law. He reacted to Boussinesq’s proposals, particularly the idea that certain mechanical equations might not unequivocally determine the evolution of a system.

He directly criticized:

- ✗ The idea of a “guiding principle” (in Boussinesq’s sense), that is some effect of the mind on the matter without any helping force.
- ✗ The very physical existence of systems with multiple spontaneous solutions.

His central postulate may be stated as: if an equation admits multiple solutions from the same initial conditions, it does not correctly reflect the real physical nature. Bertrand suspected discontinuous or hidden laws may underlie the apparent indeterminism. He then suggested that the true law of nature was different:

- Perhaps more disjoint,
- Perhaps non-continuous,
- Perhaps involving jumps in behavior. He therefore refuses to accept the idea that nature itself would allow multiple trajectories compatible with the same initial state.

For him, the plurality of solutions is not a real paradox, but rather evidence of an error in the modeling. Anyway, Bertrand's point of view can now be formally assessed. By requiring a classical directed force to break ubiquity, without needing to invent an extra-physical guiding principle, our approach allows for a rigorous justification of Bertrand's rejection of indeterminism in the name of an “invisible” (but unspecified) principle of determination.

Yet, non-Lipschitz equations are not necessarily “wrong” or “incomplete”: they are contradictory, because they lead to physically incompatible predictions. The very structure of the trajectories is enough to decide. Multiple solutions lead to geometric contradiction without recourse to hidden variables, bad modeling or unknown laws.

5. The principle of inertia as a mathematical theorem.

The Picard-Lindelöf (or Cauchy-Lipschitz) theorem, a generalization of which applies to second-order differential equations, is underlying our whole approach. It predicted the possibility of multiple solutions in the case of the Norton's dome (Lipschitz condition not verified at the vertex, infinite curvature, etc.).

We precisely studied them: the analysis revealed that those “possible” trajectories were “and-like” but not “or-like”, superposed but not mutually exclusive. Only the solution of the mass being at endless rest at the vertex is physically acceptable and not self-contradictory. Any other non-deterministic solutions are eliminable by basic physics and common sense.

Besides, it should be emphasized that the Picard-Lindelöf theorem provides a sufficient but not necessary condition for the uniqueness of the solutions to differential equation – which therefore does not make it a perfect synonym for determinism. In itself, it says nothing explicitly about physics. His Lipschitz condition is no more fulfilled in the Norton dome than it is for an infinite number of other systems considered to be truly physical or falling within Newtonian physics.

Indeed, a ball can without unrealism descend a slope as steep as a staircase, or any slope which isn't even first-order differentiable unlike the dome of Norton – as this very one has noticed. Most criticisms that judge the dome itself as an unphysical or "non-Newtonian" system could then be dismissed - especially if it can be shown to admit an unique and quite physical solution like the constant rest.

According to us, the most unphysical aspect of the dome lies in its violation of the principle of inertia by Norton's indeterministic solutions: the ball starts spontaneously, even though at no time is it subject to a net force external to the forces that ensure its equilibrium at the summit. But since the principle of inertia like Norton's solutions are all rigorously deducible from the same fundamental equation of dynamics, the real problem was to decide their validity using a third party.

For such a role, the third principle of action/reaction was of no use here. We found this arbiter of physics elsewhere, in a certain "axiom of non-ubiquity", the equivalent of the logical law of the excluded middle for motion: *a point mass or a rigid object can take one direction or the other, but not both at the same time.*

Yet, all this shows a major flaw in the three principles of Newtonian physics. Problems like Norton's dome or our revisited historical cases all converge on the idea that there is a fundamental indeterminacy between the 1st law (inertia principle) and the 2nd law (the general equation of dynamics). We need a strong argument to resolve these centuries-old conflicts that have pitted them against each other since at least 1806. Something that would allow us to consider the principle of inertia as the "winning" mathematical solution against its competitors of indeterminism.

Then, what if inertia became...*a mathematical theorem rather than remaining a dogmatic principle?* This paradigmatic shift would help us eliminate any undesirable solutions generated by the second principle through the fundamental differential equation of dynamics. Such a result has been partially achieved by the study of ubiquity phenomena. However, knowing that the principle of inertia is already a theorem for Lipschitzian forces, we lack a uniqueness theorem equivalent to Picard-Lindelöf for non-Lipschitz second-order ODEs.

This theorem must be supported by a strong definition linked to the consistency of mathematical solutions with its counterpart in the physical world. Indeed, we should dismiss as unphysical any motion that appears locally coherent (as in the half-profile of Norton's dome) but behaves differently in space as soon as we expand its field of study or change our geographical viewpoint, without affecting anything else in the dynamics applied to the mobile. This is only a moderate form of local realism.

Let us try it now. Before, note that if one considers an equation like, for all $\mathbf{y} \in \mathbf{V} \subset \mathbb{R}$:

$$\mathbf{(E)} \quad \mathbf{d^2y/dt^2 = F(y) = 0}$$

with initial conditions: $\mathbf{y(0) = 0, dy/dt(0) = v_0 \neq 0}$, the zero function $\mathbf{F}$ is obviously Lipschizian on $\mathbf{V}$, then: $\mathbf{t} \mapsto \mathbf{y(t) = v_0.t}$ is the unique solution of $\mathbf{(E)}$, which demonstrates the inertia principle for uniform rectilinear motion through a zero tangential forces field. We will then just focus on the case $\mathbf{v_0 = 0}$.

All the following results may be generalized to any $\mathbf{n > 1}$ dimensions in $\mathbb{R}^n$, given that if $\mathbf{y} \in \mathbb{R}^n$ then problem $\mathbf{(E)}$ reduces to $\mathbf{n}$ one-dimensional differential equations of the type studied below.

1) Statement of the Problem

Let $\mathbf{(E^+)}$ be the second-order autonomous differential equation of a $\mathbf{C^2}$ **function in time**, $\mathbf{y(t)}$, defined on an interval $\mathbf{V^+} \subset \mathbb{R}^+$ containing zero, $\mathbf{t \geq 0}$:

$$\mathbf{(E^+)} \quad \mathbf{d^2y/dt^2 = F^+(y)}$$

Initial conditions:

$$\mathbf{y(0) = 0},$$

$$\mathbf{dy/dt(0) = 0}$$

assuming the following:

- $\mathbf{F^+(0) = 0}$,
- $\mathbf{F^+: V^+ \to} \mathbb{R}$ is continuous and non-Lipschitzian in $\mathbf{0}$.

Definition: a well-defined solution $\mathbf{t} \mapsto \mathbf{y^+(t)}$ of $\mathbf{(E^+)}$ on $\mathbf{V^+}$ is said *contradictory* or *ubiquitous* if: there exists an interval $\mathbf{V^-} \subset \mathbb{R}^-$ including zero and an extension $\mathbf{F^-}$ of $\mathbf{F^+}$ on $\mathbf{V^-}$ such that $\mathbf{y^+}$ is incompatible with any solution $\mathbf{y^-}$ of the new ODE defined on $\mathbf{V^-}$:

$$\mathbf{(E^-)} \quad \mathbf{d^2y/dt^2 = F^-(y)}$$

Initial conditions:

$$\mathbf{y(0) = 0, dy/dt(0) = 0,}$$

in the sense that both solutions may take values in different points of $\mathbf{V= V^- \cup V^+}$ at the same time $\mathbf{t}$. Otherwise, $\mathbf{y^+}$ is said to be *consistent or non-ubiquitous*.

2) Non-Lipschitzian Theorem of Uniqueness

Under the assumptions above, the only consistent solution to equation $\mathbf{(E^+)}$ with initial conditions $\mathbf{y(0) = 0}$ and $\mathbf{dy/dt(0) = 0}$ is the trivial solution: for all $\mathbf{t}$, $\mathbf{y(t) \equiv 0}$.

In other words, the global differential equation $\mathbf{(E) : d^2y/dt^2 = F(y)}$, with $\mathbf{F = F^- \cup F^+}$ (same initial conditions), has no well-defined solution in the extended space $\mathbf{V = V^- \cup V^+}$ except **zero**.

3) Proof

- Consider $\mathbf{V^-}$ the symmetrical interval to $\mathbf{V^+}$ on $\mathbb{R}^-$, and one of the two independent transformations:

  (S) $\mathbf{F^-}$ is an extension of $\mathbf{F^+}$ on $\mathbf{V^-}$ as an antisymmetric function, i.e. $\mathbf{F^-(-y) = -F^+(y)}$ for all $\mathbf{y} \in \mathbf{V^+}$

  or:

  (A) $\mathbf{F^-}$ is an extension of $\mathbf{F^+}$ on $\mathbf{V^-}$as a Lipschitz continuous function.

Analytically, it is always possible to find the transformations **S** (by antisymmetry of $\mathbf{F^+}$ with respect to zero) and **A** (by taking $\mathbf{F^- = 0}$) that immerse $\mathbf{F^+}$ in the complete space $\mathbf{V = V^- \cup V^+} \subset \mathbb{R}$.

Case 1: Antisymmetric field F (S)

Suppose $\mathbf{y^+(t)}$ is a solution of $\mathbf{(E^+)}$ on $\mathbf{V^+}$. By antisymmetry, for all $\mathbf{t}$ in the domain of definition:

$$\mathbf{d^2y^+/dt^2 = F^+(y^+) \Rightarrow d^2(-y^+)/dt^2 = -F^+(y^+) = F^-(-y^+).}$$

Thus, $-\mathbf{y}^+(\mathbf{t})$ also solves the equation $(\mathbf{E}^-)$ with the same initial conditions ($\mathbf{y(0) = 0}$, $\mathbf{dy/dt(0) = 0}$).

Therefore, unless $\mathbf{y}^+(\mathbf{t}) \equiv \mathbf{0}$, there exists at least two simultaneous trajectories on $\mathbf{V}$ with the same initial conditions:

$$\mathbf{y_1(t) = y(t),\ y_2(t) = -y(t).}$$

Using another method considering **(E)** defined on $\mathbf{V = V^- \cup V^+}$:

$$\textbf{(E)} \quad \mathbf{d^2y/dt^2 = F(y)}$$

with $\mathbf{F = F^- \cup F^+}$ and the same initial conditions as $(\mathbf{E}^+)$ and $(\mathbf{E}^-)$. Thus:

- if $\mathbf{y} \in \mathbf{V}^+$: $\mathbf{d^2y/dt^2 = F(y)}$
- if $\mathbf{y} \in \mathbf{V}^-$: $\mathbf{d^2y/dt^2 = F(y) = -F(-y)} \Rightarrow \mathbf{d^2(-y)/dt^2 = F(-y)}$

  → Then, for all $\mathbf{y} \in \mathbf{V}$, $\mathbf{d^2|y|/dt^2 = F(|y|)}$,

  → which leads to $\mathbf{|y| = y^+}$, for any positive solution $\mathbf{y}^+$ of **(E)** on some interval **I** of time, then for all **t** in **I:** $\mathbf{y(t) = y^+(t)}$ and $\mathbf{y(t) = -\ y^+(t)}$.

Anyway, those solutions are not compatible unless they coincide, i.e., $\mathbf{y(t) \equiv 0}$.

Case 2: One-Sided Lipschitz Condition (A)

Assume $\mathbf{F}^-$ is Lipschitz on the left side $\mathbf{V}^-$. Let us define $\mathbf{y^-(t) = y(t)}$ restricted to values in $\mathbf{V}^-$. Then the initial problem $(\mathbf{E}^-)$ on $\mathbf{V}^-$:

$$\mathbf{d^2y/dt^2 = F^-(y)}, \text{ with } \mathbf{F^-(0) = 0,}$$

$$\mathbf{y(0) = 0,}$$

$$\mathbf{dy/dt(0) = 0,}$$

releases the unique solution $\mathbf{y(t) \equiv 0}$ by the classical Cauchy-Lipschitz theorem.

Suppose now that a solution $\mathbf{y^+(t)}$ of $(\mathbf{E}^+)$ exists on the right on $\mathbf{V}^+$ with $\mathbf{y^+(t) \neq 0}$ near $\mathbf{t = 0}$. Then $\mathbf{y}^+$ crosses into the region where no Lipschitz condition holds, and uniqueness cannot be recovered. But this non-zero solution does contradict the unique solution of $(\mathbf{E}^-)$, $\mathbf{y(t) \equiv 0}$, found previously, thus again $\mathbf{y^+(t) \equiv 0}$ on $\mathbf{V}^+$.

One can construct a smooth ubiquitous solution in the same way as in case 1 by noting that equation **(E)** is rewritable in the following form, valid for all **y** in **V**:

$$d^2y/dt^2 = \delta_{y\geq 0} \cdot \sqrt{|y|}$$

Here, $\delta_{y\geq 0}$ is the indicator function:

$$\delta_{y\geq 0}(y) = 1 \quad \text{if } \mathbf{y \geq 0}, \qquad \delta_{y\geq 0}(y) = 0 \quad \text{if } \mathbf{y < 0}.$$

So this defines a piecewise system:

$$\mathbf{d^2y/dt^2} = \begin{cases} \mathbf{\sqrt{y}} & \text{if } \mathbf{y \geq 0} \\ \mathbf{0} & \text{if } \mathbf{y < 0} \end{cases}$$

We construct then a piecewise solution. Let $\mathbf{y_+(t)}$ be a known non-negative $\mathbf{C^2}$ solution on $\mathbf{t \geq 0}$ to the equation:

$$\mathbf{d^2y_+/dt^2 = \sqrt{y_+(t)}}$$

with initial conditions:

$$\mathbf{y_+(0) = 0}$$

$$\mathbf{dy_+/dt(0) = 0}$$

$$\mathbf{d^2y_+/dt^2(0) = 0}$$

We now define on $\mathbf{V}$ the global function $\mathbf{y(t)}$, for all $\mathbf{t \geq 0}$, by:

on $\mathbf{V}^-$: $\quad \mathbf{y(t) = 0}$

on $\mathbf{V}^+$: $\quad \mathbf{y(t) = y_+(t)}$

Since all derivatives of $\mathbf{y_+}$ vanish at $\mathbf{t = 0}$, $\mathbf{t} \mapsto \mathbf{y(t)}$ is a $\mathbf{C^2}$ **function** on $\mathbb{R}^+$.

Verification with the Global ODE:

On $\mathbf{V}^-$:

$$\mathbf{y(t) = 0} \;\Rightarrow\; \mathbf{d^2y/dt^2 = 0}$$

$\Rightarrow$ Matches the ODE since $\delta_{y\geq 0} = 0$

On $\mathbf{V}^+$:

$$\mathbf{y(t) = y_+(t)} \;\Rightarrow\; \mathbf{d^2y/dt^2 = d^2y_+/dt^2 = \sqrt{(y_+(t))} = \sqrt{(y(t))}}$$

$\Rightarrow$ Matches the ODE since $\delta_{y\geq 0} = 1$

Hence, $\mathbf{y(t)}$ satisfies the ODE **(E)** on all $\mathbb{R}$ and is of class $\mathbf{C^2}$**.**

We constructed a smooth, global solution to a second-order non-Lipschitzian ODE by gluing together a trivial rest solution on the left with a known positive solution on the right. The solution is **ubiquitous**, in that it remains inert for $\mathbf{t < 0}$, then departs spontaneously for $\mathbf{t \geq 0}$, without violating classical smoothness conditions.

This confirms that smooth ubiquitous solutions are possible even for equations with minimal regularity.

QED.

4) Physical Interpretation

In either case, the only admissible solution is the constant rest, $\mathbf{y(t) \equiv 0}$. This result implies that if the global force field $\mathbf{F} = \mathbf{F}^- \cup \mathbf{F}^+$ is either antisymmetric (permitting mirrored trajectories) or Lipschitz on at least one side of zero, the dynamics cannot produce simultaneous multiple trajectories emanating from the same conditions at the origin.

In mechanical terms, spontaneous departures from rest without initial external input ($\mathbf{F(0) = 0}$) are ruled out under these minimal regularity or symmetry assumptions. Indeed, the given definition of consistency aims to *reject any behavior of an object likely to be modified, without adding or removing any force from the initial subsystem, i.e. by the sole artificial extension of its study space*.

Physically, this assumes that:

(1) on the one hand, ubiquities and other inconsistent behaviors are not allowed in classical reality,

(2) on the other hand, at least one of the two immersions (S) and (A) is always possible for any half-profile system.

These two postulates are reasonable in most cases. The first is commonly accepted for undeformable macroscopic objects like in the Norton's dome. As for the second, it is obvious that **(S)** is tailor-made for naturally symmetrical or symmetrizable systems, while **(A)** is better suited to irreducibly asymmetrical problems where the initial state of rest can be extended to the left.

The immersion of non-Lipschizian force fields into larger systems should only be prevented in event of a physical barrier, that in turn would exert a breaking action on the initial state of rest, or by cosmological considerations about the nature of physical space.

Admitting the assumptions above, the principle of inertia (Newton, 1687):

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

finally turns into a **theorem of inertia**:

[In any continuous and locally extendable force field] every consistent body perseveres in its state of rest, or uniform motion in a right line, unless it is compelled to change that state by [external net] forces impressed thereon.

No need then to artificially distinguish between Newtonian and non-Newtonian systems, Lipschitzian or non-Lipschizian forces, physical or unphysical idealizations, etc. most distinctions that didn't exist in the original work of the Principia, nor a priori in Nature. The inertia of bodies in Newton was intuitive, experimental, qualitative. Here it becomes structural, logical, mathematically anchored in the laws of dynamics.

The 1st more fundamental principle which could replace the law of inertia would be the following:

Principle of non-ubiquity (or geometrical consistency) of dynamic trajectories

No behavior of a rigid dynamic system can extend in several incompatible directions

statement that is more universal (applicable even to non-Newtonian, non-Lipschizian, etc. systems), neutral (formulated without mechanical hypothesis) and mathematical (concerning the geometric structure of differential solutions).

Conclusions

In this paper, it has been shown that the perpetual rest of the ball at the top of the Norton's dome is the unique mathematical solution that respects both the spherical symmetry of this problem and the principle of inertia, which thus – narrowly? – avoids a contradiction between solutions.

It is not a secret indeterminism that one would discover in the holy of holies of Newtonian physics, nor its incompleteness, but the existence of inconsistent physical solutions - in the sense of the excluded middle principle applied to physics - to eliminate from the solving of motion differential equations *over the whole dome*, rather than just one particular half-profile.

One can even consider the generalization of this approach to other physical paradoxes, like those brought to light since the 19th century, where a particle at rest in a symmetrical environment (rotational, axial, translational, etc. in one or more dimensions) starts moving spontaneously. Maybe scientists should care more about possible contradictions than about indeterminism or incompleteness, since the latter could be less serious than any structural inconsistency in Newtonian theory, which also endangers all theories built on it (fluid mechanics, electromagnetism, special relativity…).

Thousands of years of practice in engineering or construction have proven to man that mechanics was a safe bet, well before the marvelous foundation and development of physics by Arab modern science, adopted then continued by Western scholars. The elevation of a cathedral like Notre-Dame de Paris would probably not have been possible in the Middle Ages if its static elements suddenly started to move by themselves, without any apparent causality, or if fires broke out spontaneously. Its overall safety can nonetheless still be threatened by the most 'benign' actions, as we know it today…

The same is true of the sovereign edifice of Newtonianism, patiently built since the 17th century. Norton's Dome, like a competing and proud vault of indeterminism, symbolizes the fury and effectiveness of the blows that can be dealt to it. Indeed one can wonder why classical physics only eliminates this kind of "acausal" solutions indirectly, namely by considering the dome in its entirety: on a simple asymmetrical half-profile of the dome one really only sees "fire".

Then, the successive destructions and re-edifications of the "sacred cathedral" of Newtonianism do not guarantee the durability of its character: with each repair, its original charm is lost a little more. Making its first principle a solid mathematical pillar could give the doctrine a new lease of life. Yet one cannot say for how long this architecture, constantly renovated, tested, patched up...will resist before a true final collapse.

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